Transmission Lines and Electronic Signal Handling

May 24, 2019

1 Background

A great many modern physics experiments involve sending and receiving electronic signals, sometimes over long distances. An electronic signal can be analog, which is simply a voltage as a function of time (*e.g.* the basis of cable television), or digital, which consists of a series of voltage pulses. For low-speed applications over short distances, for example audio signals between a CD player and a set of speakers, one doesn't need to worry much about signal transmission - a cable can be thought of transmitting a signal instantaneously, so the voltage at one end is equal to the voltage a the other. However, electronic signals cannot propagate faster than the sped of light, roughly a meter in three nanoseconds, and at GHz frequencies we have to worry about signal transmission even over laboratory distances. For these high-speed and/or long-distance applications one must take into account the fact that the voltage on one end of a cable is not equal to the voltage at the other end, simply because it takes some time for any voltage changes (such as a pulse) to propagate along the cable. Thus we need to think of a cable as a transmission line.

It is worth noting at the beginning that essentially everything we'll talk about in this lab relating to electronic cables carries over pretty nicely to optical cables, *a.k.a.* fiber optics. Both transmit electromagnetic signals, and in both cases we will have to worry about how the signals are transmitted and whether pulses are reflected at the cable ends. Usually signal reflections from the ends of cables are undesirable, and usually they can be avoided with a small amount of care.

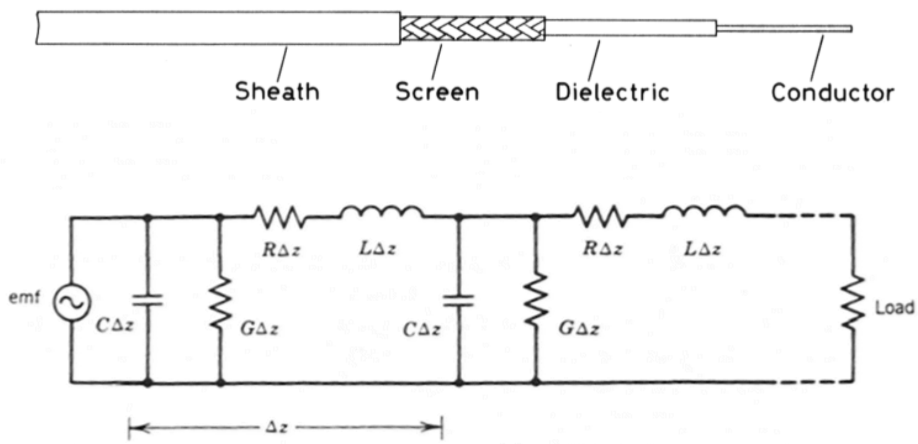

Figure 1: Schematic diagram of a coaxial cable (above), along with its equivalent circuit. (The convention here is that per unit length R is the series resistance and G is the parallel conductance, so the parallel resistance is $1/G$. For a near-ideal cable both R and G are small).

Our standard electronic transmission line is the coaxial cable, shown in Figure 1. These are like the cables used for cable television, and are often called BNC cables by physicists, after the Berkeley Nucleonics Corporation, which was a big manufacturer of coaxial cables and connectors in the early days. A coaxial cable consists of a center conductor, a dielectric spacer, and a concentric outer conductor. At low frequencies a current flows down the center conductor and returns via the outer conductor, and about all we have to worry about is the resistance of the wire conductors (R in Figure 1) and the cable capacitance, which is typically 50-150 picofarads/meter. The cable also has some inductance, but this is usually negligible at low frequencies (the inductive impedance, $Z = i\omega L$, being proportional to ω, gets smaller at low frequencies). At high frequencies we cannot ignore the inductance or capacitance of the cable, and we have to solve the full system for the voltage and current as a function of both time and position along the cable.

1.1 The Ideal (Lossless) Coaxial Cable

If we ignore the series and parallel resistances in the cable ($R \to 0$ and $G \to 0$ in Figure 1) and look at a small section in the middle of a long cable, then the voltage between the conductors is

$$V = \frac{q\Delta z}{C\Delta z} = \frac{q}{C}$$

where q is the charge per unit length on the conductors and C is the cable capacitance per unit length (see Figure 1). The rate of change of charge is the current, so

$$\Delta I = -\frac{\partial(q\Delta z)}{\partial t}$$

and thus

$$\begin{aligned} \frac{\partial I}{\partial z} &= -\frac{\partial q}{\partial t} \\ &= -C\frac{\partial V}{\partial t} \end{aligned}$$

The voltage drop along the cable comes from the cable inductance, giving

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

Differentiating both these expressions to eliminate I then yields

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V}{\partial z^2}$$

which should be recognizable as a standard wave equation. The reader can verify that the same expression is obtained for I. Thus the cable supports electromagnetic waves that propagate along the cable, at a propagation speed

$$v_{prop} = \frac{1}{\sqrt{LC}}$$

The complete solution of the wave equation yields forward and backward propagating waves

$$V(z,t) = F_{+}\left(t - \frac{z}{v}\right) + F_{-}\left(t + \frac{z}{v}\right)$$

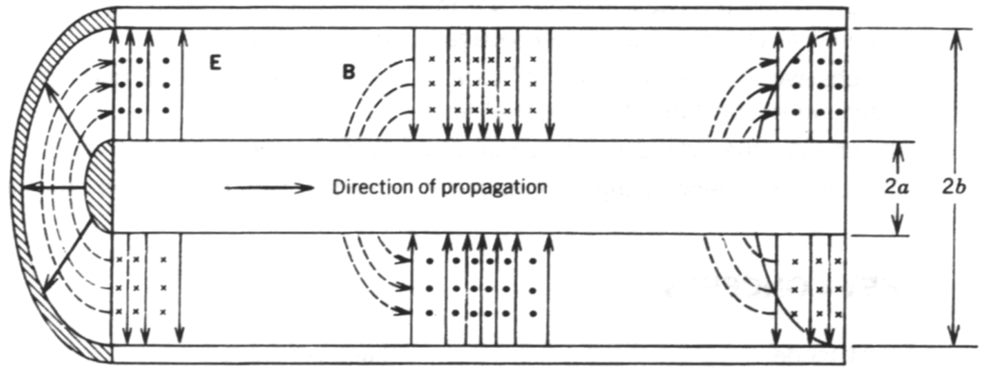

Figure 2: A section of a coaxial cable with the instantaneous electric and magnetic fields that are present when a sinusoidal signal is propagating down the cable. These fields move down the cable at the propagation speed.

where F_+ and F_- are arbitrary functions. The reader can show that with this $V(x,t)$ the corresponding current is

$$I(z,t) = \frac{V(z,t)}{Z_c}$$

where $Z_c = \sqrt{L/C}$ is called the *characteristic impedance* of the cable, and for a lossless cable Z_c is purely resistive (*i.e.* it's a real number). Figure 2 shows a cross section of a coaxial cable along with the electric and magnetic fields present when a sine wave signal is propagating down the cable.

The reader can also show (remember Ph1?) that the capacitance and inductance of the cable are

$$C = \frac{2\pi\epsilon}{\ln(b/a)} \text{ (F/m)}$$

$$L = \frac{\mu}{2\pi}\ln(b/a) \text{ (H/m)}$$

where a and b are shown in Figure 2. Thus the velocity of transmission, at least for this ideal (lossless) cable, is equal to

$$v_{prop} = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{\mu\epsilon}}$$

Note if there is no material between the (lossless) conductors, then $\mu = \mu_0$, $\epsilon = \epsilon_0$, and $v_{prop} = 1/\sqrt{\mu_0\epsilon_0} = c$, the speed of light.

1.2 The Not-so-Ideal Coaxial Cable

For the case that R and G are not negligible the equations are a bit more complex, and the wave equation for the signal voltage becomes

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG + RC)\frac{\partial V}{\partial t} + RGV \qquad (1)$$

which is the most general wave equation for any transmission line having the equivalent circuit shown in Figure 1. If we look for wave solutions to this equation, so that $V(z,t) = V(z)e^{i\omega t}$, then the general solution is

$$V(z,t) = V_1 e^{\alpha z} e^{i(\omega t + kz)} + V_2 e^{-\alpha z} e^{i(\omega t - kz)} \qquad (2)$$

where

$$\alpha = \Re(\gamma)$$
$$k = \Im(\gamma)$$

with

$$\gamma = \sqrt{(R + i\omega L)(G + i\omega C)}$$

We see the solution is a traveling wave with velocity $v = \omega/k$ that is substantially attenuated over distances large than $\sim 1/\alpha$. The solution for the current is still

$$I(z,t) = \frac{V(z,t)}{Z_c}$$

where now the characteristic impedance is given by

$$\begin{aligned} Z_c &= \sqrt{\frac{R + i\omega L}{G + i\omega C}} \\ &\approx \sqrt{\frac{L}{C}}\left[1 + i\left(\frac{G}{2\omega C} - \frac{R}{2\omega L}\right)\right] \end{aligned}$$

where the latter holds for small R, G.

Problem 1. Derive Eqn. 1 and the parameters for the wave-like solution Eqn. 2.

A major contributor to cable losses is the *skin effect*, which causes the current in the cable conductors to confine itself to a thinner and thinner layer near the conductor surface as the frequency is increased. The effective cross-sectional area of the conductor is then reduced, and the resistance subsequently increased, as the frequency increases. For a coaxial cable, this results in R being proportional to $\omega^{1/2}$, and so $\alpha \sim \omega^{1/2}$ as well. At higher frequencies, typically above 1 GHz, leakage across the dielectric can become the dominant loss mechanism, giving a nonzero G which is proportional to ω.

Whenever the cable parameters depend on frequency, we can find that the propagation velocity and/or attenuation depends on frequency, which in turn results in *pulse dispersion* or *pulse distortion*. For most coaxial cables, C and L are essentially independent of frequency up to very high frequencies, so dispersion is not usually a huge problem. The attenuation does increase rapidly with frequency, however, which tends to round out the corners of square wave pulses, since the higher frequency components of the square wave damp away most quickly.

1.3 Reflections in Cable Transmission

The above formalism for transmission down a coaxial cable included the implicit assumption that the properties of the cable, specifically C, L, etc., were independent of z, and in this case we found traveling wave solutions. When we consider the ends of a cable, we then have to add some boundary conditions to the differential equation, and doing this gives us reflections. That is, when a traveling wave pulse hits the end of the cable (where it usually is connected to some other electronic device), some part of the pulse may get reflected and thus go traveling down the cable in the opposite direction.

Reflections occur whenever a traveling wave encounters a new medium in which the speed of propagation is different. For optical media reflections occur whenever there is a change in the index of refraction. In coaxial cables one gets reflections whenever there is a change in the characteristic impedance. To see how this works in detail, consider a long length of cable connected to a load with impedance Z_L, as shown in Figure 3. We can forget about the signal generator for now, and just look at the equations and boundary conditions at the load end of the cable. We will also neglect cable losses.

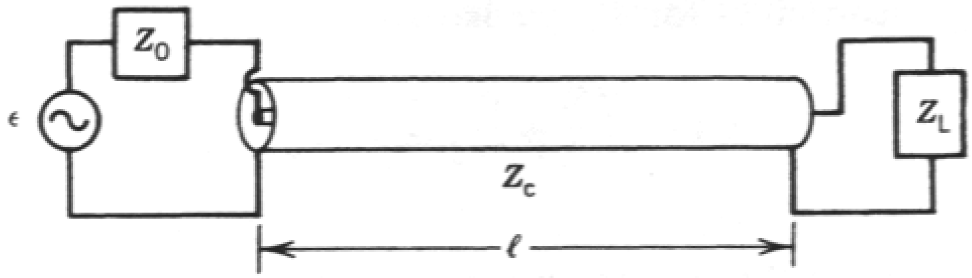

Figure 3: A signal generator (which is modeled here by a perfect voltage source ϵ in series with a "source impedance" Z_0) connected to a cable, which is in turn connected to a load with impedance Z_L.

With this the general solution for a traveling wave in the cable is

$$V(z,t) = V_1 e^{i(\omega t - kz)} + V_2 e^{i(\omega t + kz)}$$

where V_1 is the amplitude of the incident wave (wave fronts move in the $+z$ direction) and V_2 is the amplitude of the reflected wave. At the end of the cable the load impedance demands that

$$\begin{aligned} Z_L &= \frac{V(\ell,t)}{I(\ell,t)} \\ &= Z_c \left(\frac{V_1 e^{i(\omega t - kz)} + V_2 e^{i(\omega t + kz)}}{V_1 e^{i(\omega t - kz)} - V_2 e^{i(\omega t + kz)}} \right) \end{aligned}$$

(To see the sign flip in the denominator, recall from above that $\partial I/\partial z = -C\partial V/\partial t$ and integrate $\partial V/\partial t$ with respect to z. This is one of those subtle sign issues that has to be done right just once in order to get the right physics out.) Rearranging this gives

$$V_1 = e^{i2k\ell} V_2 \frac{Z_L + Z_c}{Z_L - Z_c}$$

so the ratio of the reflected wave to the incident wave is

$$\rho \equiv \frac{V_{\text{reflected}}(\ell,t)}{V_{\text{incident}}(\ell,t)} = \frac{V_2 e^{i(\omega t + kz)}}{V_1 e^{i(\omega t - kz)}} = \frac{Z_L - Z_c}{Z_L + Z_c}$$

where ρ is known as the reflection coefficient.

Problem 2. Reflections off the Source. The above treats reflections at the load end of the cable. Looking at Figure 3, we can also have leftward propagating pulses which reflect off the source. Show that the reflection coefficient in this case is simply

$$\rho = \frac{Z_0 - Z_c}{Z_0 + Z_c}$$

Hint: Kirchoff's laws tell us that

$$\epsilon(t) - Z_0 I(0,t) = V(0,t)$$

at the cable end. This relation holds when there is only an outgoing wave, and it holds again when there are three waves present: the outgoing wave, a leftward propagating wave, and a reflected wave. By superposition one can add and subtract solutions for these two cases.

1.4 Impedance Matching

When the load impedance equals the cable inpedance, $Z_L = Z_c$, then we have the desirable result that $\rho = 0$ and there is no reflected wave. In this case we say the load is *impedance matched* to the cable, and all the electromagnetic energy in the incident wave (or pulse) is absorbed in the load. Many devices are designed to use cables with a fixed Z_c, with some standards being $Z_c = 50\Omega$ or 75Ω. When both the input and output devices are matched with a cable and with each other, then there are no reflected pulses to worry about. One common exception to this practice is the oscilloscope, which has an input impedance of typically 1 MΩ, far higher than any standard coax cable. When using a 'scope to look at a signal at the end of a cable, one can *terminate* the cable by adding an additional 50Ω resistor (if $Z_c = 50\Omega$) in parallel with the 'scope. The 'scope hook-up then does not send reflections back to the source to cause trouble. (Note a 50Ω terminator is typically only used on a 'scope when one is in the high frequency/ long distance regime, in which case a cable acts like a transmission line with a 50Ω impedance. For low-speed/short distance work a cable acts like just a wire, and hooking a 50Ω resistor to one's electronic device could cause a lot of trouble all by itself.)

1.5 Reflections upon Reflections

The different cases of pulse reflection off the end of a transmission line are shown in Figure 4, where we see the voltage at the input end of a cable (the left side in Figure 3) as a function of time. At $t = 0$ a step function signal was generated at this end of the cable, with amplitude V_0. While the voltage step propagates down the cable the first time, there are no reflections, so the voltage stays fixed at V_0. After a time $t = 2\tau$, where τ is the time for a pulse to travel the length of the cable, we see the effects of the reflected signal. When the load impedance is $R = \infty$ (top graph), the step reflects without changing sign, so after $t = 2\tau$ the reflected signal adds to the input voltage, giving $V \approx 2V_0$ (slightly less if the signal is attenuated in the cable).

When $R = 0$ the step changes sign upon reflections (remember Ph12?), and so the reflected signal destructively interferes with the input voltage, so after $t = 2\tau$ we have $V \approx 0$ (second graph). When $R = Z_c$ (bottom graph) there is no reflected step. If the reflected step reflects again off the left side of the cable and makes another round trip, then the voltage after $t = 4\tau$ will change yet again. All these effects will be seen in the lab.

2 LABORATORY EXERCISES

The first step in the lab is to check out the signal generator and oscilloscope to make sure these things work the way you think they should, in the absence of any complications coming from transmission lines. Check with a TA on where to find the various pieces of hardware you will need, as well as help finding the more useful features of the digital electronics among all the many menu options.

Exercise 1. Use the signal generator to make a square wave signal with an amplitude of one volt and a frequency of 2 MHz (50 % duty cycle). Send the output directly into the 'scope using a short (less than two meters) BNC cable. Use the measurement option of the 'scope to measure the signal period, amplitude, and the rise time of the square wave leading edge. With the measurements displayed on the screen, make a hardcopy of the square wave signal. Given that the signal generator only goes up to 21.5 MHz, how does your measured risetime compare with expectations? Next connect the signal generator to the 'scope with the spool of 100-feet RG-174 cable, just to observe the signal degradation. Make another hardcopy of this signal.

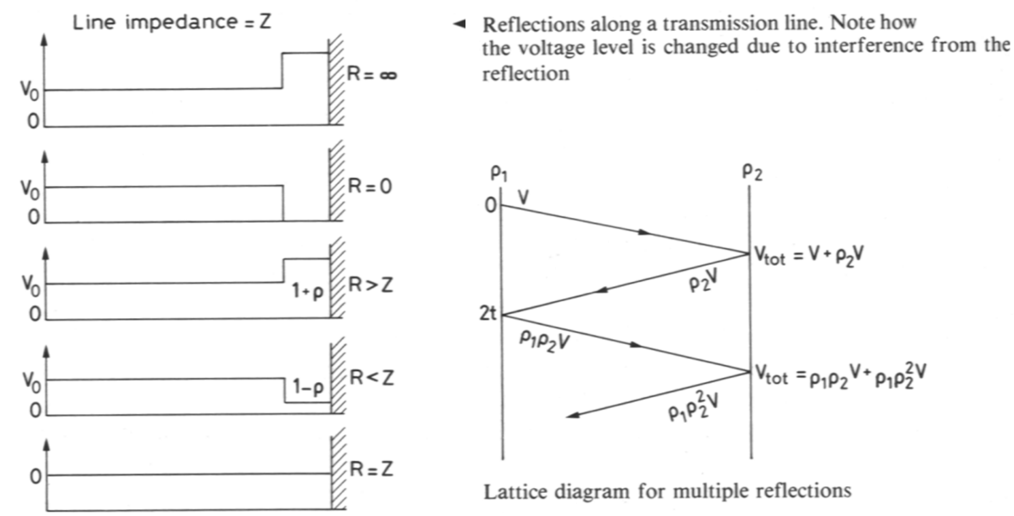

Figure 4: On the left we see different cases of a step function voltage signal propagating down a line and reflecting from its end, just after the edge reflected. With multiple reflections (right) the resulting signal can be quite complex.

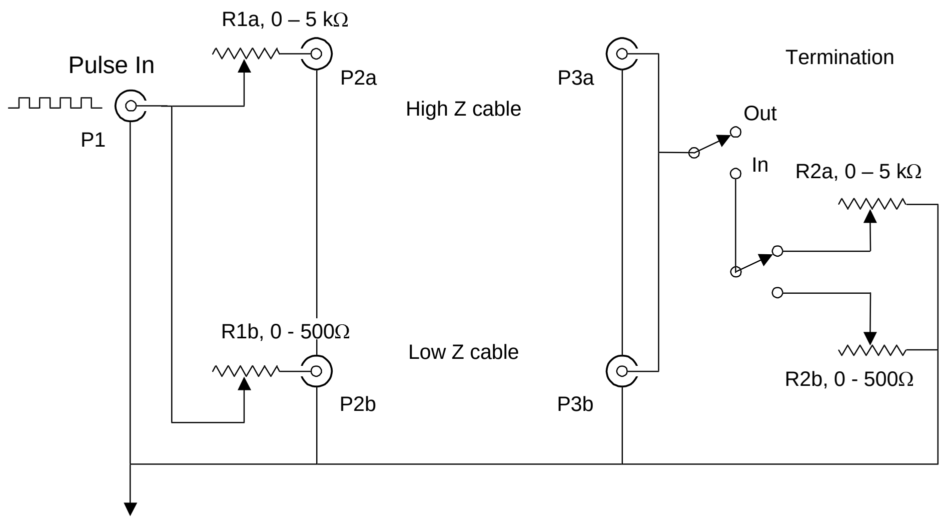

Figure 5: Schematic diagram of the connection panel for the coaxial cable measurements.

The next step is to add a transmission line using the connection panel provided (See Figure 5). Use the signal generator to make a 10 kHz square wave signal (with 50 % duty cycle) with an amplitude of 10 volts, and connect it to the Pulse In input of the panel. Connect the P2a and P3a connectors with the 500-feet RG-59 B/U cable. Use the 10X oscilloscope probe to look at point P2a by clipping onto the wires on the back of the panel. (See the Appendix below for a description of what the 'scope probe does for you.)

Exercise 2. Short the far end of the cable by setting the termination switches to hook up R2a, and set the resistance to zero ohms. (You can use an ohmmeter to check any of the resistance values - be sure to disconnect the input signal first.) Observe the signal at P2a as you vary the input impedance R1a. Make hardocpies when R1a is equal to 0Ω, $5k\Omega$ (the two extremes), around $\sim 25\Omega$ (optimize the value to produce a single short pulse), and $\sim 200\Omega$. Make sure in your hardcopies that the voltage and time scales are clearly displayed, so you'll actually know what's plotted when you look at it later. You might want to use the cursor function on the 'scope to add some markers to the display. *Clearly label the plots* and tape them into your notebook. Explain the different waveforms qualitatively. (You may be *brief* in your explanations. Start by explaining the amplitude and sign of the voltage when $0 < t < 2\tau$ for each of the four plots. Next explain the voltages when $2\tau < t < 4\tau$, etc.)

Exercise 3. With the far end of the cable still shorted, set R1a so that the input impedance matches Z_c, so there are no reflections at the input. Measure R1a at this setting using an ohmmeter. Measure the width of the short pulse and derive the pulse transmission velocity, given that the cable length is 4.6 meters. Measure the baseline signals before and after the short pulse, and from this estimate the attenuation constant α for the cable.

Exercise 4. Set the far end of the cable to an open circuit by setting the termination switch to Out, and again observe the signal at P2a as you vary R1a. Make hardcopies at roughly the same four values as above, and again qualitatively explain the observed signals.

Exercise 5. Set the termination switch to In, and observe the signal at P2a as both R1a and R2a are changed. Play with the setting in order to get a good impedance match at the termination end of the cable, and then measure the impedance-matched R2a value, which is a good estimate of Z_c. Compare this with your impedance-matched value of R1a (from Exercise 3). With care you should get nearly the same resistance for both, but you may see the difference that comes about because R1a is in series with the input impedance of the signal generator (see the discussion of Figure 3 above). If the two values don't agree very well, try both measurements again, and this time crank the gain on the 'scope to zoom in on the relevant part of the signal. Although there will be some ambiguity in what constitutes zero reflected pulse, you can estimate the uncertainty in your measurement by setting each resistor value to points that are clearly a bit too high, and then clearly a bit too low, and measuring these values in addition to the impedance-matched value.

As a final step with this cable, you should look at the signal at the far end of the cable, P3a, as R1a and R2a are varied. Note that a good pulse shape is obtained if reflections are suppressed at either end of the cable, as you would expect.

Exercise 6. Do similar measurements using the RG-58/U (100 feet) and RG-174 (100 feet) cables, and use all your measurements to complete the following table:

	$Z_c(\Omega)$	v/c	C (pf/meter)	α (meter^{-1})
RG-59B/U				
RG-58/U				
RG-174				

(Note if you measure the characteristic impedance of the RG-62/U cable by looking at reflections from both ends of the cable, like in Exercise 5, you should clearly see a difference that comes from the output impedance of the signal generator.)

References

[1] W. R. Leo, *Techniques for Nuclear and Particle Physics Experiments, 2nd Revised Edition*, 1994 (Springer-Verlag).

[2] D. W. Preston and E. R. Dietz, *The Art of Experimental Physics*, 1991 (John Wiley & Sons).

A The Oscilloscope Probe

When using an oscilloscope to look at an electronic signal one often has to worry about whether the 'scope unintentionally affects the signal - an effect called *circuit loading.* A typical 'scope has an input resistance of about 1MΩ and an input capacitance of about 20 pf, as shown in Figure 6. In other words, the minimum effect of hooking up a 'scope will be roughly like attaching the monitored circuit point to a 1MΩ resistor to ground, in parallel with a 20 pf capacitor. When working with very high speed digital electronics even this load could change the circuit performance, in which case you can't hook up a ' scope directly. But for a great many applications 1MΩ and 20 pdf together don't draw much current, and the 'scope gives a pretty accurate reading of whatever signal is present tin the circuit. (Occasionally the author has found, when debugging, a circuit, that the darn thing works when the 'scope is attached looking at the signal, but then stops working when the 'scope is removed. Replacing the 'scope with a small capacitor can then sometimes fix the circuit! About the only time this works is when the circuit gets stuck in some unwanted high-frequency oscillation, since even a small capacitor to ground will suck the energy out of a high-frequency signal.)

Circuit loading can get much worse when a cable is inserted between the circuit and the 'scope, since a typical BNC cable has a capacitance of around 50150 picofarads/meter. When working with low frequency circuits, say ¡1 MHz, circuit loading may not matter even with a good length of cable, since the DC resistance is still equal to the 1MΩ resistance of the 'scope, and the impedance of the cable capacitance ($Z = 1/i\omega C$) might still be several kiloOhms, and this is maybe not a problem. But if your signals are at frequencies above 1 MHz, chances are a 'scope with a simple BNC cable will cause trouble. For such circumstances one uses an *oscilloscope probe*, which is a device designed to minimize unintentional circuit loading.

One of the most common oscilloscope probes is the *10X passive probe*, which we use in this lab, and a circuit diagram is shown in Figure 6. The basic idea of this kind of 'scope probe is to put a large resistor, typically 9MΩ, right at the tip of the probe, so the cable capacitance is isolated from the circuit. There is always some capacitance a the probe tip, but some effort goes into making it small. Then after a fixed length of low-capacitance cable there is an equalizer box which connects to the 'scope input. Note the probe resistance and the 'scope resistance make a resistor divider that reduces the signal amplitude by a factor of ten, which is why it's called a 10X probe. By reducing the signal amplitude one greatly reduces the circuit loading.

The equalizer box contains a small capacitor and perhaps a resistor (there are different versions), and is designed to work together with the cable and the 'scope to produce a constant 10X reduction in signal amplitude, independent of frequency. The details of the design depend on the cable and the 'scope. Some probes are designed to work only with specific 'scopes, some probes can be adjusted for different 'scopes. A properly adjusted probe is said to be *frequency compensated*. Without compensation a square wave signal (which contains lots of different frequencies) would appear as a distorted square wave on the 'scope.

In addition to the common 10X passive probe, one also runs across 1X passive probes. These are basically the same as 10X probes, but with a much smaller tip resistor and a low-capacitance cable. This probe provides essentially no reduction in signal, but of course there is greater circuit loading (but less loading than from a bare BNC cable). Finally, you can buy *active probes*, which, as their name implies, have high input impedance amplifiers in their tips. Active probes can give much better performance than passive probes, which is needed for very demanding applications(*e.g.* cell phone receivers and associated electronics, etc.).

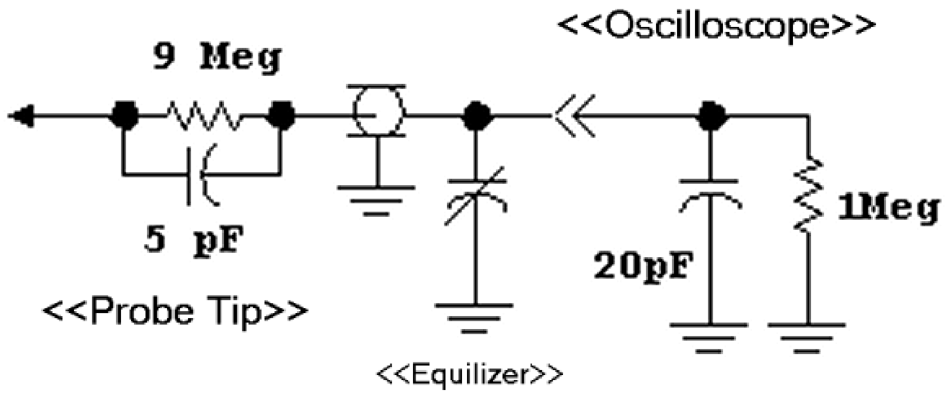

Figure 6: Schematic diagram of a typical oscilloscope 10X probe. The probe tip contains a large resistor in parallel with a small stray capacitance. The oscilloscope (right part of drawing) looks to the outside world like a resistor and capacitor in parallel.