Contents

[1]See MTW Chap. 16.
[2]See MTW Chap. 17.

Chapter 24

Fundamental Concepts of General Relativity

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Box 24.1
Reader's Guide

- This chapter relies significantly on
  - The special relativity portions of Chap. 1.
  - Chapter 22, on the transition from special relativity to general relativity.
- This chapter is a foundation for the applications of general relativity theory in Chaps. 24–26.

24.1 Overview

Newton's theory of gravity is logically incompatible with the special theory of relativity: Newtonian gravity presumes the existence of a universal, frame-independent 3-dimensional space in which lives the Newtonian potential Φ, and a universal, frame-independent time t with respect to which the propagation of Φ is instantaneous. Special relativity, by contrast, insists that the concepts of time and of 3-dimensional space are frame-dependent, so that instantaneous propagation of Φ in one frame would mean non-instantaneous propagation in another.

The most straightforward way to remedy this incompatibility is to retain the assumption that gravity is described by a scalar field Φ, but modify Newton's instantaneous, action-at-a-distance field equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 4\pi G\rho \tag{24.1}$$

(where G is Newton's gravitation constant and ρ is the mass density) to read

$$\vec{\nabla}^2\Phi \ \equiv \ g^{\alpha\beta}\Phi_{;\alpha\beta} = -4\pi G T^\mu{}_\mu \ , \tag{24.2}$$

where $\vec{\nabla}^2$ is the squared gradient, or d'Alembertian in Minkowski spacetime and $T^\mu{}_\mu$ is the trace (contraction on its slots) of the stress-energy tensor. This modified field equation at first sight is attractive and satisfactory (but see Ex. 24.1, below): (*i*) It satisfies Einstein's Principle of Relativity in that it is expressed as a geometric, frame-independent relationship between geometric objects; and (*ii*) in any Lorentz frame it takes the form [with factors of $c =$ (speed of light) restored]

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = \frac{4\pi G}{c^2}(T^{00} - T^{xx} - T^{yy} - T^{zz}) \ , \tag{24.3}$$

which, in the kinds of situation contemplated by Newton [energy density predominantly due to rest mass, $T^{00} \cong \rho c^2$; stress negligible compared to rest mass-energy, $|T^{jk}| \ll \rho c^2$; and $1/c \times$ (time rate of change of Φ) negligible compared to spatial gradient of Φ], reduces to the Newtonian field equation (24.1).

Not surprisingly, most theoretical physicists in the decade following Einstein's formulation of special relativity (1905–1915) presumed that gravity would be correctly describable, within the framework of special relativity, by this type of modification of Newton's theory, or something resembling it. For a brief historical account see Chap. 13 of Pais (1982). To Einstein, by contrast, it seemed clear as early as 1907 that the correct description of gravity should involve a generalization of special relativity rather than an incorporation into special relativity: Since an observer in a local, freely falling reference frame near the earth should not feel any gravitational acceleration at all, local freely falling frames (*local inertial frames*) should in some sense be the domain of special relativity, and gravity should somehow be described by the *relative acceleration* of such frames.

Although the seeds of this idea were in Einstein's mind as early as 1907 (see the discussion of the equivalence principle in Einstein, 1907), it required eight years for him to bring them to fruition. A first crucial step, which took half the eight years, was for Einstein to conquer his initial aversion to Minkowski's (1908) geometric formulation of special relativity, and to realize that a curvature of Minkowski's 4-dimensional spacetime is the key to understanding the relative acceleration of freely falling frames. The second crucial step was to master the mathematics of differential geometry, which describes spacetime curvature, and using that mathematics to formulate a logically self-consistent theory of gravity. This second step took an additional four years and culminated in Einstein's (1915, 1916) general theory of relativity. For a historical account of Einstein's eight-year struggle toward general relativity see, e.g., Part IV of Pais (1982); and for selected quotations from Einstein's technical papers during this eight-year period, which tell the story of his struggle in his own words, see Sec. 17.7 of MTW.

It is remarkable that Einstein was led, not by experiment, but by philosophical and aesthetic arguments, to reject the incorporation of gravity into special relativity [Eqs. (24.2) and (24.3) above], and insist instead on describing gravity by curved spacetime. Only after the full formulation of his general relativity did experiments begin to confirm that he was right and that the advocates of special-relativistic gravity were wrong, and only more than 50 years after general relativity was formulated did the experimental evidence become extensive and strong. For detailed discussions see, e.g., Will (1981, 1986), and Part 9 of MTW.

The mathematical tools, the diagrams, and the phrases by which we describe general relativity have changed somewhat in the seventy years since Einstein formulated his theory; and, indeed, we can even assert that we understand the theory more deeply than did Einstein. However, the basic ideas are unchanged; and general relativity's claim to be the most elegant and aesthetic of physical theories has been reinforced and strengthened by our growing insights.

General relativity is not merely a theory of gravity. Like special relativity before it, the general theory is a framework within which to formulate all the laws of physics, classical and quantum—but now with gravity included. However, there is one remaining, crucial, gaping hole in this framework: It is incapable of functioning, indeed it fails completely, when conditions become so extreme that space and time themselves must be quantized. In those extreme conditions general relativity must be married in some deep, as-yet-ill-understood way, with quantum theory, to produce an all-inclusive quantum theory of gravity—a theory which, one may hope, will be a "theory of everything." To this we shall return, briefly, in Chaps. 24 and 26.

In this chapter we present, in modern language, the foundations of general relativity. Our presentation will rely heavily on the concepts, viewpoint, and formalism developed in Chaps. 1 and 22.

We shall begin in Sec. 24.2 with a discussion of three concepts that are crucial to Einstein's viewpoint on gravity: a local Lorentz frame (the closest thing there is, in the presence of gravity, to special relativity's "global" Lorentz frame), the extension of the principle of relativity to deal with gravitational situations, and Einstein's equivalence principle by which one can "lift" laws of physics out of the flat spacetime of special relativity and into the curved spacetime of general relativity. In Sec. 24.3 we shall see how gravity prevents the meshing of local Lorentz frames to form global Lorentz frames, and shall infer from this that spacetime must be curved. In Sec. 24.4 we shall lift into curved spacetime the law of motion for free test particles, and in Sec. 24.5 we shall see how spacetime curvature pushes two freely moving test particles apart and shall use this phenomenon to make contact between spacetime curvature and the Newtonian "tidal gravitational field" (gradient of the Newtonian gravitational acceleration). In Sec. 24.6 we shall study a number of mathematical and geometric properties of the tensor field that embodies spacetime curvature: the Riemann tensor. In Sec. 24.7 we shall examine "curvature coupling delicacies" which plague the lifting of laws of physics from flat spacetime to curved spacetime. In Sec. 24.8 we shall meet the Einstein field equation, which describes the manner in which spacetime curvature is produced by the total stress-energy tensor of all matter and nongravitational fields. In Ex. 24.12 we shall examine in some detail how Newton's laws of gravity arise as a weak-gravity limit of general relativity. Finally, in Sec. 24.9 we shall examine the conservation laws for energy, momentum, and angular momentum of gravitating bodies that live in "asymptotically flat" regions of spacetime.

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EXERCISES

Exercise 24.1 *Example: A Special Relativistic, Scalar-Field Theory of Gravity*
Equation (24.2) is the field equation for a special relativistic theory of gravity with gravitational potential Φ. To complete the theory one must describe the forces that the field Φ produces on matter.

(a) One conceivable choice for the force on a test particle of rest mass m is the following generalization of the familiar Newtonian expression:

$$\nabla_{\vec{u}}\vec{p} = -m\vec{\nabla}\Phi \; ; \quad \text{i.e.,} \; \frac{dp^\alpha}{d\tau} = -m\Phi^{,\alpha} \; \text{ in a Lorentz frame,} \tag{24.4}$$

where τ is proper time along the particle's world line, $\vec{p}$ is the particle's 4-momentum, $\vec{u}$ is its 4-velocity, and $\vec{\nabla}\Phi$ is the spacetime gradient of the gravitational potential. Show that this equation of motion reduces, in a Lorentz frame and for low particle velocities, to the standard Newtonian equation of motion. Show, however, that this equation of motion is flawed in that the gravitational field will alter the particle's rest mass—in violation of extensive experimental evidence that the rest mass of an elementary particle is unique and conserved.

(b) Show that the above equation of motion, when modified to read

$$\nabla_{\vec{u}}\vec{p} = -(\mathbf{g} + \vec{u}\otimes\vec{u})\cdot m\vec{\nabla}\Phi \; ;$$
$$\text{i.e.,} \; \frac{dp^\alpha}{d\tau} = -(g^{\alpha\beta} + u^\alpha u^\beta)m\Phi_{,\beta} \; \text{ in a Lorentz frame,} \tag{24.5}$$

preserves the particle's rest mass. In this equation of motion $\vec{u}$ is the particle's 4-velocity, and $\mathbf{g} + \vec{u}\otimes\vec{u}$ projects $\vec{\nabla}\Phi$ into the "3-space" orthogonal to the particle's world line; cf. Fig. 22.6.

(c) Show, by treating a zero-rest-mass particle as the limit of a particle of finite rest mass ($\vec{p} = m\vec{u}$ and $\zeta = \tau/m$ finite as τ and m go to zero), that the above theory predicts that in any Lorentz reference frame $p^\alpha e^\Phi$ (with $\alpha = 0, 1, 2, 3$) are constant along the zero-rest-mass particle's world line. Explain why this prediction implies that there will be no deflection of light around the limb of the sun, which conflicts severely with experiments that were done *after* Einstein formulated his general theory of relativity. (There was no way, experimentally, to rule out the above theory in the epoch, ca. 1914, when Einstein was doing battle with his colleagues over whether gravity should be treated within the framework of special relativity or should be treated as a geometric extension of special relativity.)

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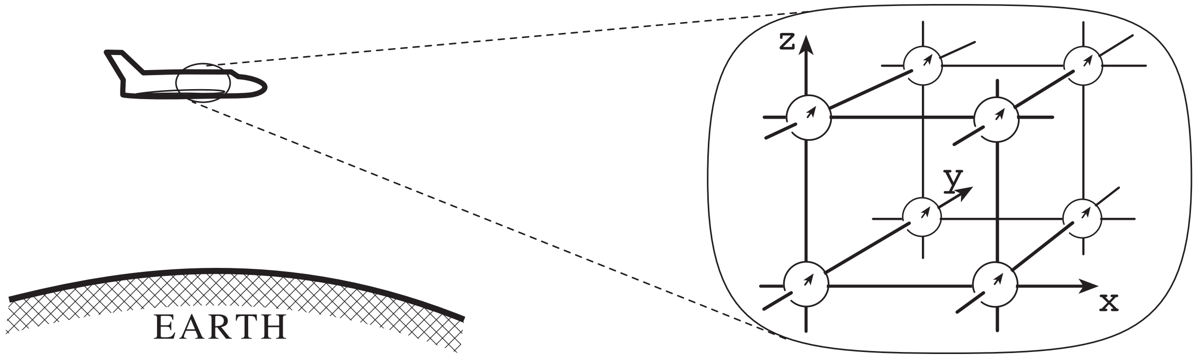

Fig. 24.1: A local inertial frame (local Lorentz frame) inside a space shuttle that is orbiting the earth.

24.2 Local Lorentz Frames, the Principle of Relativity, and Einstein's Equivalence Principle

One of Einstein's greatest insights was to recognize that special relativity is valid not globally, but only locally, inside locally freely falling (inertial) reference frames. Figure 24.1 shows a specific example of a *local inertial frame*: The interior of a space shuttle in earth orbit, where an astronaut has set up a freely falling (from his viewpoint "freely floating") latticework of rods and clocks. This latticework is constructed by all the rules appropriate to a special relativistic, inertial (Lorentz) reference frame (Sec. 1.2): (*i*) the latticework moves freely through spacetime so no forces act on it, and its rods are attached to gyroscopes so they do not rotate; (*ii*) the measuring rods are orthogonal to each other, with their intervals of length uniform compared, e.g., to the wavelength of light (orthonormal lattice); (*iii*) the clocks are densely packed in the lattice, they tick uniformly relative to ideal atomic standards (they are ideal clocks), and they are synchronized by the Einstein light-pulse process. However, there is one crucial change from special relativity: The latticework must be *small enough* that one can neglect the effects of inhomogeneities of gravity (which general relativity will associate with spacetime curvature; and which, for example, would cause two freely floating particles, one nearer the earth than the other, to gradually move apart even though initially they are at rest with respect to each other). The necessity for smallness is embodied in the word "local" of "local inertial frame", and we shall quantify it with ever greater precision as we move on through this chapter.

We shall use the phrases *local Lorentz frame* and *local inertial frame* interchangeably to describe the above type of synchronized, orthonormal latticework; and the spacetime coordinates t, x, y, z that the latticework provides (in the manner of Sec. 1.2) we shall call, interchangeably, *local Lorentz coordinates* and *local inertial coordinates*.

Since, in the presence of gravity, inertial reference frames must be restricted to be local, the inertial-frame version of the *principle of relativity* must similarly be restricted to say: *All the* **local**, *nongravitational laws of physics are the same in every* **local** *inertial frame, everywhere and everywhen in the universe.* Here, by "local" laws we mean those laws, classical or quantum, which can be expressed entirely in terms of quantities confined to (measurable within) a local inertial frame; and the exclusion of gravitational laws from this version of the principle of relativity is necessary because gravity is to be described by a curvature of spacetime which (by definition, see below) cannot show up in a local inertial frame. This version of the principle of relativity can be described in operational terms using precisely the same language as for the special relativistic version (Secs. 22.2.1 and 22.2.2): If two different observers, in two different local Lorentz frames, in different (or the same) regions of the universe and epochs of the universe, are given identical written instructions for a self-contained physics experiment (an experiment that can be performed within the confines of the local Lorentz frame), then their two experiments must yield the same results, to within their experimental accuracies.

It is worth emphasizing that the principle of relativity is asserted to hold everywhere and everywhen in the universe: the local laws of physics must have the same form in the early universe, a fraction of a second after the big bang, as they have on earth today, and as they have at the center of the sun or inside a black hole.

It is reasonable to expect that *the specific forms that the local, nongravitational laws of physics take in general relativistic local Lorentz frames are the same as they take in the (global) Lorentz frames of special relativity*. The assertion that this is so is a modern version of *Einstein's equivalence principle*. In the next section we will use this principle to deduce some properties of the general relativistic spacetime metric; and in Sec. 24.7 we will use it to deduce the explicit forms of some of the nongravitational laws of physics in curved spacetime.

24.3 The Spacetime Metric, and Gravity as a Curvature of Spacetime

The Einstein equivalence principle guarantees that nongravitational physics within a local Lorentz frame can be described using a spacetime metric **g**, which gives for the invariant interval between neighboring events with separation vector $\vec{\xi} = \Delta x^\alpha \partial/\partial x^\alpha$, the standard special relativistic expression

$$\vec{\xi}^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta = (\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \ . \tag{24.6}$$

Correspondingly, in a local Lorentz frame the components of the spacetime metric take on their standard special-relativity values

$$g_{\alpha\beta} = \eta_{\alpha\beta} \equiv \{-1 \text{ if } \alpha=\beta=0 \ , \ \ +1 \text{ if } \alpha=\beta=(x, \text{ or } y, \text{ or } z), \ \ 0 \text{ otherwise}\} \ . \tag{24.7}$$

Turn, now, to a first look at the gravity-induced constraints on the size of a local Lorentz frame: Above the earth set up, initially, a family of local Lorentz frames scattered over the entire region from two earth radii out to four earth radii, with all the frames initially at rest with respect to the earth [Fig. 24.2(a)]. From experience—or, if you prefer, from Newton's theory of gravity which after all is quite accurate near earth—we know that as time passes these frames will all fall toward the earth. If (as a pedagogical aid) we drill holes through the earth to let the frames continue falling after reaching the earth's surface, the frames will all pass through the earth's center and fly out the earth's opposite side.

(a) (b)

Fig. 24.2: (a) A family of local Lorentz frames, all momentarily at rest above the earth's surface. (b) A family of local, 2-dimensional Euclidean coordinate systems on the earth's surface. The nonmeshing of Lorentz frames in (a) is analogous to the nonmeshing of Euclidean coordinates in (b) and motivates attributing gravity to a curvature of spacetime.

Obviously, two adjacent frames, which initially were at rest with respect to each other, acquire a relative velocity during their fall, which causes them to interpenetrate and pass through each other as they cross the earth's center. Gravity is the cause of this relative velocity.

If these two adjacent frames could be meshed to form a larger Lorentz frame, then as time passes they would always remain at rest relative to each other. Thus, a meshing to form a larger Lorentz frame is impossible. The gravity-induced relative velocity prevents it. In brief: *Gravity prevents the meshing of local Lorentz frames to form global Lorentz frames*.

This situation is closely analogous to the nonmeshing of local, 2-dimensional, Euclidean coordinate systems on the surface of the earth [Figure 24.2(b)]: The curvature of the earth prevents a Euclidean mesh—thereby giving grief to map makers and surveyors. This analogy suggested to Einstein, in 1912, a powerful new viewpoint on gravity: Just as the curvature of space prevents the meshing of local Euclidean coordinates on the earth's surface, so it must be that a curvature of spacetime prevents the meshing of local Lorentz frames in the spacetime above the earth—or anywhere else in spacetime, for that matter. And since it is already known that gravity is the cause of the nonmeshing of Lorentz frames, it must be that *gravity is a manifestation of spacetime curvature*.

To make this idea more quantitative, consider, as a pedagogical tool, the 2-dimensional metric of the earth's surface expressed in terms of a spherical polar coordinate system and in "line-element" form:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 \ . \tag{24.8}$$

Here R is the radius of the earth, or equivalently the "radius of curvature" of the earth's surface. This line element, rewritten in terms of the alternative coordinates

$$x \equiv R\phi \ , \quad y \equiv R\left(\frac{\pi}{2} - \theta\right) \ , \tag{24.9}$$

has the form

$$ds^2 = \cos^2(y/R)dx^2 + dy^2 = dx^2 + dy^2 + \mathrm{O}(y^2/R^2)dx^2 \ , \tag{24.10}$$

where $\mathrm{O}(y^2/R^2)$ means "terms of order y^2/R^2" or smaller. Notice that the metric coefficients have the standard Euclidean form $g_{jk} = \delta_{jk}$ all along the equator ($y = 0$); but as one moves away from the equator, they begin to differ from Euclidean by fractional amounts of $O(y^2/R^2) = O[y^2/(\text{radius of curvature of earth})^2]$. Thus, local Euclidean coordinates can be meshed and remain Euclidean all along the equator—or along any other great circle—, but the earth's curvature forces the coordinates to cease being Euclidean when one moves off the chosen great circle, thereby causing the metric coefficients to differ from δ_{jk} by amounts $\Delta g_{jk} = O[(\text{distance from great circle})^2 \ / \ (\text{radius of curvature})^2]$.

Turn next to a specific example of curved spacetime: that of a "$k = 0$ Friedmann model" for our expanding universe (to be studied in depth in Chap. 26 below). In spherical coordinates (η, χ, θ, ϕ), the 4-dimensional metric of this curved spacetime, described as a line element, takes the form

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)] \ , \tag{24.11}$$

where a, the "expansion factor of the universe," is a monotonic increasing function of the "time" coordinate η. This line element, rewritten near $\chi = 0$ in terms of the alternative coordinates

$$t = \int_0^\eta a d\eta + \frac{1}{2}\chi^2 \frac{da}{d\eta} \ , \quad x = a\chi \sin\theta\cos\phi \ , \quad y = a\chi\sin\theta\sin\phi \ , \quad z = a\chi\cos\theta \ , \tag{24.12}$$

takes the form [cf. Ex. 24.2]

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta + O\left(\frac{x^2+y^2+z^2}{\mathcal{R}^2}\right) dx^\alpha dx^\beta \ , \tag{24.13}$$

where $\mathcal{R}$ is a quantity which, by analogy with the radius of curvature R of the earth's surface, can be identified as a "radius of curvature" of spacetime:

$$\frac{1}{\mathcal{R}^2} = O\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) \quad \text{where } \dot{a} \equiv \left(\frac{da}{dt}\right)_{x=y=z=0} , \quad \ddot{a} \equiv \left(\frac{d^2a}{dt^2}\right)_{x=y=z=0} . \tag{24.14}$$

From the form of the metric coefficients in Eq. (24.14) we see that all along the world line $x = y = z = 0$ the coordinates are precisely Lorentz, but as one moves away from that world line they cease to be Lorentz, and the metric coefficients begin to differ from $\eta_{\alpha\beta}$ by amounts $\Delta g_{\alpha\beta} = O[(\text{distance from the chosen world line})^2/(\text{radius of curvature of spacetime})^2]$. This is completely analogous to our equatorial Euclidean coordinates on the earth's surface. The curvature of the earth's surface prevented our local Euclidean coordinates from remaining Euclidean as we moved away from the equator; here the curvature of spacetime prevents our local Lorentz coordinates from remaining Lorentz as we move away from our chosen world line.

Notice that our chosen world line is that of the spatial origin of our local Lorentz coordinates. Thus, we can think of those coordinates as provided by a spatially tiny latticework of rods and clocks, like that of Figure 24.1; and the latticework remains locally Lorentz for all time (as measured by its own clocks), but it ceases to be locally Lorentz when one moves a finite spatial distance (in its own frame) away from its spatial origin. (This is analogous to the local Euclidean coordinates on the Earth's equator: they remain Euclidean all along the equator [Eq. (24.10)], going all around the world, but they deviate from Euclidean when one moves away from the equator.)

This behavior is generic. One can show that, if any freely falling observer, anywhere in spacetime, sets up a little latticework of rods and clocks in accord with our standard rules and keeps the latticework's spatial origin on his or her free-fall world line, then the coordinates provided by the latticework will be locally Lorentz, with metric coefficients

$$\boxed{g_{\alpha\beta} = \left\{ \begin{matrix} \eta_{\alpha\beta} + \mathrm{O}\left(\frac{\delta_{jk}x^j x^k}{\mathcal{R}^2}\right) \\ \eta_{\alpha\beta} \quad \text{at spatial origin} \end{matrix} \right\} \text{in a local Lorentz frame,}} \tag{24.15}$$

where $\mathcal{R}$ is the radius of curvature of spacetime. Notice that because the deviations of the metric from $\eta_{\alpha\beta}$ are second order in the distance from the spatial origin, the first derivatives of the metric coefficients are of first order, $g_{\alpha\beta,\mu} = \mathrm{O}(x^j/\mathcal{R}^2)$. This, plus the vanishing of the commutation coefficients in our coordinate basis, implies that the connection coefficients of the local Lorentz frame's coordinate basis are

$$\boxed{\Gamma^{\alpha}{}_{\beta\gamma} = \left\{ \begin{matrix} \mathrm{O}\left(\frac{\sqrt{\delta_{jk}x^j x^k}}{\mathcal{R}^2}\right) \\ 0 \quad \text{at spatial origin} \end{matrix} \right\} \text{in a local Lorentz frame.}} \tag{24.16}$$

It is instructive to compare Eq. (24.15) for the metric in the local Lorentz frame of a freely falling observer in curved spacetime with Eq. (22.83) for the metric in the proper reference frame of an accelerated observer in flat spacetime. Whereas the spacetime curvature in (24.15) produces corrections to $g_{\alpha\beta} = \eta_{\alpha\beta}$ of *second* order in distance from the world line, the acceleration and spatial rotation of the reference frame in (22.83) produces corrections of *first* order. This remains true when one studies accelerated observers in curved spacetime (Chap. 24). In their proper reference frames the metric coefficients $g_{\alpha\beta}$ will contain both the first-order terms of (22.83) due to acceleration and rotation, and the second-order terms of (24.15) due to spacetime curvature.

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EXERCISES

Exercise 24.2 *Derivation: Local Lorentz Frame in Friedman Universe*
By inserting the coordinate transformation (24.12) into the Friedman metric (24.11), derive the metric (24.13), (24.14) for a local Lorentz frame.

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24.4 Free-fall Motion and Geodesics of Spacetime

In order to make more precise the concept of spacetime curvature, we will need to study quantitatively the relative acceleration of neighboring, freely falling particles.[1] Before we

[1]See MTW pp. 244–247, 312–324

can carry out such a study, however, we must understand quantitatively the motion of a single freely falling particle in curved spacetime. That is the objective of this section.

In a global Lorentz frame of flat, special relativistic spacetime a free particle moves along a straight world line, i.e., a world line with the form

$$(t, x, y, z) = (t_o, x_o, y_o, z_o) + (p^0, p^x, p^y, p^z)\zeta \; ; \quad \text{i.e., } x^\alpha = x^\alpha_o + p^\alpha \zeta \; . \tag{24.17}$$

Here p^α are the Lorentz-frame components of the particle's 4-momentum; ζ is the affine parameter such that $\vec{p} = d/d\zeta$, i.e., $p^\alpha = dx^\alpha/d\zeta$ [Eq. (1.18) *ff*]; and x^α_o are the coordinates of the particle when its affine parameter is $\zeta = 0$. The straight-line motion (24.17) can be described equally well by the statement that the Lorentz-frame components p^α of the particle's 4-momentum are constant, i.e., are independent of ζ

$$\frac{dp^\alpha}{d\zeta} = 0 \; . \tag{24.18}$$

Even nicer is the frame-independent description, which says that as the particle moves it parallel-transports its tangent vector $\vec{p}$ along its world line

$$\boxed{\nabla_{\vec{p}}\vec{p} = 0 \; , \quad \text{or, equivalently } p^\alpha{}_{;\beta} p^\beta = 0 \; .} \tag{24.19}$$

For a particle of nonzero rest mass m, which has $\vec{p} = m\vec{u}$ and $\zeta = \tau/m$ with $\vec{u} = d/d\tau$ its 4-velocity and τ its proper time, Eq. (24.19) is equivalent to $\nabla_{\vec{u}}\vec{u} = 0$. This is the form of the particle's law of motion discussed in Eq. (22.89).

This description of the motion is readily carried over into curved spacetime using the equivalence principle: Let $\mathcal{P}(\zeta)$ be the world line of a freely moving particle in curved spacetime. At a specific event $\mathcal{P}_o = \mathcal{P}(\zeta_o)$ on that world line introduce a local Lorentz frame (so the frame's spatial origin, like the particle, passes through $\mathcal{P}_o$ as time progresses). Then the equivalence principle tells us that the particle's law of motion must be the same in this local Lorentz frame as it is in the global Lorentz frame of special relativity:

$$\left(\frac{dp^\alpha}{d\zeta}\right)_{\zeta=\zeta_o} = 0 \; . \tag{24.20}$$

More powerful than this local-Lorentz-frame description of the motion is a description that is frame-independent. We can easily deduce such a description from Eq. (24.20). Since the connection coefficients vanish at the origin of the local Lorentz frame where (24.20) is being evaluated [cf. Eq. (24.16)], (24.20) can be written equally well, in our local Lorentz frame, as

$$0 = \left(\frac{dp^\alpha}{d\zeta} + \Gamma^\alpha{}_{\beta\gamma} p^\beta \frac{dx^\gamma}{d\zeta}\right)_{\zeta=\zeta_o} = \left((p^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\beta\gamma} p^\beta)\frac{dx^\gamma}{d\zeta}\right)_{\zeta=\zeta_o} = (p^\alpha{}_{;\gamma} p^\gamma)_{\zeta=\zeta_o} \; . \tag{24.21}$$

Thus, as the particle passes through the spatial origin of our local Lorentz coordinate system, the components of the directional derivative of its 4-momentum along itself vanish. Now, if two 4-vectors have components that are equal in one basis, their components are guaranteed [by the tensorial transformation law (22.19)] to be equal in all bases, and correspondingly the two vectors, viewed as frame-independent, geometric objects, must be equal. Thus, since Eq. (24.21) says that the components of the 4-vector $\nabla_{\vec{p}}\vec{p}$ and the zero vector are equal in our chosen local Lorentz frame, it must be true that

$$\boxed{\nabla_{\vec{p}}\vec{p} = 0\ .} \tag{24.22}$$

at the moment when the particle passes through the point $\mathcal{P}_o = \mathcal{P}(\zeta_o)$. Moreover, since $\mathcal{P}_o$ is an arbitrary point (event) along the particle's world line, it must be that (24.22) is a geometric, frame-independent *equation of motion* for the particle, valid everywhere along its world line. Notice that this geometric, frame-independent equation of motion $\nabla_{\vec{p}}\vec{p} = 0$ in curved spacetime is precisely the same as that [Eq. (24.19)] for flat spacetime. We shall generalize this conclusion to other laws of physics in Sec. 24.7 below.

Our equation of motion (24.22) for a freely moving point particle says, in words, that the particle *parallel transports* its 4-momentum along its world line. In any curved manifold, not just in spacetime, the relation $\vec{\nabla}_{\vec{p}}\vec{p} = 0$ is called the *geodesic equation*, and the curve to which $\vec{p}$ is the tangent vector is called a *geodesic*. On the surface of a sphere such as the earth, the geodesics are the great circles; they are the unique curves along which local Euclidean coordinates can be meshed, keeping one of the two Euclidean coordinates constant along the curve [cf. Eq. (24.10)], and they are the trajectories generated by an airplane's inertial guidance system, which tries to fly the plane along the straightest trajectory it can. Similarly, in spacetime the trajectories of freely falling particles are geodesics; they are the unique curves along which local Lorentz coordinates can be meshed, keeping the three spatial coordinates constant along the curve and letting the time vary, thereby producing a local Lorentz reference frame [Eqs. (24.15) and (24.16)], and they are also the spacetime trajectories along which inertial guidance systems will guide a spacecraft.

The geodesic equation guarantees that the square of the 4-momentum will be conserved along the particle's world line; in slot-naming index notation,

$$(g_{\alpha\beta}p^\alpha p^\beta)_{;\gamma}p^\gamma = 2g_{\alpha\beta}p^\alpha p^\beta{}_{;\gamma}p^\gamma = 0\ . \tag{24.23}$$

(Here the standard rule for differentiating products has been used; this rule follows from the definition (22.27) of the frame-independent directional derivative of a tensor; it also can be deduced in a local Lorentz frame where $\Gamma^\alpha{}_{\mu\nu} = 0$ so each gradient with a ";" reduces to a partial derivative with a ",".) Also in Eq. (24.23) the term involving the gradient of the metric has been discarded since it vanishes [Eq. (22.40)], and the two terms involving derivatives of p^α and p^β, being equal, have been combined. In index-free notation the frame-independent relation (24.23) says

$$\nabla_{\vec{p}}(\vec{p}\cdot\vec{p}) = 2\vec{p}\cdot\nabla_{\vec{p}}\vec{p} = 0\ . \tag{24.24}$$

This is a pleasing result, since the square of the 4-momentum is the negative of the particle's squared rest mass, $\vec{p}\cdot\vec{p} = -m^2$, which surely should be conserved along the particle's free-fall world line! Note that, as in flat spacetime, so also in curved, for a particle of finite rest mass the free-fall trajectory (the geodesic world line) is timelike, $\vec{p}\cdot\vec{p} = -m^2 < 0$, while for a zero-rest-mass particle it is null, $\vec{p}\cdot\vec{p} = 0$. Spacetime also supports spacelike geodesics, i.e., curves with tangent vectors $\vec{p}$ that satisfy the geodesic equation (24.22) and are spacelike, $\vec{p} \cdot \vec{p} > 0$. Such curves can be thought of as the world lines of freely falling "tachyons," i.e., faster-than-light particles—though it seems unlikely that such particles really exist in Nature. Note that the constancy of $\vec{p} \cdot \vec{p}$ along a geodesic implies that a geodesic can never change its character: if initially timelike, it will always remain timelike; if initially null, it will remain null; if initially spacelike, it will remain spacelike.

When studying the motion of a particle with finite rest mass, one often uses as the tangent vector to the geodesic the particle's 4-velocity $\vec{u} = \vec{p}/m$ rather than the 4-momentum, and correspondingly one uses as the parameter along the geodesic the particle's proper time $\tau = m\zeta$ rather than ζ (recall: $\vec{u} = d/d\tau$; $\vec{p} = d/d\zeta$). In this case the geodesic equation becomes

$$\boxed{\nabla_{\vec{u}}\vec{u} = 0 \;;} \tag{24.25}$$

cf. Eq. (22.89). Similarly, for spacelike geodesics, one often uses as the tangent vector $\vec{u} = d/ds$, where s is proper distance (square root of the invariant interval) along the geodesic; and the geodesic equation then assumes the same form (24.25) as for a timelike geodesic.

The geodesic world line of a freely moving particle has three very important properties:

(i) When written in a coordinate basis, the geodesic equation $\nabla_{\vec{p}}\vec{p} = 0$ becomes the following differential equation for the particle's world line $x^\alpha(\zeta)$ in the coordinate system [Ex. 24.3]

$$\boxed{\frac{d^2x^\alpha}{d\zeta^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta} \;.} \tag{24.26}$$

Here $\Gamma^\alpha{}_{\mu\nu}$ are the connection coefficients of the coordinate system's coordinate basis. [Equation (22.91) was a special case of this.] Note that these are four coupled equations ($\alpha = 0, 1, 2, 3$) for the four coordinates x^α as functions of affine parameter ζ along the geodesic. If the initial position, x^α at $\zeta = 0$, and initial tangent vector (particle momentum), $p^\alpha = dx^\alpha/d\zeta$ at $\zeta = 0$, are specified, then these four equations will determine uniquely the coordinates $x^\alpha(\zeta)$ as a function of ζ along the geodesic.

(ii) Consider a spacetime that possesses a symmetry, which is embodied in the fact that the metric coefficients in some coordinate system are independent of one of the coordinates x^A. Associated with that symmetry there will be a conserved quantity $p_A \equiv \vec{p} \cdot \partial/\partial x^A$ associated with free-particle motion. Exercise 24.4 derives this result and develops a familiar example.

(iii) Among all timelike curves linking two events $\mathcal{P}_0$ and $\mathcal{P}_1$ in spacetime, those whose proper time lapse (timelike length) is stationary under small variations of the curve are timelike geodesics; see Ex. 24.5. In other words, timelike geodesics are the curves that satisfy the action principle (24.30) below. Now, one can always send a photon from $\mathcal{P}_0$ to $\mathcal{P}_1$ by bouncing it off a set of strategically located mirrors, and that photon path is the limit of a timelike curve as the curve becomes null. Therefore, there exist timelike curves from $\mathcal{P}_0$ to $\mathcal{P}_1$ with vanishingly small length, so the geodesics cannot minimize the proper time lapse. This means that the curve of *maximal* proper time lapse (length) is a geodesic, and that any other geodesics will have a length that is a "saddle point" (stationary under variations of the path but not a maximum or a minimum).

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EXERCISES

Exercise 24.3 *Derivation: Geodesic equation in an arbitrary coordinate system.*
Show that in an arbitrary coordinate system $x^\alpha(\mathcal{P})$ the geodesic equation (24.22) takes the form (24.26).

Exercise 24.4 *Derivation: Constant of Geodesic Motion in a Spacetime with Symmetry*

(a) Suppose that in some coordinate system the metric coefficients are independent of some specific coordinate x^A: $g_{\alpha\beta,A} = 0$. [Example: in spherical polar coordinates t, r, θ, ϕ in flat spacetime $g_{\alpha\beta,\phi} = 0$, so we could set $x^A = \phi$.] Show that

$$p_A \equiv \vec{p} \cdot \frac{\partial}{\partial x^A} \tag{24.27}$$

is a constant of the motion for a freely moving particle [p_ϕ = (conserved z-component of angular momentum) in above, spherically symmetric example]. [*Hint*: Show that the geodesic equation can be written in the form

$$\frac{dp_\alpha}{d\zeta} - \Gamma_{\mu\alpha\nu} p^\mu p^\nu = 0 \,, \tag{24.28}$$

where $\Gamma_{\mu\alpha\nu}$ is the covariant Christoffel symbol of Eqs. (22.38), (22.39).] Note the analogy of the constant of the motion p_A with Hamiltonian mechanics: there, if the Hamiltonian is independent of x^A then the generalized momentum p_A is conserved; here, if the metric coefficients are independent of x^A, then the covariant component p_A of the momentum is conserved. For an elucidation of the connection between these two conservation laws, see the Hamiltonian formulation of geodesic motion in Exercise 25.2 of MTW.

(b) As an example, consider a particle moving freely through a time-independent, Newtonian gravitational field. In Ex. 24.12 below we shall learn that such a gravitational field can be described in the language of general relativity by the spacetime metric

$$ds^2 = -(1 + 2\Phi)dt^2 + (\delta_{jk} + h_{jk})dx^j dx^k \,, \tag{24.29}$$

where $\Phi(x,\ y,\ z)$ is the time-independent Newtonian potential and h_{jk} are contributions to the metric that are independent of the time coordinate t and have magnitude of order $|\Phi|$. That the gravitational field is weak means $|\Phi| \ll 1$ (or, in cgs units, $|\Phi/c^2| \ll 1$). The coordinates being used are Lorentz, aside from tiny corrections of order $|\Phi|$; and, as this exercise and Ex. 24.12 show, they coincide with the coordinates of the Newtonian theory of gravity. Suppose that the particle has a velocity $v^j \equiv dx^j/dt$ through this coordinate system that is less than or of order $|\Phi|^{\frac{1}{2}}$ and thus small compared to the speed of light. Because the metric is independent of the time coordinate t, the component p_t of the particle's 4-momentum must be conserved along its world line. Since, throughout physics, the conserved quantity associated with time-translation invariance is always the energy, we expect that p_t, when evaluated accurate to first order in $|\Phi|$, must be equal to the particle's conserved Newtonian energy, $E = m\Phi + \frac{1}{2}mv^j v^k \delta_{jk}$, aside from some multiplicative and additive constants. Show that this, indeed, is true, and evaluate the constants.

Exercise 24.5 *Problem: Action principle for geodesic motion*
Show, by introducing a specific but arbitrary coordinate system, that among all timelike world lines that a particle could take to get from event $\mathcal{P}_0$ to $\mathcal{P}_1$, the one or ones whose proper time lapse is stationary under small variations of path are the free-fall geodesics. In other words, an action principle for a timelike geodesic $\mathcal{P}(\lambda)$ [i.e., $x^\alpha(\lambda)$ in any coordinate system x^α] is

$$\boxed{\delta \int_{\mathcal{P}_0}^{\mathcal{P}_1} d\tau = \int_0^1 \left(g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{\frac{1}{2}} d\lambda = 0 \,,} \tag{24.30}$$

where λ is an arbitrary parameter which, by definition, ranges from 0 at $\mathcal{P}_0$ to 1 at $\mathcal{P}_1$. [Note: unless, after the variation, you choose the arbitrary parameter λ to be "affine" ($\lambda = a\tau + b$ where a and b are constants), your equation for $d^2x^\alpha/d\lambda^2$ will not look quite like (24.26).]

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24.5 Relative Acceleration, Tidal Gravity, and Spacetime Curvature

Now that we understand the motion of an individual freely falling particle in curved spacetime, we are ready to study the effects of gravity on the relative motions of such particles.[2] Before doing so in general relativity, let us recall the Newtonian discussion of the same problem:

24.5.1 Newtonian Description of Tidal Gravity

Consider, as shown in Fig. 24.3(a), two point particles, A and B, falling freely through 3-dimensional Euclidean space under the action of an external Newtonian potential Φ (i.e., a potential generated by other masses, not by the particles themselves). At Newtonian time $t = 0$ the particles are separated by only a small distance and are moving with the same velocity $\mathbf{v}_A = \mathbf{v}_B$. As time passes, however, the two particles, being at slightly different locations in space, experience slightly different gravitational potentials Φ and gravitational accelerations $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ and thence develop slightly different velocities, $\mathbf{v}_A \neq \mathbf{v}_B$. To quantify this, denote by $\boldsymbol{\xi}$ the vector separation of the two particles in Euclidean 3-space. The components of $\boldsymbol{\xi}$ on any Euclidean basis [e.g., that of Fig. 24.3(a)] are $\xi^j = x^j_B - x^j_A$, where x^j_I is the coordinate location of particle I. Correspondingly, the rate of change of ξ^j

[2] See MTW pp. 29–37, 218–224, 265–275

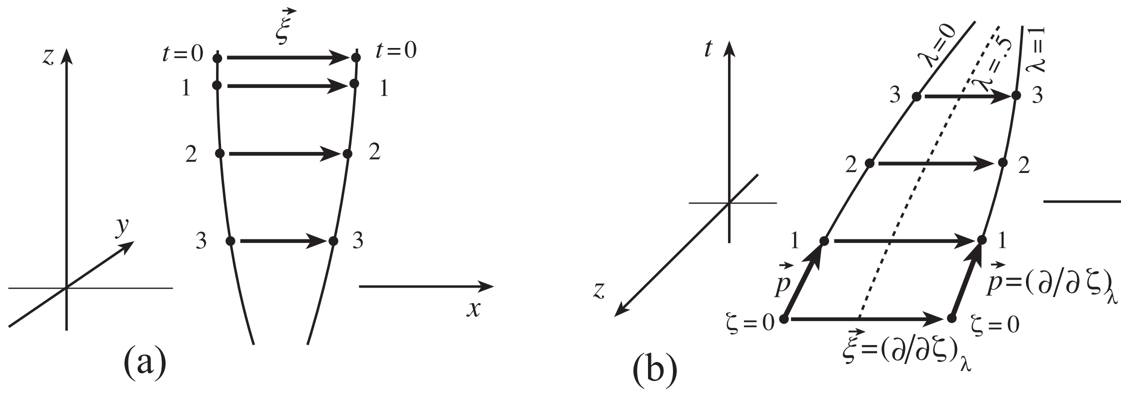

Fig. 24.3: The effects of tidal gravity on the relative motions of two freely falling particles. Diagram (a) depicts this in a Euclidean 3-space diagram using Newton's theory of gravity. Diagram (b) depicts it in a spacetime diagram using Einstein's theory of gravity, general relativity.

with respect to Newtonian time is $d\xi^j/dt = v_B^j - v_A^j$; i.e., the relative velocity of the two particles is the difference of their two velocities. The second time derivative of the relative separation, i.e., the relative acceleration of the two particles, is thus given by

$$\frac{d^2\xi^j}{dt^2} = \frac{d^2x_B^j}{dt^2} - \frac{d^2x_A^j}{dt^2} = -\left(\frac{\partial\Phi}{\partial x^j}\right)_B + \left(\frac{\partial\Phi}{\partial x^j}\right)_A = -\frac{\partial^2\Phi}{\partial x^j\partial x^k}\xi^k , \tag{24.31}$$

accurate to first order in the separation ξ^k. This equation gives the components of the relative acceleration in an arbitrary Euclidean basis. Rewritten in geometric, basis-independent language this equation says

$$\boxed{\frac{d^2\boldsymbol{\xi}}{dt^2} = -\mathcal{E}(\ldots, \boldsymbol{\xi}) \; ; \quad \text{i.e.,} \; \frac{d^2\xi^j}{dt^2} = -\mathcal{E}^j{}_k\xi^k \, ,} \tag{24.32}$$

where $\mathcal{E}$ is a symmetric, second-rank tensor, called the *Newtonian tidal gravitational field*:

$$\boxed{\mathcal{E} = \boldsymbol{\nabla\nabla}\Phi = -\boldsymbol{\nabla}\mathbf{g} \; ; \quad \text{i.e.,} \; \mathcal{E}_{jk} = \frac{\partial^2\Phi}{\partial x^j\partial x^k} \; \text{ in Euclidean coordinates.}} \tag{24.33}$$

The name "tidal gravitational field" comes from the fact that this is the field which, generated by the moon and the sun, produces the tides on the earth's oceans. Note that, since this field is the gradient of the Newtonian gravitational acceleration **g**, it is a quantitative measure of the inhomogeneities of Newtonian gravity.

Equation (24.31) shows quantitatively how the tidal gravitational field produces the relative acceleration of our two particles. As a specific application, one can use it to compute, in Newtonian theory, the relative accelerations and thence relative motions of two neighboring local Lorentz frames as they fall toward and through the center of the earth [Fig. 24.2(a) and associated discussion].

24.5.2 Relativistic Description of Tidal Gravity

Turn attention, now, to the general relativistic description of the relative motions of two free particles. As shown in Fig. 24.3(b), the particles, labeled A and B, move along geodesic world lines with affine parameters ζ and 4-momentum tangent vectors $\vec{p} = d/d\zeta$. The origins of ζ along the two world lines can be chosen however we wish, so long as events with the same ζ on the two world lines, $\mathcal{P}_A(\zeta)$ and $\mathcal{P}_B(\zeta)$ are close enough to each other that we can perform power-series expansions in their separation, $\vec{\xi}(\zeta) = \mathcal{P}_B(\zeta) - \mathcal{P}_A(\zeta)$, and keep only the leading terms. As in our Newtonian analysis, we require that the two particles initially have vanishing relative velocity, $\nabla_{\vec{p}}\vec{\xi} = 0$, and we shall compute the tidal-gravity-induced relative acceleration $\nabla_{\vec{p}}\nabla_{\vec{p}}\vec{\xi}$.

As a tool in our calculation, we shall introduce into spacetime a two-dimensional surface which contains our two geodesics A and B, and also contains an infinity of other geodesics in between and alongside them; and on that surface we shall introduce two coordinates, ζ =(affine parameter along each geodesic) and λ=(a parameter that labels the geodesics); see Fig. 24.3(b). Geodesic A will carry the label $\lambda = 0$; geodesic B will be $\lambda = 1$; $\vec{\xi} \equiv (\partial/\partial\lambda)_{\zeta=\mathrm{const}}$ will be a vector field which, evaluated on geodesic A (i.e., at $\lambda = 0$), is equal to the separation vector we wish to study; and the vector field $\vec{p} = (\partial/\partial\zeta)_{\lambda=\mathrm{const}}$ will be a vector field which, evaluated on any geodesic (A, B, or other curve of constant λ), is equal to the 4-momentum of the particle which moves along that geodesic. Our identification of $(\partial/\partial\lambda)_{\zeta=\mathrm{const}}(\lambda = 0)$ with the separation vector $\vec{\xi}$ between geodesics A and B is the leading term in a power series expansion; it is here that we require, for good accuracy, that the geodesics be close together and be so parametrized that $\mathcal{P}_A(\zeta)$ is close to $\mathcal{P}_B(\zeta)$.

Our objective is to compute the relative acceleration of particles B and A, $\nabla_{\vec{p}}\nabla_{\vec{p}}\vec{\xi}$ evaluated at $\lambda = 0$. The quantity $\nabla_{\vec{p}}\vec{\xi}$, which we wish to differentiate a second time in that computation, is one of the terms in the following expression for the commutator of the vector fields $\vec{p}$ and $\vec{\xi}$ [Eq. (22.41)]:

$$\boxed{[\vec{p}, \vec{\xi}] = \nabla_{\vec{p}}\vec{\xi} - \nabla_{\vec{\xi}}\vec{p}\,.} \tag{24.34}$$

Because $\vec{p} = (\partial/\partial\zeta)_\lambda$ and $\vec{\xi} = (\partial/\partial\lambda)_\zeta$, these two vector fields commute, and Eq. (24.34) tells us that $\nabla_{\vec{p}}\vec{\xi} = \nabla_{\vec{\xi}}\vec{p}$. Correspondingly, the relative acceleration of our two particles can be expressed as

$$\nabla_{\vec{p}}\nabla_{\vec{p}}\vec{\xi} = \nabla_{\vec{p}}\nabla_{\vec{\xi}}\vec{p} = (\nabla_{\vec{p}}\nabla_{\vec{\xi}} - \nabla_{\vec{\xi}}\nabla_{\vec{p}})\vec{p}\,. \tag{24.35}$$

Here the second equality results from adding on, for use below, a term that vanishes because $\nabla_{\vec{p}}\vec{p} = 0$ (geodesic equation).

This first part of our calculation was performed efficiently using index-free notation. The next step will be easier if we introduce indices as names for slots. Then expression (24.35) takes the form

$$(\xi^\alpha{}_{;\beta}p^\beta)_{;\gamma}p^\gamma = (p^\alpha{}_{;\gamma}\xi^\gamma)_{;\delta}p^\delta - (p^\alpha{}_{;\gamma}p^\gamma)_{;\delta}\xi^\delta\,, \tag{24.36}$$

which can be evaluated by using the rule for differentiating products and then renaming indices and collecting terms; the result is

$$(\xi^\alpha{}_{;\beta}p^\beta)_{;\gamma}p^\gamma = (p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma})\xi^\gamma p^\delta + p^\alpha{}_{;\gamma}(\xi^\gamma{}_{;\delta}p^\delta - p^\gamma{}_{;\delta}\xi^\delta)\,. \tag{24.37}$$

The second term in this expression vanishes, since it is just the commutator of $\vec{\xi}$ and $\vec{p}$ [Eq. (24.34)] written in slot-naming index notation, and as we noted above, $\vec{\xi}$ and $\vec{p}$ commute. The remaining first term,

$$(\xi^\alpha{}_{;\beta}p^\beta)_{;\gamma}p^\gamma = (p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma})\xi^\gamma p^\delta \ , \tag{24.38}$$

reveals that *the relative acceleration of the two particles is caused by noncommutation of the two slots of a double gradient* (slots here named γ and δ). In the flat spacetime of special relativity the two slots would commute and there would be no relative acceleration. Spacetime curvature prevents them from commuting and thereby causes the relative acceleration.

Now, one can show that $p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma}$ is linear in p^α; see Ex. 24.6. Therefore, there must exist a fourth rank tensor field $\mathbf{R}(\_\,,\_\,,\_\,,\_)$ such that

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = -R^\alpha{}_{\beta\gamma\delta}p^\beta \tag{24.39}$$

for any vector field $\vec{p}(\mathcal{P})$. The tensor $\mathbf{R}$ can be regarded as responsible for the failure of gradients to commute, so it must be some aspect of spacetime curvature. It is called the *Riemann curvature tensor.*

Inserting Eq. (24.39) into Eq. (24.38) and writing the result in both slot-naming index notation and abstract notation, we obtain

$$\boxed{(\xi^\alpha{}_{;\beta}p^\beta)_{;\gamma}p^\gamma = -R^\alpha{}_{\beta\gamma\delta}p^\beta\xi^\gamma p^\delta \ , \quad \nabla_{\vec{p}}\nabla_{\vec{p}}\vec{\xi} = -\mathbf{R}(\ \ldots\ , \vec{p}, \vec{\xi}, \vec{p})\ .} \tag{24.40}$$

This is the *equation of relative acceleration* for freely moving test particles. It is also called the *equation of geodesic deviation*, because it describes the manner in which spacetime curvature $\mathbf{R}$ forces geodesics that are initially parallel (the world lines of freely moving particles with zero initial relative velocity) to deviate from each other; cf. Fig. 24.3(b).

24.5.3 Comparison of Newtonian and Relativistic Descriptions

It is instructive to compare this relativistic description of the relative acceleration of freely moving particles with the Newtonian description. For this purpose we shall consider a region of spacetime, such as our solar system, in which the Newtonian description of gravity is highly accurate; and there we shall study the relative acceleration of two free particles from the viewpoint of a local Lorentz frame in which the particles are both initially at rest.

In the Newtonian description, the transformation from a Newtonian universal reference frame (e.g., that of the center of mass of the solar system) to the chosen local Lorentz frame is achieved by introducing new Euclidean coordinates that are uniformly accelerated relative to the old ones, with just the right uniform acceleration to annul the gravitational acceleration at the center of the local Lorentz frame. This transformation adds a spatially homogeneous constant to the Newtonian acceleration $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ but leaves unchanged the tidal field $E = \boldsymbol{\nabla\nabla}\Phi$. Correspondingly, the Newtonian equation of relative acceleration in the local Lorentz frame retains its standard Newtonian form, $d^2\xi^j/dt^2 = -\mathcal{E}^j{}_k\xi^k$ [Eq. (24.32)], with the components of the tidal field computable equally well in the original universal reference frame, or in the local Lorentz frame, from the standard relation $\mathcal{E}^j{}_k = \mathcal{E}_{jk} = \partial^2\Phi/\partial x^j\partial x^k$.

As an aid in making contact between the relativistic and the Newtonian descriptions, we shall convert over from using the 4-momentum $\vec{p}$ as the tangent vector and ζ as the parameter along the particles' world lines to using the 4-velocity $\vec{u} = \vec{p}/m$ and the proper time $\tau = m\zeta$; this conversion brings the relativistic equation of relative acceleration (24.40) into the form

$$\nabla_{\vec{u}}\nabla_{\vec{u}}\vec{\xi} = -\mathbf{R}(\ldots, \vec{u}, \vec{\xi}, \vec{u}) \; . \tag{24.41}$$

Because the particles are (momentarily) at rest near the origin of the local Lorentz frame, their 4-velocities are $\vec{u} \equiv d/d\tau = \partial/\partial t$, which implies that the components of their 4-velocities are $u^0 = 1$, $u^j = 0$, and their proper times τ are equal to coordinate time t, which in turn coincides with the time t of the Newtonian analysis: $\tau = t$. In the relativistic analysis, as in the Newtonian, the separation vector $\vec{\xi}$ will have only spatial components, $\xi^0 = 0$ and $\xi^j \neq 0$. [If this were not so, we could make it so by a readjustment of the origin of proper time for particle B; cf. Fig. 24.3(b).] These facts, together with the vanishing of all the connection coefficients and derivatives of them ($\Gamma^j{}_{k0,0}$) that appear in $(\xi^j{}_{;\beta}u^\beta)_{;\gamma}u^\gamma$ at the origin of the local Lorentz frame [cf. Eqs. (24.15) and (24.16)], imply that the local Lorentz components of the equation of relative acceleration (24.41) take the form

$$\frac{d^2\xi^j}{dt^2} = -R^j{}_{0k0}\xi^k \; . \tag{24.42}$$

By comparing this with the Newtonian equation of relative acceleration (24.32) we infer that, *in the Newtonian limit, in the local rest frame of the two particles,*

$$\boxed{R^j{}_{0k0} = \mathcal{E}_{jk} = \frac{\partial^2\Phi}{\partial x^j \partial x^k} \; .} \tag{24.43}$$

Thus, *the Riemann curvature tensor is the relativistic generalization of the Newtonian tidal field*. This conclusion and the above equations make quantitative the statement that *gravity is a manifestation of spacetime curvature*.

Outside a spherical body with weak (Newtonian) gravity, such as the Earth, the Newtonian potential is $\Phi = -GM/r$, where G is Newton's gravitation constant, M is the body's mass and r is the distance from its center. If we introduce Cartesian coordinates with origin at the body's center and with z-axis through the point at which the Riemann tensor is to be measured, then Φ in these coordinates is $\Phi = -GM/(z^2 + x^2 + y^2)^{\frac{1}{2}}$, and on the z-axis the only nonzero $R^j{}_{0k0}$, as computed from Eq. (24.43), are

$$\boxed{R^z{}_{0z0} = \frac{-2GM}{r^3}, \quad R^x{}_{0x0} = R^y{}_{0y0} = \frac{+GM}{r^3} \; .} \tag{24.44}$$

Correspondingly, for two particles separated from each other in the radial (z) direction, the relative acceleration (24.42) is $d^2\xi^j/dt^2 = +(2GM/r^3)\xi^j$; i.e., the particles are pulled apart by the body's tidal gravitational field. Similarly, for two particles separated from each other in a horizontal direction (in the x–y plane), $d^2\xi^j/dt^2 = -(GM/r^3)\xi^j$; i.e., the particles are pushed together by the body's tidal gravitational field. There thus is a radial tidal stretch and a lateral tidal squeeze; and the lateral squeeze has half the strength of the radial stretch but occurs in two laterial dimensions compared to the one radial dimension. These stretch and squeeze, produced by the sun and moon, are responsible for the tides on the earth's oceans.

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EXERCISES

Exercise 24.6 *Derivation: Linearity of Commutator of Double Gradient*

(a) Let a and b be scalar fields with arbitrary but smooth dependence on location in spacetime, and $\vec{A}$ and $\vec{B}$ be tensor fields. Show that

$$(aA^\alpha + bB^\alpha)_{;\gamma\delta} - (aA^\alpha + bB^\alpha)_{;\delta\gamma} = a(A^\alpha{}_{;\gamma\delta} - A^\alpha{}_{;\delta\gamma}) + b(B^\alpha{}_{;\gamma\delta} - B^\alpha{}_{;\delta\gamma}) \ . \qquad (24.45)$$

[*Hint*: The double gradient of a scalar field commutes, as one can easily see in a local Lorentz frame.]

(b) Use Eq. (24.45) to show that (i) the commutator of the double gradient is independent of how the differentiated vector field varies from point to point, and depends only on the value of the field at the location where the commutator is evaluated, and (ii) the commutator is linear in that value. Thereby conclude that there must exist a fourth rank tensor field $\mathbf{R}$ such that Eq. (24.39) is true for any vector field $\vec{p}$.

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24.6 Properties of the Riemann Curvature Tensor

We now pause, in our study of the foundations of general relativity, to examine a few properties of the Riemann curvature tensor $\mathbf{R}$.[3]

We begin, as a tool for deriving other things, by evaluating the components of the Riemann tensor at the spatial origin of a local Lorentz frame; i.e. at a point where $\Gamma^\alpha{}_{\beta\gamma}$ vanishes but its derivatives do not. For any vector field $\vec{p}$ a straightforward computation reveals

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = (\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma})p^\beta \ . \qquad (24.46)$$

By comparing with Eq. (24.39), we can read off the local-Lorentz components of Riemann:

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} \quad \text{at spatial origin of a local Lorentz frame.} \qquad (24.47)$$

From this expression we infer that, at a spatial distance $\sqrt{\delta_{ij}x^i x^j}$ from the origin of a local Lorentz frame, the connection coefficients and the metric have magnitudes

$$\Gamma^\alpha{}_{\beta\gamma} = O(R^\alpha{}_{\mu\nu\lambda}\sqrt{\delta_{ij}x^i x^j}) \ , \quad g_{\alpha\beta} - \eta_{\alpha\beta} = O(R^\mu{}_{\nu\lambda\rho}\delta_{ij}x^i x^j) \ , \quad \text{in a local Lorentz frame.} \qquad (24.48)$$

[3]See MTW pp. 273–288, 324–327.

Comparison with Eqs. (24.15) and (24.16) shows that the radius of curvature of spacetime (a concept defined only semiquantitatively) is of order the inverse square root of the components of the Riemann tensor in a local Lorentz frame:

$$\boxed{\mathcal{R} = \mathrm{O}\left(\frac{1}{|R^\alpha{}_{\beta\gamma\delta}|^{\frac{1}{2}}}\right) \quad \text{in a local Lorentz frame.}} \tag{24.49}$$

By comparison with Eq. (24.44), we see that at radius r outside a weakly gravitating body of mass M, the radius of curvature of spacetime is

$$\boxed{\mathcal{R} \sim \left(\frac{r^3}{GM}\right)^{\frac{1}{2}} = \left(\frac{c^2 r^3}{GM}\right)^{\frac{1}{2}} ,} \tag{24.50}$$

where the factor c in the second expression makes the formula valid in conventional units. For further discussion see Ex. 24.7.

From the components (24.47) of the Riemann tensor in a local Lorentz frame, together with the vanishing of the connection coefficients at the origin and the standard expressions (10.37), (10.38) for the connection coefficients in terms of the metric components, one easily can show that

$$\boxed{R_{\alpha\beta\gamma\delta} = \frac{1}{2}(g_{\alpha\delta,\beta\gamma} + g_{\beta\gamma,\alpha\delta} - g_{\alpha\gamma,\beta\delta} - g_{\beta\delta,\alpha\gamma}) \quad \text{in a local Lorentz frame.}} \tag{24.51}$$

From these expressions, plus the commutation of partial derivatives $g_{\alpha\gamma,\beta\delta} = g_{\alpha\gamma,\delta\beta}$ and the symmetry of the metric one easily can show that in a local Lorentz frame the components of the Riemann tensor have the following symmetries:

$$\boxed{R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \ , \quad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \ , \quad R_{\alpha\beta\gamma\delta} = +R_{\gamma\delta\alpha\beta}} \tag{24.52}$$

(antisymmetry in first pair of indices, antisymmetry in second pair of indices, and symmetry under interchange of the pairs). When one computes the value of the tensor on four vectors, $\mathbf{R}(\vec{A}, \vec{B}, \vec{C}, \vec{D})$ using component calculations in this frame, one trivially sees that these symmetries produce corresponding symmetries under interchange of the vectors inserted into the slots, and thence under interchange of the slots themselves. This is always the case: any symmetry that the components of a tensor exhibit in a special basis will induce the same symmetry on the slots of the geometric, frame-independent tensor. The resulting symmetries for $\mathbf{R}$ are given by Eq. (24.52) with the "Escher mind-flip" [Sec. 1.5.3] in which the indices switch from naming components in a special frame to naming slots: *The Riemann tensor is antisymmetric under interchange of its first two slots, antisymmetric under interchange of the last two, and symmetric under interchange of the two pairs.*

One additional symmetry can be verified, by calculation in the local Lorentz frame [i.e., from Eq. (24.51)]:

$$\boxed{R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0 \ .} \tag{24.53}$$

(Note that this cyclic symmetry is the same as occurs in the Maxwell equations (22.70) or (1.65), and also the same as occurs in the commutator identities $\left[\vec{B},[\vec{C},\vec{D}\,]\right]+\left[\vec{C},[\vec{D},\vec{B}\,]\right]+\left[\vec{D},[\vec{B},\vec{C}\,]\right]=0$.) One can show that the full set of symmetries (24.52) and (24.53) reduces the number of independent components of the Riemann tensor, in 4-dimensional spacetime, from $4^4=256$ to "just" 20.

Of these 20 independent components, 10 are contained in the *Ricci curvature tensor*—which is the contraction of the Riemann tensor on its first and third slots

$$\boxed{R_{\alpha\beta} \;\equiv\; R^{\mu}{}_{\alpha\mu\beta}\;,} \tag{24.54}$$

and which by the symmetries (24.52) and (24.53) of Riemann is itself symmetric

$$\boxed{R_{\alpha\beta}=R_{\beta\alpha}\;.} \tag{24.55}$$

The other 10 independent components of Riemann are contained in the Weyl curvature tensor, which we will not study here; see, e.g., pp. 325 and 327 of MTW. The contraction of the Ricci tensor on its two slots,

$$\boxed{R\equiv R^{\alpha}{}_{\alpha}\;,} \tag{24.56}$$

is called the *curvature scalar*.

One often needs to know the components of the Riemann curvature tensor in some non-local-Lorentz basis. Exercise 24.8 derives the following equation for them in an arbitrary basis:

$$\boxed{R^{\alpha}{}_{\beta\gamma\delta}=\Gamma^{\alpha}{}_{\beta\delta,\gamma}-\Gamma^{\alpha}{}_{\beta\gamma,\delta}+\Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta}-\Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma}-\Gamma^{\alpha}{}_{\beta\mu}c_{\gamma\delta}{}^{\mu}\;.} \tag{24.57}$$

Here $\Gamma^{\alpha}{}_{\beta\gamma}$ are the connection coefficients in the chosen basis, $\Gamma^{\alpha}{}_{\beta\gamma,\delta}$ is the result of letting the basis vector $\vec{e}_\delta$ act as a differential operator on $\Gamma^{\alpha}{}_{\beta\gamma}$, as though $\Gamma^{\alpha}{}_{\beta\gamma}$ were a scalar, and $c_{\gamma\delta}{}^{\mu}$ are the basis vectors' commutation coefficients. Calculations with this equation are usually very long and tedious, and so are carried out using symbolic-manipulation software on a computer.

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EXERCISES

Exercise 24.7 *Example: Orders of magnitude of the radius of curvature of spacetime*
With the help of the Newtonian limit (24.43) of the Riemann curvature tensor, show that near the earth's surface the radius of curvature of spacetime has a magnitude $\mathcal{R}\sim$ (1 astronomical unit) $\equiv$ (distance from sun to earth). What is the radius of curvature of spacetime near the sun's surface? near the surface of a white-dwarf star? near the surface of a neutron star? near the surface of a one-solar-mass black hole? in intergalactic space?

Exercise 24.8 *Derivation: Components of Riemann tensor in an arbitrary basis*
By evaluating expression (24.39) in an arbitrary basis (which might not even be a coordinate basis), derive Eq. (24.57) for the components of the Riemann tensor. In your derivation

keep in mind that commas denote partial derivations *only* in a coordinate basis; in an arbitrary basis they denote the result of letting a basis vector act as a differential operator; cf. Eq. (22.32).

Exercise 24.9 *Problem: Curvature of the surface of a sphere*
On the surface of a sphere such as the earth introduce spherical polar coordinates in which the metric, written as a line element, takes the form

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{24.58}$$

where a is the sphere's radius.

(a) Show (first by hand and then by computer) that the connection coefficients for the coordinate basis $\{\partial/\partial\theta, \partial/\partial\phi\}$ are

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \ , \quad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta \ , \quad \text{all others vanish.} \tag{24.59}$$

(b) Show that the symmetries (24.52) and (24.53) of the Riemann tensor guarantee that its only nonzero components in the above coordinate basis are

$$R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} \ . \tag{24.60}$$

(c) Show, first by hand and then by computer, that

$$R_{\theta\phi\theta\phi} = a^2\sin^2\theta \ . \tag{24.61}$$

(d) Show that in the basis

$$\{\vec{e}_{\hat\theta}, \vec{e}_{\hat\phi}\} = \left\{\frac{1}{a}\frac{\partial}{\partial\theta}, \frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}\right\} \ , \tag{24.62}$$

the components of the metric, the Riemann tensor, the Ricci tensor, and the curvature scalar are

$$g_{\hat j\hat k} = \delta_{jk} \ , \quad R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{1}{a^2}, \quad R_{\hat j\hat k} = \frac{1}{a^2}g_{\hat j\hat k} \ , \quad R = \frac{2}{a^2} \ . \tag{24.63}$$

The first of these implies that the basis is orthonormal; the rest imply that the curvature is independent of location on the sphere, as it should be by spherical symmetry. [The θ-dependence in the coordinate components of Riemann, Eq. (24.61), like the θ-dependence in the metric component $g_{\phi\phi}$, is a result of the θ-dependence in the length of the coordinate basis vector $\vec{e}_\phi$: $|\vec{e}_\phi| = a\sin\theta$.)

Exercise 24.10 *Problem: Geodesic Deviation on a Sphere*
Consider two neighboring geodesics (great circles) on a sphere of radius a, one the equator and the other a geodesic slightly displaced from the equator (by $\Delta\theta = b$) and parallel to it at $\phi = 0$. Let $\vec{\xi}$ be the separation vector between the two geodesics, and note that at $\phi = 0$, $\vec{\xi} = b\partial/\partial\theta$. Let l be proper distance along the equatorial geodesic, so $d/dl = \vec{u}$ is its tangent vector.

(a) Show that $l = a\phi$ along the equatorial geodesic.

(b) Show that the equation of geodesic deviation (24.40) reduces to

$$\frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta \; , \quad \frac{d^2\xi^\phi}{d\phi^2} = 0 \; . \tag{24.64}$$

(c) Solve this, subject to the above initial conditions, to obtain

$$\xi^\theta = b\cos\phi \; , \quad \xi^\phi = 0 \; . \tag{24.65}$$

Verify, by drawing a picture, that this is precisely what one would expect for the separation vector between two great circles.

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24.7 Curvature Coupling Delicacies in the Equivalence Principle, and Some Nongravitational Laws of Physics in Curved Spacetime[4]

If one knows a local, special relativistic, nongravitational law of physics in geometric, frame-independent form [for example, the expression for the stress-energy tensor of a perfect fluid in terms of its 4-velocity $\vec{u}$ and its rest-frame mass-energy density ρ and pressure P

$$\mathbf{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\mathbf{g} \tag{24.66}$$

Eq. (22.59)], then the equivalence principle guarantees that in general relativity the law will assume the same geometric, frame-independent form. One can see that this is so by the same method as we used to derive the general relativistic equation of motion $\nabla_{\vec{p}}\vec{p} = 0$ for free particles [Eq. (24.22) and associated discussion]: (i) rewrite the special relativistic law in terms of components in a global Lorentz frame [$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + Pg^{\alpha\beta}$], ($ii$) then infer from the equivalence principle that this same component form of the law will hold, unchanged, in a local Lorentz frame in general relativity, and (iii) then deduce that this component law is the local Lorentz frame version of the original geometric law [$\mathbf{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\mathbf{g}$], now lifted into general relativity.

Thus, *when the local, nongravitational laws of physics are known in frame-independent form, one need not distinguish between whether they are special relativistic or general relativistic.*

In this conclusion the word *local* is crucial: The equivalence principle is strictly valid only at the spatial origin of a local Lorentz frame; and, correspondingly, it is in danger of failure for any law of physics that cannot be formulated solely in terms of quantities which reside at the

[4]See MTW Chap. 16.

spatial origin—i.e., along a timelike geodesic. For the above example, $\mathbf{T} = (\rho + P)\vec{u}\otimes\vec{u} + P\mathbf{g}$, there is no problem; and for the local law of conservation of 4-momentum $\vec{\nabla}\cdot\mathbf{T} = 0$ there is no problem. However, for the global law of conservation of 4-momentum

$$\int_{\partial\mathcal{V}} T^{\alpha\beta} d\Sigma_\beta = 0 \tag{24.67}$$

[Eq. (1.96) and Fig. 1.17], there is serious trouble: This law is severely nonlocal, since it involves integration over a finite, closed 3-surface $\partial\mathcal{V}$ in spacetime. Thus, the equivalence principle fails for it. The failure shows up especially clearly when one notices (as we discussed in Sec. 22.3.4) that the quantity $T^{\alpha\beta}d\Sigma_\beta$ which the integral is trying to add up over $\partial\mathcal{V}$ has one empty slot, named α; i.e., it is a vector. This means that to compute the integral (24.67) we must transport the contributions $T^{\alpha\beta}d\Sigma_\beta$ from the various tangent spaces in which they normally live, to the tangent space of some single, agreed upon location, where they are to be added. By what rule should the transport be done? In special relativity one uses parallel transport, so the components of the vector are held fixed in any global Lorentz frame. However, it turns out that spacetime curvature makes parallel transport dependent on the path of the transport (and correspondingly, a vector is changed by parallel transport around a closed curve). As a result, the integral $\int_{\partial\mathcal{V}} T^{\alpha\beta}d\Sigma_\beta$ depends not only on the common location to which one transports each surface element's contribution in order to add them, it also depends on the path of the transport, which in general is quite arbitrary. This dependence makes the integral ill defined and correspondingly causes a breakdown, in general relativity, in the global law of 4-momentum conservation.

Another instructive example is the law by which a freely moving particle transports its spin angular momentum. The spin angular momentum is readily defined in the momentary local Lorentz rest frame of the particle's center of mass; there it is a 4-vector with vanishing time component, and with space components given by the familiar integral

$$S_i = \int_{\text{interior of body}} \epsilon_{ijk} x^j T^{k0} dx\, dy\, dz \ , \tag{24.68}$$

where T^{k0} are the components of the momentum density. In special relativity the law of angular momentum conservation (e.g., MTW Sec. 5.11) guarantees that the Lorentz-frame components S^α of this spin angular momentum remain constant, so long as no external torques act on the particle. This conservation law can be written in special relativistic, frame-independent notation, as Eq. (22.87), specialized to a non-accelerated particle:

$$\nabla_{\vec{u}}\vec{S} = 0 \ ; \tag{24.69}$$

i.e., the spin vector $\vec{S}$ is parallel transported along the world line of the particle (which has 4-velocity $\vec{u}$). If this were a *local* law of physics, it would take this same form, unchanged, in general relativity, i.e., in curved spacetime. Whether the law is local or not depends, clearly, on the size of the particle. If the particle is vanishingly small in its own rest frame, then the law is local and (24.69) will be valid in general relativity. However, if the particle has finite size, the law (24.69) is in danger of failing—and, indeed it does fail if the particle's finite size is accompanied by a finite quadrupole moment. In that case, the coupling of the quadrupole moment $\mathcal{I}_{\alpha\beta}$ to the curvature of spacetime $R^\alpha{}_{\beta\gamma\delta}$ produces a torque on the "particle", causing a breakdown in (24.69):

$$\boxed{S^\alpha{}_{;\mu} u^\mu = \epsilon^{\alpha\beta\gamma\delta} \mathcal{I}_{\beta\mu} R^\mu{}_{\nu\gamma\zeta} u_\delta u^\nu u^\zeta \ .} \tag{24.70}$$

The earth is a good example: the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$ produced at earth by the moon and sun couples to the earth's centrifugal-flattening-induced quadrupole moment $\mathcal{I}_{\mu\nu}$; and the resulting torque (24.70) causes the earth's spin axis to precess relative to the distant stars, with a precession period of 26,000 years—sufficiently fast to show up clearly in historical records as well as in modern astronomical measurements. For details see, e.g., Ex. 16.4 of MTW.

This example illustrates the fact that, if a small amount of nonlocality is present in a physical law, then when lifted from special relativity into general relativity, the law will acquire a small *curvature-coupling* modification.

What is the minimum amount of nonlocality that can produce curvature-coupling modifications in physical laws? As a rough rule of thumb, the minimum amount is double gradients: Because the connection coefficients vanish at the origin of a local Lorentz frame, the local Lorentz components of a single gradient are the same as the components in a global Lorentz frame, e.g., $A^\alpha{}_{;\beta} = \partial A^\alpha / \partial x^\beta$. However, because spacetime curvature prevents the spatial derivatives of the connection coefficients from vanishing at the origin of a local Lorentz frame, any law that involves double gradients is in danger of acquiring curvature-coupling corrections when lifted into general relativity. As an example, it turns out that the wave equation for the electromagnetic vector 4-potential, which in Lorenz gauge takes the form $A^{\alpha;\mu}{}_\mu = 0$ in flat spacetime, becomes in curved spacetime

$$\boxed{A^{\alpha;\mu}{}_\mu = R^{\alpha\mu} A_\mu \ ,} \tag{24.71}$$

where $R^{\alpha\mu}$ is the Ricci curvature tensor; see Ex. 24.11 below. [Note: in Eq. (24.71), and always, all indices that follow the semicolon represent differentiation slots; i.e., $A^{\alpha;\mu}{}_\mu \equiv A^{\alpha;\mu}{}_{;\mu}$.]

The curvature-coupling ambiguities that occur when one lifts slightly nonlocal laws from special relativity into general relativity using the equivalence principle are very similar to "factor-ordering ambiguities" that occur when one lifts a Hamiltonian into quantum mechanics from classical mechanics using the correspondence principle. In the equivalence principle the curvature coupling can be regarded as due to the fact that double gradients, which commute in special relativity, do not commute in general relativity. In the correspondence principle the factor ordering difficulties result from the fact that quantities that commute classically [e.g., position x and momentum p] do not commute quantum mechanically [$\hat{x}\hat{p} \neq \hat{p}\hat{x}$], so when the products of such quantities appear in a classical Hamiltonian one does not know their correct order in the quantum Hamiltonian [does xp become $\hat{x}\hat{p}$, or $\hat{p}\hat{x}$, or $\frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$?].

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EXERCISES

Exercise 24.11 *Example: Curvature coupling in electromagnetic wave equation*

Since the Maxwell equations, written in terms of the classically measureable electromagnetic field tensor **F** [Eqs. (22.70) or (1.65)], involve only single gradients, it is reasonable to expect them to be lifted into curved spacetime without curvature-coupling additions. Assume that this is true.

It can be shown that: (*i*) if one writes the electromagnetic field tensor **F** in terms of a 4-vector potential $\vec{A}$ as

$$F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta} \ , \tag{24.72}$$

then half of the curved-spacetime Maxwell equations, $F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$ [Eqs. (22.70)] are automatically satisfied; (*ii*) **F** is unchanged by gauge transformations in which a gradient is added to the vector potential, $\vec{A} \to \vec{A} + \vec{\nabla}\psi$; and (*iii*) by such a gauge transformation one can impose the Lorentz-gauge condition $\vec{\nabla} \cdot \vec{A} = 0$ on the vector potential.

Show that, when the charge-current 4-vector vanishes, $\vec{J} = 0$, the other half of the Maxwell equations, $F^{\alpha\beta}{}_{;\beta} = 0$ [Eqs. (22.70)] become, in Lorenz gauge and in curved spacetime, the wave equation with curvature coupling, Eq. (24.71).

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24.8 The Einstein Field Equation[5]

One crucial issue remains to be studied in this overview of the foundations of general relativity: What is the physical law that determines the curvature of spacetime? Einstein's search for that law, his *Einstein field equation*, occupied a large fraction of his efforts during the years 1913, 1914, and 1915. Several times he thought he had found it, but each time his proposed law turned out to be fatally flawed; for some flavor of his struggle see the excerpts from his writings in Sec. 17.7 of MTW.

In this section we shall briefly examine one segment of Einstein's route toward his field equation: the segment motivated by contact with Newtonian gravity.

The Newtonian potential Φ is a close analog of the general relativistic spacetime metric **g**: From Φ we can deduce everything about Newtonian gravity, and from **g** we can deduce everything about spacetime curvature. In particular, by differentiating Φ twice we can obtain the Newtonian tidal field E [Eq. (24.33)], and by differentiating the components of **g** twice we can obtain the components of the relativistic generalization of E: the components of the Riemann curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$ [Eq. (24.51) in a local Lorentz frame; Eq. (24.57) in an arbitrary basis].

In Newtonian gravity Φ is determined by Newton's field equation

$$\boxed{\boldsymbol{\nabla}^2\Phi = 4\pi G\rho \ ,} \tag{24.73}$$

which can be rewritten in terms of the tidal field $\mathcal{E}_{jk} = \partial^2\Phi/\partial x^j \partial x^k$ as

$$\mathcal{E}^j{}_j = 4\pi G\rho \ . \tag{24.74}$$

[5] See MTW Chap. 17.

Note that this equates a piece of the tidal field, its trace, to the density of mass. By analogy we can expect the Einstein field equation to equate a piece of the Riemann curvature tensor (the analog of the Newtonian tidal field) to some tensor analog of the Newtonian mass density. Further guidance comes from the demand that in nearly Newtonian situations, e.g., in the solar system, the Einstein field equation should reduce to Newton's field equation. To exploit that guidance, we can (*i*) write the Newtonian tidal field for nearly Newtonian situations in terms of general relativity's Riemann tensor, $\mathcal{E}_{jk} = R_{j0k0}$ [Eq. (24.43); valid in a local Lorentz frame], (*ii*) then take the trace and note that by its symmetries $R^0{}_{000} = 0$ so that $\mathcal{E}^j{}_j = R^\alpha{}_{0\alpha 0} = R_{00}$, and (*iii*) thereby infer that the Newtonian limit of the Einstein equation should read, in a local Lorentz frame,

$$R_{00} = 4\pi G\rho \; . \tag{24.75}$$

Here R_{00} is the time-time component of the Ricci curvature tensor—which can be regarded as a piece of the Riemann tensor. An attractive proposal for the Einstein field equation should now be obvious: Since the equation should be geometric and frame-independent, and since it must have the Newtonian limit (24.75), it presumably should say $R_{\alpha\beta} = 4\pi G\times$ (a second-rank symmetric tensor that generalizes the Newtonian mass density ρ). The obvious required generalization of ρ is the stress-energy tensor $T_{\alpha\beta}$, so

$$R_{\alpha\beta} = 4\pi G T_{\alpha\beta} \; . \tag{24.76}$$

Einstein flirted extensively with this proposal for the field equation during 1913–1915. However, it, like several others he studied, was fatally flawed. When expressed in a coordinate system in terms of derivatives of the metric components $g_{\mu\nu}$, it becomes (because $R_{\alpha\beta}$ and $T_{\alpha\beta}$ both have ten independent components) ten independent differential equations for the ten $g_{\mu\nu}$. This is too many equations: By an arbitrary change of coordinates, $x^\alpha_{\rm new} = F^\alpha(x^0_{\rm old}, x^1_{\rm old}, x^2_{\rm old}, x^3_{\rm old})$ involving four arbitrary functions F^0, F^1, F^2, F^3, one should be able to impose on the metric components four arbitrary conditions, analogous to gauge conditions in electromagnetism (for example, one should be able to set $g_{00} = -1$ and $g_{0j} = 0$ everywhere); and correspondingly, the field equations should constrain only six, not ten of the components of the metric (the six g_{ij} in our example).

In November 1915 Einstein (1915), and independently Hilbert (1915) [who was familiar with Einstein's struggle as a result of private conversations and correspondence] discovered the resolution of this dilemma: Because the local law of 4-momentum conservation guarantees $T^{\alpha\beta}{}_{;\beta} = 0$ independent of the field equation, if we replace the Ricci tensor in (24.76) by a constant (to be determined) times some new curvature tensor $G^{\alpha\beta}$ that is also automatically divergence free independent of the field equation ($G^{\alpha\beta}{}_{;\beta} \equiv 0$), then the new field equation $G^{\alpha\beta} = \kappa T^{\alpha\beta}$ (with κ = constant) will not constrain all ten components of the metric. Rather, in a coordinate system the four equations $[G^{\alpha\beta} - \kappa T^{\alpha\beta}]_{;\beta} = 0$ with $\alpha = 0, 1, 2, 3$ will automatically be satisfied; they will not constrain the metric components in any way, and there will remain in the field equation only six independent constraints on the metric components, precisely the desired number.

It turns out, in fact, that from the Ricci tensor and the scalar curvature one can construct a curvature tensor $G^{\alpha\beta}$ with the desired property:

$$\boxed{G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\ .} \tag{24.77}$$

Today we call this the *Einstein curvature tensor*. That it has vanishing divergence, independently of how one chooses the metric,

$$\boxed{\vec{\nabla} \cdot \mathbf{G} \equiv 0\ ,} \tag{24.78}$$

is called the *contracted Bianchi identity* , since it can be obtained by contracting the following *Bianchi identity* on the tensor $\epsilon_\alpha{}^{\beta\mu\nu}\epsilon_\nu{}^{\gamma\delta\epsilon}$ (Sec. 13.5 of MTW):

$$\boxed{R^\alpha{}_{\beta\gamma\delta;\epsilon} + R^\alpha{}_{\beta\delta\epsilon;\gamma} + R^\alpha{}_{\beta\epsilon\gamma;\delta} = 0\ .} \tag{24.79}$$

[This Bianchi identity holds true for the Riemann curvature tensor of any and every "manifold", i.e. of any and every smooth space; it is derived most easily by introducing a local Lorentz frame, by showing from (24.57) that in such a frame the components $R_{\alpha\beta\gamma\delta}$ of Riemann have the form (24.51) plus corrections that are quadratic in the distance from the origin, by then computing the left side of (24.79), with index α down, at the origin of that frame and showing it is zero, and by then arguing that because the origin of the frame was an arbitrary event in spacetime, and because the left side of (24.79) is the component of a tensor, the left side viewed as a frame-independent geometric object must vanish at all events in the manifold. For an extensive discussion of the Bianchi identities (24.79) and (24.78) see, e.g., Chap. 15 of MTW.]

The Einstein field equation, then, should equate a multiple of $T^{\alpha\beta}$ to the Einstein tensor $G^{\alpha\beta}$:

$$G^{\alpha\beta} = \kappa T^{\alpha\beta}\ . \tag{24.80}$$

The proportionality factor κ is determined from the Newtonian limit: By rewriting the field equation (24.80) in terms of the Ricci tensor

$$R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = \kappa T^{\alpha\beta}\ , \tag{24.81}$$

then taking the trace to obtain $R = -\kappa g_{\mu\nu}T^{\mu\nu}$, then inserting this back into (24.81), we obtain

$$R^{\alpha\beta} = \kappa(T^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}T^{\mu\nu})\ . \tag{24.82}$$

In nearly Newtonian situations and in a local Lorentz frame, the mass-energy density $T^{00} \cong \rho$ is far greater than the momentum density T^{j0} and also far greater than the stress T^{jk}; and correspondingly, the time-time component of the field equation (24.82) becomes

$$R^{00} = \kappa(T^{00} - \frac{1}{2}\eta^{00}\eta_{00}T^{00}) = \frac{1}{2}\kappa T^{00} = \frac{1}{2}\kappa\rho\ . \tag{24.83}$$

By comparing with the correct Newtonian limit (24.75) and noting that in a local Lorentz frame $R_{00} = R^{00}$, we see that

$$\kappa = 8\pi G\ . \tag{24.84}$$

Quantity	Conventional Units	Geometrized Units
speed of light, c	$2.998 \times 10^8\,\mathrm{m\ sec}^{-1}$	one
Newton's gravitation constant, G	$6.673 \times 10^{-11}\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{sec}^{-2}$	one
G/c^2	$7.425 \times 10^{-28}\,\mathrm{m\,kg}^{-1}$	one
c^5/G	$3.629 \times 10^{52}\,\mathrm{W}$	one
$c^2/\sqrt{G}$	3.479×10^{24} gauss cm	one
	$= 1.160 \times 10^{24}$ volts	
Planck's reduced constant $\hbar$	$1.055 \times 10^{-34}\ \mathrm{kg\ m^2 s^{-1}}$	$(1.616 \times 10^{-35}\ \mathrm{m})^2$
sun's mass, $M_\odot$	1.989×10^{30} kg	1.477 km
sun's radius, $R_\odot$	6.960×10^8 m	6.960×10^8 m
earth's mass, $M_\oplus$	5.977×10^{24} kg	4.438 mm
earth's radius, $R_\oplus$	6.371×10^6 m	6.371×10^6 m
Hubble constant H_o	$65 \pm 25\ \mathrm{km\ sec^{-1} Mpc^{-1}}$	$[(12 \pm 5) \times 10^9\ \mathrm{lt\ yr}]^{-1}$
density to close universe, $\rho_{\rm crit}$	$9^{+11}_{-5} \times 10^{-27}\ \mathrm{kg\ m^{-3}}$	$7^{+8}_{-3} \times 10^{-54}\ \mathrm{m^{-2}}$

Table 24.1: Some useful quantities in conventional and geometrized units. *Note*: 1 Mpc = 10^6 parsecs (pc), 1 pc = 3.026 light year ("lt yr"), 1 lt yr = 0.946×10^{16} m, 1 AU = 1.49×10^{11} m. For other useful astronomical constants see C. W. Allen, *Astrophysical Quantities*.

By now the reader must be accustomed to our use of geometrized units in which the speed of light is unity. Just as that has simplified greatly the mathematical notation in Chapters 1, 22 and 23, so also future notation will be greatly simplified if we set Newton's gravitation constant to unity. This further geometrization of our units corresponds to equating mass units to length units via the relation

$$\boxed{1 = \frac{G}{c^2} = 7.42 \times 10^{-28} \frac{\mathrm{m}}{\mathrm{kg}}\ ; \quad \text{i.e., } 1\ \mathrm{kg} = 7.42 \times 10^{-28}\ \mathrm{m}\ .} \tag{24.85}$$

Any equation can readily be converted from conventional units to geometrized units by removing all factors of c and G; and it can readily be converted back by inserting whatever factors of c and G one needs in order to make both sides of the equation dimensionally correct. The caption of Table 24.1 lists a few important numerical quantities in both conventional units and geometrized units. (SI units are badly suited to dealing with relativistic electrodynamics; for this reason J. D. Jackson has insisted on switching from SI to Gaussian units in the last 1/3 of the 1999 edition of his classic textbook, and we do the same in the relativity portions of this book and in Table 24.1.)

In geometrized units the Einstein field equation (24.80), with $\kappa = 8\pi G = 8\pi$ [Eq. (24.84)], assumes the following standard form, to which we shall appeal extensively in coming chapters:

$$\boxed{G^{\mu\nu} = 8\pi T^{\mu\nu}\ ; \quad \text{i.e., } \mathbf{G} = 8\pi \mathbf{T}\ .} \tag{24.86}$$

24.9 Weak Gravitational Fields

The foundations of general relativity are all now in our hands. In this concluding section of the chapter, we shall explore their predictions for the properties of weak gravitational fields, beginning with the Newtonian limit of general relativity and then moving on to more general situations.

24.9.1 Newtonian Limit of General Relativity

A general relativistic gravitational field (spacetime curvature) is said to be *weak* if there exist "nearly globally Lorentz" coordinate systems in which the metric coefficients differ only slightly from unity:

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \;, \quad \text{with} \;\; |h_{\alpha\beta}| \ll 1 \;.} \tag{24.87}$$

The Newtonian limit requires that gravity be weak in this sense throughout the system being studied. It further requires a slow-motion constraint, which has three aspects: (i) The sources of the gravity must have slow enough motions that, with some specific choice of the nearly globally Lorentz coordinates,

$$|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}| \;; \tag{24.88}$$

(ii) the sources' motions must be slow enough that in this frame the momentum density is very small compared to the energy density

$$|T^{j0}| \ll T^{00} \equiv \rho \;; \tag{24.89}$$

and (iii) any particles on which the action of gravity is to be studied must move with low velocities; i.e., must have 4-velocities satisfying

$$|u^j| \ll u^0 \;. \tag{24.90}$$

Finally, the Newtonian limit requires that the stresses in the gravitating bodies be very small compared to their mass densities

$$|T^{jk}| \ll T^{00} \equiv \rho \;. \tag{24.91}$$

When conditions (24.87)–(24.91) are all satisfied, then at leading nontrivial order in the small dimensionless quantities $|h_{\alpha\beta}|$, $|h_{\alpha\beta,t}|/|h_{\alpha\beta,j}|$, $|T^{j0}|/T^{00}$, $|u^j|/u^0$, and $|T^{jk}|/T^{00}$ the laws of general relativity reduce to those of Newtonian theory.

The details of this reduction are an exercise for the reader [Ex. 24.12]; here we give an outline:

The low-velocity constraint $|u^j|/u^0 \ll 1$ on the 4-velocity of a particle, together with its normalization $u^\alpha u^\beta g_{\alpha\beta}$ and the near flatness of the metric (24.87), implies that

$$u^0 \cong 1 \;, \quad u^j \cong v^j \equiv \frac{dx^j}{dt} \;. \tag{24.92}$$

Since $u^0 = dt/d\tau$, the first of these relations implies that in our nearly globally Lorentz coordinate system the coordinate time is very nearly equal to the proper time of our slow-speed particle. In this way, we recover the "universal time" of Newtonian theory. The universal, Euclidean space is that of our nearly Lorentz frame, with $h_{\mu\nu}$ completely ignored because of its smallness. These universal time and universal Euclidean space become the arena in which Newtonian physics is formulated.

Equation (24.92) for the components of a particle's 4-velocity, together with $|v^j| \ll 1$ and $|h_{\mu\nu}| \ll 1$, imply that the geodesic equation for a freely moving particle at leading nontrivial order is

$$\frac{dv^j}{dt} \cong \frac{1}{2} h_{00,j} \quad \text{where } \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \, . \tag{24.93}$$

(Because our spatial coordinates are Cartesian, we can put the spatial index j up on one side of the equation and down on the other without creating any danger of error.)

By comparing Eq. (24.93) with Newton's equation of motion for the particle, we deduce that h_{00} must be related to the Newtonian gravitational potential by

$$h_{00} = -2\Phi \, , \tag{24.94}$$

so the spacetime metric in our nearly globally Lorentz coordinate system must be

$$\boxed{ds^2 = -(1 + 2\Phi)dt^2 + (\delta_{jk} + h_{jk})dx^j dx^k + 2h_{0j} dt\, dx^j \, .} \tag{24.95}$$

Because gravity is weak, only those parts of the Einstein tensor that are linear in $h_{\alpha\beta}$ are significant; quadratic and higher-order contributions can be ignored. Now, by the same mathematical steps as led us to Eq. (24.51) for the components of the Riemann tensor in a local Lorentz frame, one can show that linearized Riemann tensor in our nearly global Lorentz frame have that same form, i.e. (setting $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) \, . \tag{24.96}$$

From this equation and the slow-motion constraint $|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}|$, we infer that the space-time-space-time components of Riemann are

$$\boxed{R_{j0k0} = -\frac{1}{2} h_{00,jk} = \Phi_{,jk} = \mathcal{E}_{jk} \, .} \tag{24.97}$$

In the last step we have used Eq. (24.94). We have thereby recovered the relation between the Newtonian tidal field $\mathcal{E}_{jk} \equiv \Phi_{,jk}$ and the Relativistic tidal field R_{j0k0}. That relation can now be used, via the train of arguments in the preceding section, to show that the Einstein field equation $G^{\mu\nu} = 8\pi T^{\mu\nu}$ reduces to the Newtonian field equation $\nabla^2\Phi = 4\pi T^{00} \equiv 4\pi\rho$.

This analysis leaves the details of h_{0j} and h_{jk} unknown, because the Newtonian limit is insensitive to them.

24.9.2 Linearized Theory

There are many systems in the universe that have weak gravity, but for which the slow-motion approximations (24.88)–(24.90) and/or weak-stress approximation (24.91) fail. Examples are electromagnetic fields and high-speed particles. For such systems we need a generalization of Newtonian theory that drops the slow-motion and weak-stress constraints, but keeps the weak-gravity constraint

$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \, , \quad \text{with } \ |h_{\alpha\beta}| \ll 1 \, .} \tag{24.98}$$

The obvious generalization is a linearization of general relativity in $h_{\alpha\beta}$, with no other approximations being made—the so-called *linearized theory of gravity.* In this subsection we shall develop it.

In formulating linearized theory we can regard the metric pertubation $h_{\mu\nu}$ as a gravitational field that lives in flat spacetime, and correspondingly we can carry out our mathematics as though we were in special relativity. In other words, linearized theory can be regarded as a field theory of gravity in flat spacetime—the type of theory that Einstein toyed with then rejected (Sec. 24.1 above).

In linearized theory, the Riemann tensor takes the form (24.96), but we have no right to simplify it further into the form (24.97), so we must follow a different route to the Einstein field equation:

Contracting the first and third indices in (24.96), we obtain the linearized Ricci tensor $R_{\mu\nu}$, contracting once again we obtain the scalar curvature R, and then from Eq. (24.77) we obtain for the Einstein tensor and the Einstein field equation

$$\begin{aligned} 2G_{\mu\nu} &= h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\beta}{}^{\beta}) \\ &= 16\pi T_{\mu\nu} \ . \end{aligned} \tag{24.99}$$

Here all indices that follow the comma are partial-derivative indices, and

$$h \equiv \eta^{\alpha\beta} h_{\alpha\beta} \tag{24.100}$$

is the "trace" of the metric perturbation. We can simplify the field equation (24.99) by reexpressing it in terms of the quantity

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \ .} \tag{24.101}$$

One can easily check that this quantity has the opposite trace to that of $h_{\mu\nu}$ ($\bar{h} \equiv \bar{h}_{\alpha\beta}\eta^{\alpha\beta} = -h$), so it is called the *trace-reversed metric perturbation.* In terms of it, the field equation (24.99) becomes

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta,}{}^{\alpha\beta} + \bar{h}_{\mu\alpha,\nu}{}^{\alpha} + \bar{h}_{\nu\alpha,\mu}{}^{\alpha} = 16\pi T_{\mu\nu} \ . \tag{24.102}$$

We can simplify this field equation further by specializing our coordinates. We introduce a new nearly globally Lorentz coordinate system that is related to the old one by

$$\boxed{x^{\alpha}_{\rm new}(\mathcal{P}) = x^{\alpha}_{\rm old}(\mathcal{P}) + \xi_{\mu}(\mathcal{P}) \ ,} \tag{24.103}$$

where ξ_μ is a very small vectorial displacement of the coordinate grid. This change of coordinates via four arbitrary functions ($\alpha = 0, 1, 2, 3$) produces a change of the functional form of the metric perturbation $h_{\alpha\beta}$ to

$$\boxed{h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \ ,} \tag{24.104}$$

[Ex. 24.13] and a corresponding change of the trace-reversed metric pertubation. This is linearized theory's analog of a *gauge transformation* in electromagnetic theory. Just as an electromagnetic gauge alters the vector potential $A_\mu^{\rm new} = A_\mu^{\rm old} - \psi_{,\mu}$, so the linearized-theory gauge change alters $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$; and just as the force-producing electromagnetic field tensor $F_{\mu\nu}$ is unaffected by an electromagnetic gauge change, so the tidal-force-producing linearized Riemann tensor is left unaffected by the gravitational gauge change.

By a special choice of the four functions ξ^α, we can impose the following four gauge conditions on $\bar{h}_{\mu\nu}$:

$$\boxed{\bar{h}_{\mu\nu,}{}^{\nu} = 0 \ .} \tag{24.105}$$

These, obviously, are linearized theory's analog of the electromagnetic Lorentz gauge condition $A_\mu,{}^\mu = 0$, so they are called the *gravitational Lorenz gauge*. Just as the flat-spacetime Maxwell equations take the remarkably simple wave-equation form $A_{\mu,\alpha}{}^\alpha = 4\pi J_\mu$ in Lorenz gauge, so also the linearized Einstein equation (24.102) takes the corresponding simple wave-equation form in gravitational Lorenz gauge:

$$\boxed{-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi T_{\mu\nu} \ .} \tag{24.106}$$

By the same method as one uses in electromagnetic theory, one can solve this gravitational field equation for the field $\bar{h}_{\mu\nu}$ produced by an arbitrary stress-energy-tensor source:

$$\boxed{\bar{h}_{\mu\nu}(t, \mathbf{x}) = \int \frac{4T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}'])}{|\mathbf{x} - \mathbf{x}'|} dV_{x'}} \tag{24.107}$$

The quantity in the numerator is the stress-energy source evaluated at the "retarded time" $t' = t - |\mathbf{x} - \mathbf{x}'|$. This equation for the field, and the wave equation (24.106) that underlies it, show explicitly that dynamically changing distributions of stress-energy must generate *gravitational waves*, which propagate outward from their source at the speed of light (Einstein, 1918). We shall study these gravitational waves in Chap. 25.

24.9.3 Gravitational Field Outside a Stationary, Linearized Source

Let us specialize to a time-independent source (so $T_{\mu\nu,t} = 0$ in our chosen nearly globally Lorentz frame), and compute its external gravitational field as a power series in 1/(distance to source). We place our origin of coordinates at the source's center of mass, so

$$\int x^j T^{00} dV_x = 0 \ , \tag{24.108}$$

and in the same manner as in electromagnetic theory, we expand

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{x^j x^{j'}}{r^3} + \ldots \ , \tag{24.109}$$

where $r \equiv |\mathbf{x}|$ is the distance of the field point from the source's center of mass. Inserting Eq. (24.109) into the general solution (24.107) of the Einstein equation and taking note of the conservation laws $T^{\alpha j}{}_{,j} = 0$, we obtain for the source's external field

$$\boxed{\bar{h}_{00} = \frac{4M}{r} + {\rm O}\left(\frac{1}{r^3}\right) \ , \quad \bar{h}_{0j} = -\frac{2\epsilon_{jkm} S^k x^m}{r^3} + {\rm O}\left(\frac{1}{r^3}\right) \ , \quad \bar{h}_{ij} = {\rm O}\left(\frac{1}{r^3}\right) \ ;} \tag{24.110}$$

Here M and S^k are the source's mass and angular momentum:

$$\boxed{M \equiv \int T^{00} dV_x \; , \quad S_k \equiv \int \epsilon_{kab} x^a T^{0b} dV_x \; .} \tag{24.111}$$

see Ex. 24.14. This expansion in $1/r$, as in the electromagnetic case, is a multipolar expansion. At order $1/r$ the field is spherically symmetric and the monopole moment is the source's mass M. At order $1/r^2$ there is a "magnetic-type dipole moment", the source's spin angular momentum S_k. These are the leading-order moments in two infinite sets: the "mass multipole" moments (analog of electric moments), and the "mass-current multipole" moments (analog of magnetic moments). For details on all the higher order moments, see, e.g., Thorne (1980).

The metric perturbation can be computed by reversing the trace reversal, $h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \eta_{\alpha\beta}\bar{h}$. Thereby we obtain for the spacetime metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ at linear order, outside the source,

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4\epsilon_{jkm} S^k x^m}{r^3} dt dx^j + \left(1 + \frac{2M}{r}\right) \delta_{jk} dx^j dx^k + \mathcal{O}\left(\frac{1}{r^3}\right) dx^\alpha dx^\beta \; .} \tag{24.112}$$

In spherical polar coordinates, with the polar axis along the direction of the source's angular momentum, the leading order terms take the form

$$\boxed{ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4S}{r} \sin^2\theta dt d\phi + \left(1 + \frac{2M}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \; ,} \tag{24.113}$$

where $S \equiv |\mathbf{S}|$ is the magnitude of the source's angular momentum.

This is a very important result. It tells us that we can "read off" the mass M and angular momentum S^k from the asymptotic form of the source's metric. In the next chapter we shall devise, from the metric (24.113), physical measurements that one can make outside the source to determine its mass and angular momentum. As one would expect from Newtonian theory, the mass M will show up as the source of a "gravitational acceleration" that can be measured via Kepler's laws for an orbiting particle. It will turn out that the angular-momentum term in the metric shows up physically via a *dragging of inertial frames* that causes inertial-guidance gyroscopes near the body to precess relative to the "distant stars".

For a time-independent body with *strong* internal gravity (e.g. a black hole), the distant gravitational field will have the same general form (24.112), (24.113) as for a weakly gravitating body, but the constants M and S^k that appear in the metric will not be expressible as the integrals (24.111) over the body's interior. Nevertheless, they will be measureable by the same techniques as for a weakly gravitating body (Kepler's laws and frame dragging), and they can be interpreted as the body's total mass and angular momentum.

24.9.4 Conservation Laws for Mass, Momentum and Angular Momentum

Consider a static (unmoving) sphere $\mathcal{S}$ surrounding our time-independent source of gravity, with such a large radius r that the O$(1/r^3)$ corrections in $\bar{h}_{\mu\nu}$ and in the metric [Eqs. (24.111)–(24.113)] can be ignored. Suppose that a small amount of mass-energy E (as measured in the sphere's and source's rest frame) is injected through the sphere, into the source. Then the special relativistic law of mass-energy conservation tells us that the source's mass $M = \int T^{00} dV_x$ will increase by $\Delta M = E$. Similarly, if an energy flux T^{0j} flows through the sphere, the source's mass will change by

$$\boxed{\frac{dM}{dt} = -\int_{\mathcal{S}} T^{0j} d\Sigma_j \ ,} \tag{24.114}$$

where $d\Sigma_j$ is the sphere's outward-pointing surface-area element, and the minus sign is due to the fact that $d\Sigma_j$ points outward, not inward. Since M is the mass that appears in the source's asymptotic gravitational field $\bar{h}_{\mu\nu}$ and metric $g_{\alpha\beta}$, this conservation law can be regarded as describing how the source's gravitating mass changes when energy is injected into it.

From the special relativistic law for angular momentum conservation, we deduce a similar result: A flux $\epsilon_{ijk} x^j T^{km}$ of angular momentum through the sphere produces the following change in the angular momentum S_k that appears in the source's asymptotic field $\bar{h}_{\mu\nu}$ and metric:

$$\boxed{\frac{dS_i}{dt} = -\int_{\mathcal{S}} \epsilon_{ijk} x^j T^{km} d\Sigma_m \ .} \tag{24.115}$$

There is also a conservation law for a gravitationally measured linear momentum. That linear momentum does not show up in the asymptotic field and metric that we wrote down above [Eqs. (24.111)–(24.113)] because our coordinates were chosen to be attached to the source's center of mass—i.e., they are the Lorentz coordinates of the source's rest frame. However, if linear momentum P_j is injected through our sphere $\mathcal{S}$ and becomes part of the source, then the source's center of mass will start moving, and the asymptotic metric will acquire a new term

$$\delta g_{0j} = -4P_j/r \ , \tag{24.116}$$

where (after the injection)

$$P_j = P^j = \int T^{0j} dV_x \tag{24.117}$$

[see Eq. (24.107) with $\bar{h}^{0j} = -\bar{h}_{0j} = -h_{0j} = -\delta g_{0j}$; also see Ex 24.14b]. More generally, the rate of change of the source's total linear momentum (the P_j term in the asymptotic g_{0j}) is the integral of the inward flux of momentum (inward component of the stress tensor) across the sphere:

$$\boxed{\frac{dP_j}{dt} = -\int_{\mathcal{S}} T^{jk} d\Sigma_j \ .} \tag{24.118}$$

For a time-independent source with *strong* internal gravity, not only does the asymptotic metric, far from the source, have the same form (24.112), (24.113), (24.116) as for a weakly gravitating source; the conservation laws (24.114), (24.115), (24.118) for its gravitationally measured mass, angular momentum and linear momentum continue to hold true. The sphere $\mathcal{S}$, of course, must be placed far from the source, in a region where gravity is very weak, so linearized theory will be valid in the vicinity of $\mathcal{S}$. When this is done, then the special relativistic description of inflowing mass, angular momentum and energy is valid at $\mathcal{S}$, and the linearized Einstein equations, applied in the vicinity of $\mathcal{S}$ (and not extended into the strong-gravity region), turn out to guarantee that the M, S_j and P_j appearing in the asympototic metric evolve in accord with the conservation laws (24.114), (24.115), (24.118).

For strongly gravitating sources, these conservation laws owe their existence to the spacetime's asymptotic time-translation, rotation, and space-translation symmetries. In generic, strong-gravity regions of spacetime there are no such symmetries, and correspondingly no integral conservation laws for energy, angular momentum, or linear momentum.

If a strongly gravitating source is dynamical rather than static, it will emit gravitational waves (Chap. 25). The amplitudes of those waves, like the influence of the source's mass, die out as $1/r$ far from the source, so spacetime retains its asymptotic time-translation, rotation and space-translation symmetries. These symmetries continue to enforce integral conservation laws on the gravitationally measured mass, angular momentum and linear momentum [Eqs. (24.114), (24.115), (24.118)], but with the new requirement that one include, in the fluxes through $\mathcal{S}$, contributions from the gravitational waves' energy, angular momentum and linear momentum; see Chap. 25.

For a more detailed and rigorous derivation and discussion of these asymptotic conservation laws, see Chaps. 18 and 19 of MTW.

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EXERCISES

Exercise 24.12 *Derivation: Newtonian limit of general relativity*
Consider a system that can be covered by a nearly globally Lorentz coordinate system in which the Newtonian-limit constraints (24.87)–(24.91) are satisfied. For such a system, flesh out the details of the text's derivation of the Newtonian limit. More specifically:

(a) Derive Eq. (24.92) for the components of the 4-velocity of a particle.

(b) Show that the geodesic equation reduces to Eq. (24.93).

(c) Show that to linear order in the metric perturbation $h_{\alpha\beta}$ the components of the Riemann tensor take the form (24.96).

(d) Show that in the slow-motion limit the space-time-space-time components of Riemann take the form (24.97).

Exercise 24.13 *Derivation: Gauge Transformations in Linearized Theory*

(a) Show that the "infinitesimal" coordinate transformation (24.103) produces the change (24.104) of the linearized metric perturbation.

(b) Exhibit a differential equation for the ξ^α that brings the metric perturbation into gravitational Lorenz gauge, i.e. that makes $h^{\rm new}_{\mu\nu}$ obey the Lorenz gauge condtion (24.105)

(c) Show that in gravitational Lorenz gauge, the Einstein field equation (24.102) reduces to (24.106).

Exercise 24.14 *Derivation: External Field of Stationary, Linearized Source*
Derive Eqs. (24.110) for the trace reversed metric perturbation outside a stationary (time-independent), linearized source of gravity. More specifically:

(a) First derive $\bar h_{00}$. In your derivation identify a dipolar term of the form $4D_j x^j/r^3$, and show that by placing the origin of coordinates on the center of mass, Eq. (24.108), one causes the dipole moment D_j to vanish.

(b) Next derive $\bar h_{0j}$. The two terms in (24.109) should give rise to two terms. The first of these is $4P_j/r$ where P_j is the source's linear momentum. Show, using the gauge condition $\bar h^{0\mu}_{\ ,\mu} = 0$ [Eq. (24.105)] that if the momentum is nonzero, then the mass dipole term of part (a) must have a nonzero time derivative, which violates our assumption of stationarity. Therefore, for this source the linear momentum must vanish. Show that the second term gives rise to the $\bar h_{0j}$ of Eq. (24.110). [Hint: you will have to add a perfect divergence, $(T^{0a'}x^{j'}x^{m'})_{,a'}$ to the integrand.]

(c) Finally derive $\bar h_{ij}$. [Hint: Show that $T^{ij} = (T^{ia}x^i)_{,a}$ and thence that the volume integral of T^{ij} vanishes; and similarly for $T^{ij}x^k$.]

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Bibliographic Note

For a superb, detailed historical account of Einstein's intellectual struggle to formulate the laws of general relativity, see Pais (1982). For Einstein's papers of that era, in the original German and in English translation, with detailed annotations and explanations by editors with strong backgrounds in both physics and history of science, see Einstein (1989–2002). For some key papers of that era by other major contributors besides Einstein, in English translation, see Einstein, Lorentz, Minkowski and Weyl (1923).

This chapter's pedagogical approach to presenting the fundamental concepts of general relativity is strongly influenced by MTW (Misner, Thorne and Wheeler 1973), where readers will find much greater detail. See, especially, Chap. 8 for the the mathematics (differential geometry) of curved spacetime, or Chaps. 9–14 for far greater detail; Chap. 16 for the Einstein equivalence principle and how to lift laws of physics into curved spacetime; Chap. 17 for the Einstein field equations and many different ways to derive them; Chap. 18 for

Box 24.2
Important Concepts in Chapter 23

- Local Lorentz frame, Sec. 24.2
  - Nonmeshing of local Lorentz frames due to spacetime curvature, Sec. 24.3
  - Metric and connection coefficients in Local Lorentz frame, Eqs. (24.15) and (24.16)
- Principle of relativity, Sec. 24.2
- Motion of a freely falling particle: geodesic with $\nabla_{\vec p}\vec p = 0$, Sec. 24.4
  - Geodesic equation in any coordinate system, Eq. (24.26)
  - Conserved quantity associated with symmetry of the spacetime, Ex. 24.4
  - Action principle for geodesic, Ex. 24.5
- Tidal Gravity and Spacetime Curvature
  - Newtonian Tidal field $\mathcal{E}_{ij} = \partial^2\Phi/\partial x^i \partial x^j$, Sec. 24.5.1
  - Riemann curvature as tidal field; equation of geodesic deviation, Sec. 24.5.1
  - Connection of relavistic and Newtonian tidal fields, $R_{j0k0} = \mathcal{E}_{jk}$, Sec. 24.5.2
  - Tidal field outside the Earth or other spherical, gravitating body, Eq. (24.44).
- Properties of Riemann tensor (symmetries, Ricci Tensor, Curvature scalar, how to compute its components, radius of curvature of spacetime), Sec. 24.6
- Einstein's equivalence principle: Lifting the laws of physics into curved spacetime, Secs. 24.2 and 24.7
  - Curvature coupling effects, Sec. 24.7
  - Breakdown of global conservation laws for energy and momentum, Sec. 24.7
- The Einstein field equation, Sec. 24.8
  - Its connection to Newton's field equation, Sec. 24.8
  - Einstein tensor and its vanishing divergence, Sec. 24.8
- Geometrized units, Eq. (24.85) and Table 24.1
- Newtonian limit of general relativity, Sec. 24.9.1
  - Conditions for validity: weak gravity, slow motion, small stresses, Sec. 24.9.1
- Linearized theory, Sec. 24.9.2
  - Gravitational Lorenz gauge, Eq. (24.105)
  - Wave equation for metric pertubation, with stress-energy tensor as source, Eqs. (24.106), (24.107)
- Metric outside a stationary, linearized source, Sec. 24.9.3
  - Roles of mass and angular momentum in metric, Sec. 24.9.3
  - Integral conservation laws for source's mass, linear momentum and angular momentum, Sec. 24.9.4

weak gravitational fields (the Newtonian limit and Linearized Theory); and Chaps. 19 and 20 for the metric outside a stationary, linearized source and for the source's conservation laws for mass, momentum, and angular momentum.

For a superb, elementary introduction to the fundamental concepts of general relativity from a viewpoint that is somewhat less mathematical than this chapter or MTW, see Hartle (2003). We also recommend, at a somewhat elementary level, Schutz (1985), and at a more advanced level, Carroll (2004), and at a very advanced and mathematical level, Wald (1984).

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