Contents

Part I

FOUNDATIONS

In this book, a central theme will be a *Geometric Principle: The laws of physics must all be expressible as geometric (coordinate-independent and reference-frame-independent) relationships between geometric objects, which represent physical entitities.*

There are three different conceptual frameworks for the classical laws of physics, and correspondingly three different geometric arenas for the laws; see Fig. 1. *General Relativity* is the most accurate classical framework; it formulates the laws as geometric relationships between geometric objects in the arena of *curved 4-dimensional spacetime*. *Special Relativity* is the limit of general relativity in the complete absence of gravity; its arena is *flat, 4-dimensional Minkowski spacetime*[1]. *Newtonian Physics* is the limit of general relativity when (i) gravity is weak but not necessarily absent, (ii) relative speeds of particles and materials are small compared to the speed of light c, and (iii) all stresses (pressures) are small compared to the total density of mass-energy; its arena is *flat, 3-dimensional Euclidean space* with time separated off and made universal (by contrast with relativity's reference-frame-dependent time).

In Parts II–VI of this book (statistical physics, optics, elasticity theory, fluid mechanics, plasma physics) we shall confine ourselves to the Newtonian formulations of the laws (plus special relativistic formulations in portions of Track 2), and accordingly our arena will be flat Euclidean space (plus flat Minkowski spacetime in portions of Track 2). In Part VII, we shall extend many of the laws we have studied into the domain of strong gravity (general relativity), i.e., the arena of curved spacetime.

In Parts II and III (statistical physics and optics), in addition to confining ourselves to flat space (plus flat spacetime in Track 2), we shall avoid any sophisticated use of curvilinear coordinates; i.e., when using coordinates in nontrivial ways, we shall confine ourselves to Cartesian coordinates in Euclidean space (and Lorentz coordinates in Minkowski spacetime).

Part I of this book contains just two chapters. ***Chapter 1 is an introduction to the geometric viewpoint on Newtonian physics, and to all the tools of differential geometry that we shall need in Parts II and III for Newtonian Physics in its arena, 3-dimensional Euclidean space. Chapter 2 introduces the geometric view-***

[1]so-called because it was Hermann Minkowski (1908) who identified the special relativistic invariant interval as defining a metric in spacetime, and who elucidated the resulting geometry of flat spacetime.

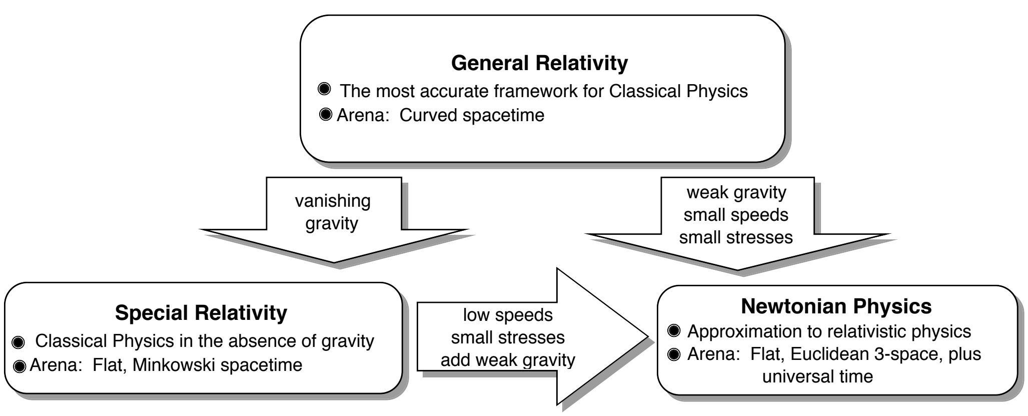

Fig. 1: The three frameworks and arenas for the classical laws of physics, and their relationship to each other.

point on Special Relativistic Physics, and extends our differential geometric tools into special relativity's arena, flat Minkowski spacetime.

In Parts IV, V, and VI, when studying elasticity theory, fluid mechanics, and plasma physics, we will use curvilinear coordinates in nontrivial ways. As a foundation for this, at the beginning of Part IV we will extend our flat-space differential geometric tools to curvilinear coordinate systems (e.g. cylindrical and spherical coordinates). Finally, at the beginning of Part VII, we shall extend our geometric tools to the arena of curved spacetime.

Throughout this book we shall pay close attention to the relationship between classical physics and quantum physics. Indeed, we shall often find it powerful to use quantum mechanical language or formalism when discussing and analyzing classical phenomena. This quantum power in classical domains arises from the fact that quantum physics is primary and classical physics is secondary. Classical physics arises from quantum physics, not conversely. The relationship between quantum frameworks and arenas for the laws of physics, and classical frameworks, is sketched in Fig. 2.

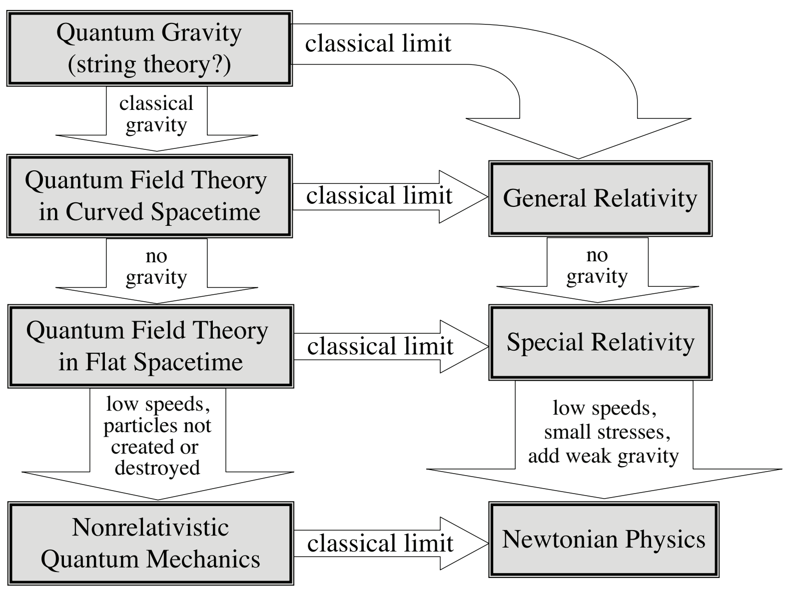

Fig. 2: The relationship of the three frameworks for classical physics (on right) to four frameworks for quantum physics (on left). Each arrow indicates an approximation. All other frameworks are approximations to the ultimate laws of quantum gravity (whatever they may be — perhaps a variant of string theory).

Chapter 1

Newtonian Physics: Geometric Viewpoint

Version 1101.1.K by Kip, 15 September 2011

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Box 1.1
Reader's Guide

- This chapter is a foundation for almost all of this book.
- Much of the material in this chapter will already be familiar to many readers, but the geometric viewpoint on the material may not be familiar. *All readers should make sure they understand this geometric viewpoint.* To do so, they should browse if not read the entire chapter, making sure, especially, to look at Secs. 1.1.1, 1.3, 1.5, 1.7, and 1.8.
- The stress tensor, introduced and discussed in Sec. 1.9 will play an important role in Kinetic Theory (Chap. 3), and a crucial role in Elasticity Theory (Part IV), Fluid Mechanics (Part V) and Plasma Physics (Part VI).
- The integral and differential conservation laws derived and discussed in Secs. 1.8 and 1.9 will play major roles throughout this book.
- The Box labeled **T2** is advanced material ("Track 2") that can be skipped in an elementary or short course, or on a first reading of this book.

1.1 Introduction

1.1.1 The Geometric Viewpoint on the Laws of Physics

In this book, we shall adopt a different viewpoint on the laws of physics than that in most elementary and intermediate texts. In most textbooks, physical laws are expressed in terms of quantities (locations in space, momenta of particles, etc.) that are measured in some coordinate system. For example, Newtonian vectorial quantities are expressed as triplets of numbers, e.g., $\mathbf{p} = (p_x, p_y, p_z) = (1, 9, -4)$, representing the components of a particle's momentum on the axes of a spatial coordinate system; and tensors are expressed as arrays of numbers, e.g.

$$\mathsf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{1.1}$$

for the moment of inertia tensor.

By contrast, in this book, we shall express all physical quantities and laws in a **geometric form**, i.e. a form that is **independent of any coordinate system or basis vectors**. For example, a particle's velocity $\mathbf{v}$ and the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ that it encounters will be vectors described as arrows that live in the 3-dimensional, flat Euclidean space of everyday experience. They require no coordinate system or basis vectors for their existence or description—though sometimes coordinates will be useful.

We shall insist that the Newtonian laws of physics all obey a *Geometric Principle*: they are all geometric relationships between geometric objects, expressible without the aid of any coordinates or bases. An example is the Lorentz force law $md\mathbf{v}/dt = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ — a (coordinate-free) relationship between the geometric (coordinate-independent) vectors $\mathbf{v}$, $\mathbf{E}$ and $\mathbf{B}$ (and the particle's mass m and charge q). As another example, a body's moment of inertia tensor I can be viewed as a vector-valued linear function of vectors (a coordinate-independent, basis-independent geometric object!). Insert into the tensor I the body's angular velocity vector $\mathbf{\Omega}$ and you will get out the body's angular momentum vector: $\mathbf{L} = \mathsf{I}(\mathbf{\Omega})$. No coordinates or basis vectors are needed for this law of physics, nor is any description of I as a matrix-like entity with components I_{ij}. Components are secondary; they only exist after one has chosen a set of basis vectors. Components are an impediment to a clear and deep understanding of the laws of physics. The coordinate-free, component-free description is deeper, and—once one becomes accustomed to it—much more clear and understandable.

By adopting this geometric viewpoint, we shall gain great conceptual power, and often also computational power. For example, when we ignore experiment and simply ask what forms the laws of physics can possibly take (what forms are allowed by the requirement that the laws be geometric), we shall find that there is remarkably little freedom. Coordinate independence and basis independence strongly constrain the laws of physics. This power, together with the elegance of the geometric formulation, suggests that in some deep (ill-understood) sense, Nature's physical laws *are* geometric and have nothing whatsoever to do with coordinates or components or vector bases.

1.1.2 Purposes of this Chapter

The principal purpose of this foundational chapter is to teach the reader this geometric viewpoint — and in the process, help some readers ***unlearn*** *the more complex and obfuscatory viewpoints on physics that are currently burdened with.*

The mathematical foundation for our geometric viewpoint is *differential geometry* (also called "tensor analysis" by physicists). Differential geometry can be thought of as an extension of the vector analysis with which all readers should be familiar. *A second purpose of this chapter is to develop differential geometry in a simple form well adapted to Newtonian classical physics.*

1.1.3 Overview of This Chapter

In this chapter, we shall we shall lay the geometric foundations for the Newtonian laws of physics in flat Euclidean space. We begin in Sec. 1.2 by introducing some foundational geometric concepts: points, scalars, vectors, inner products of vectors, distance between points. Then in Sec. 1.3, we introduce the concept of a tensor as a linear function of vectors, and we develop a number of differential geometric tools: the tools of coordinate-free tensor algebra. In Sec. 1.4, we illustrate our tensor-algebra tools by using them to describe—without any coordinate system—the kinematics of a charged point particle that moves through Euclidean space, driven by electric and magnetic forces.

In Sec. 1.5, we introduce, for the first time, Cartesian coordinate systems and their basis vectors, and also the components of vectors and tensor on those basis vectors; and we explore how to express geometric relationships in the language of components. In Sec. 1.6, we deduce how the components of vectors and tensors transform, when one rotates one's Cartesian coordinate axes. (These are the transformation laws that most physics textbooks use to define vectors and tensors.)

In Sec. 1.7, we introduce directional derivatives and gradients of vectors and tensors, thereby moving from tensor algebra to true differential geometry (in Euclidean space). We also introduce the Levi-Civita tensor and use it to define curls and cross products, and we learn how to use *index gymnastics* to derive, quickly, formulae for multiple cross products. In Sec. 1.8 we use the Levi-Civita tensor to define vectorial areas and scalar volumes, and integration over surfaces. These concepts then enable us to formulate, in geometric, coordinate-free ways, *integral and differential conservation laws.* In Sec. 1.9 we discuss, in particular, the law of momentum conservation, formulating it in a geometric way with the aid of a geometric object called the *stress tensor*. As important examples, we use this geometric conservation law to derive and discuss the equations of Newtonian fluid dynamics, and the interaction between a charged medium and an electromagnetic field. We conclude in Sec. 1.10 with some concepts from special relativity that we shall need in our discussions of Newtonian physics.

1.2 Foundational Concepts

The arena for the Newtonian laws is a spacetime composed of the familiar 3-dimensional Euclidean space of everyday experience (which we shall call *3-space*), and a universal time t. We shall denote points (locations) in 3-space by capital script letters such as $\mathcal{P}$ and $\mathcal{Q}$. These points and the 3-space in which they live require no coordinates for their definition.

A *scalar* is a single number that we associate with a point, $\mathcal{P}$, in 3-space. We are interested in scalars that represent physical quantities, e.g., temperature T. When a scalar is a function of location $\mathcal{P}$ in space, e.g. $T(\mathcal{P})$, we call it a *scalar field*.

A *vector* in Euclidean 3-space can be thought of as a straight arrow that reaches from one point, $\mathcal{P}$, to another, $\mathcal{Q}$ (e.g., the arrow $\Delta\mathbf{x}$ of Fig. 1.1a). Equivalently, $\Delta\mathbf{x}$ can be thought of as a direction at $\mathcal{P}$ and a number, the vector's length. Sometimes we shall select one point $\mathcal{O}$ in 3-space as an "origin" and identify all other points, say $\mathcal{Q}$ and $\mathcal{P}$, by their vectorial separations $\mathbf{x}_\mathcal{Q}$ and $\mathbf{x}_\mathcal{P}$ from that origin.

The Euclidean distance $\Delta\sigma$ between two points $\mathcal{P}$ and $\mathcal{Q}$ in 3-space can be measured with a ruler and so, of course, requires no coordinate system for its definition. (If one does have a Cartesian coordinate system, it can be computed by the Pythagorean formula, a precursor to the "invariant interval" of flat spacetime, Sec. **??**.) This distance $\Delta\sigma$ is also the length $|\Delta\mathbf{x}|$ of the vector $\Delta\mathbf{x}$ that reaches from $\mathcal{P}$ to $\mathcal{Q}$, and the square of that length is denoted

$$|\Delta\mathbf{x}|^2 \equiv (\Delta\mathbf{x})^2 \equiv (\Delta\sigma)^2 \ . \tag{1.2}$$

Of particular importance is the case when $\mathcal{P}$ and $\mathcal{Q}$ are neighboring points and $\Delta\mathbf{x}$ is a differential (infinitesimal) quantity $d\mathbf{x}$. By traveling along a sequence of such $d\mathbf{x}$'s, laying them down tail-at-tip, one after another, we can map out a curve to which these $d\mathbf{x}$'s are tangent (Fig. 1.1b). The curve is $\mathcal{P}(\lambda)$, with λ a parameter along the curve; and the infinitesimal vectors that map it out are $d\mathbf{x} = (d\mathcal{P}/d\lambda)d\lambda$.

The product of a scalar with a vector is still a vector; so if we take the change of location $d\mathbf{x}$ of a particular element of a fluid during a (universal) time interval dt, and multiply it by $1/dt$, we obtain a new vector, the fluid element's velocity $\mathbf{v} = d\mathbf{x}/dt$, at the fluid element's location $\mathcal{P}$. Performing this operation at every point $\mathcal{P}$ in the fluid defines the velocity field $\mathbf{v}(\mathcal{P})$. Similarly, the sum (or difference) of two vectors is also a vector and so taking the difference of two velocity measurements at times separated by dt and multiplying by $1/dt$ generates the acceleration $\mathbf{a} = d\mathbf{v}/dt$. Multiplying by the fluid element's (scalar) mass m

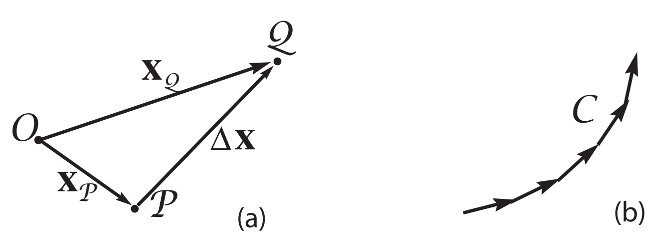

Fig. 1.1: (a) A Euclidean 3-space diagram depicting two points $\mathcal{P}$ and $\mathcal{Q}$, their vectorial separations $\vec{x}_\mathcal{P}$ and $\vec{x}_\mathcal{Q}$ from the (arbitrarily chosen) origin $\mathcal{O}$, and the vector $\Delta\mathbf{x} = \mathbf{x}_\mathcal{Q} - \mathbf{x}_\mathcal{P}$ connecting them. (b) A curve $\mathcal{C}$ generated by laying out a sequence of infinitesimal vectors, tail-to-tip.

gives the force $\mathbf{F} = m\mathbf{a}$ that produced the acceleration; dividing an electrically produced force by the fluid element's charge q gives another vector, the electric field $\mathbf{E} = \mathbf{F}/q$, and so on. We can define inner products [Eq. (1.4a) below] of pairs of vectors at a point (e.g., force and displacement) to obtain a new scalar (e.g., work), and cross products [Eq. (1.21a)] of vectors to obtain a new vector (e.g., torque). By examining how a differentiable scalar field changes from point to point, we can define its gradient [Eq. (1.15b)]. In this fashion, which should be familiar to the reader and will be elucidated and generalized below, we can construct all of the standard scalars and vectors of Newtonian physics. What is important is that *these physical quantities require no coordinate system for their definition.* They are geometric (coordinate-independent) objects residing in Euclidean 3-space at a particular time.

It is a fundamental (though often ignored) principle of physics that *the Newtonian physical laws are all expressible as geometric relationships between these types of geometric objects, and these relationships do not depend upon any coordinate system or orientation of axes, nor on any reference frame* (on any purported velocity of the Euclidean space in which the measurements are made).[1] We shall call this the *Geometric Principle* for the laws of physics, and we shall use it throughout this book. It is the Newtonian analog of Einstein's Principle of Relativity (Sec. **??** below).

1.3 Tensor Algebra Without a Coordinate System

In preparation for developing our geometric view of physical laws, we now introduce, in a coordinate-free way, some fundamental concepts of differential geometry: tensors, the inner product, the metric tensor, the tensor product, and contraction of tensors.

We have already defined a vector $\mathbf{A}$ as a straight arrow from one point, say $\mathcal{P}$, in our space to another, say $\mathcal{Q}$. Because our space is flat, there is a unique and obvious way to transport such an arrow from one location to another, keeping its length and direction unchanged.[2] Accordingly, we shall regard vectors as unchanged by such transport. This enables us to ignore the issue of where in space a vector actually resides; it is completely determined by its direction and its length.

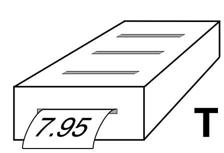

Fig. 1.2: A rank-3 tensor T.

A *rank-n tensor* T is, by definition, a real-valued, linear function of n vectors. Pictorially we shall regard T as a box (Fig. 1.2) with n slots in its top, into which are inserted n vectors,

[1]By changing the velocity of Euclidean space, one adds a constant velocity to all particles, but this leaves the laws, e.g. Newton's $\mathbf{F} = m\mathbf{a}$, unchanged.

[2]This is not so in curved spaces, as we shall see in Sec. 24.7.

and one slot in its end, out of which rolls computer paper with a single real number printed on it: the value that the tensor T has when evaluated as a function of the n inserted vectors. Notationally we shall denote the tensor by a bold-face sans-serif character T

$$\mathsf{T}(\underbrace{\_\,,\_\,,\_\,,\_}) \qquad (1.3a)$$

$\nwarrow$ n slots in which to put the vectors.

If T is a rank-3 tensor (has 3 slots) as in Fig. 1.2, then its value on the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ will be denoted $\mathsf{T}(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Linearity of this function can be expressed as

$$\mathsf{T}(e\mathbf{E} + f\mathbf{F}, \mathbf{B}, \mathbf{C}) = e\mathsf{T}(\mathbf{E}, \mathbf{B}, \mathbf{C}) + f\mathsf{T}(\mathbf{F}, \mathbf{B}, \mathbf{C}) \ , \qquad (1.3b)$$

where e and f are real numbers, and similarly for the second and third slots.

We have already defined the *squared length* $(\mathbf{A})^2 \equiv \mathbf{A}^2$ of a vector $\mathbf{A}$ as the squared distance (in 3-space) or interval (in spacetime) between the points at its tail and its tip. The *inner product* $\mathbf{A} \cdot \mathbf{B}$ of two vectors is defined in terms of the squared length by

$$\mathbf{A} \cdot \mathbf{B} \equiv \frac{1}{4}\left[(\mathbf{A} + \mathbf{B})^2 - (\mathbf{A} - \mathbf{B})^2\right] \ . \qquad (1.4a)$$

In Euclidean space, this is the standard inner product, familiar from elementary geometry.

Because the inner product $\mathbf{A} \cdot \mathbf{B}$ is a linear function of each of its vectors, we can regard it as a tensor of rank 2. When so regarded, the inner product is denoted $\mathsf{g}(\_\,,\_)$ and is called the *metric tensor*. In other words, the metric tensor g is that linear function of two vectors whose value is given by

$$\mathsf{g}(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B} \ . \qquad (1.4b)$$

Notice that, because $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$, the metric tensor is *symmetric* in its two slots; i.e., one gets the same real number independently of the order in which one inserts the two vectors into the slots:

$$\mathsf{g}(\mathbf{A}, \mathbf{B}) = \mathsf{g}(\mathbf{B}, \mathbf{A}) \qquad (1.4c)$$

With the aid of the inner product, we can regard any vector $\mathbf{A}$ as a tensor of rank one: The real number that is produced when an arbitrary vector $\mathbf{C}$ is inserted into $\mathbf{A}$'s slot is

$$\mathbf{A}(\mathbf{C}) \equiv \mathbf{A} \cdot \mathbf{C} \ . \qquad (1.4d)$$

Second-rank tensors appear frequently in the laws of physics—often in roles where one sticks a single vector into the second slot and leaves the first slot empty thereby producing a single-slotted entity, a vector. A familiar example is a rigid body's (Newtonian) moment-of-inertia tensor $\mathsf{I}(\_\,,\_)$. Insert the body's angular velocity vector $\boldsymbol{\Omega}$ into the second slot, and you get the body's angular momentum vector $\mathbf{J}(\_) = \mathsf{I}(\_\,,\boldsymbol{\Omega})$. Another example is the stress tensor of a solid, a fluid, a plasma, or a field (Sec. 1.9 below).

From three (or any number of) vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ we can construct a tensor, their *tensor product* (also called *outer product* in contradistinction to the inner product $\mathbf{A} \cdot \mathbf{B}$), defined as follows:

$$\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}(\mathbf{E}, \mathbf{F}, \mathbf{G}) \equiv \mathbf{A}(\mathbf{E})\mathbf{B}(\mathbf{F})\mathbf{C}(\mathbf{G}) = (\mathbf{A} \cdot \mathbf{E})(\mathbf{B} \cdot \mathbf{F})(\mathbf{C} \cdot \mathbf{G}) \ . \qquad (1.5a)$$

Here the first expression is the notation for the value of the new tensor, $\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C}$ evaluated on the three vectors $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$; the middle expression is the ordinary product of three real numbers, the value of $\mathbf{A}$ on $\mathbf{E}$, the value of $\mathbf{B}$ on $\mathbf{F}$, and the value of $\mathbf{C}$ on $\mathbf{G}$; and the third expression is that same product with the three numbers rewritten as scalar products. Similar definitions can be given (and should be obvious) for the tensor product of any two or more tensors of any rank; for example, if T has rank 2 and S has rank 3, then

$$\mathsf{T}\otimes\mathsf{S}(\mathbf{E},\mathbf{F},\mathbf{G},\mathbf{H},\mathbf{J}) \equiv \mathsf{T}(\mathbf{E},\mathbf{F})\mathsf{S}(\mathbf{G},\mathbf{H},\mathbf{J}) \ . \tag{1.5b}$$

One last geometric (i.e. frame-independent) concept we shall need is *contraction*. We shall illustrate this concept first by a simple example, then give the general definition. From two vectors $\mathbf{A}$ and $\mathbf{B}$ we can construct the tensor product $\mathbf{A}\otimes\mathbf{B}$ (a second-rank tensor), and we can also construct the scalar product $\mathbf{A}\cdot\mathbf{B}$ (a real number, i.e. a *scalar*, i.e. a *rank-0 tensor*). The process of contraction is the construction of $\mathbf{A}\cdot\mathbf{B}$ from $\mathbf{A}\otimes\mathbf{B}$

$$\text{contraction}(\mathbf{A}\otimes\mathbf{B}) \equiv \mathbf{A}\cdot\mathbf{B} \ . \tag{1.6a}$$

One can show fairly easily using component techniques (Sec. 1.5 below) that any second-rank tensor T can be expressed as a sum of tensor products of vectors, $\mathsf{T} = \mathbf{A}\otimes\mathbf{B} + \mathbf{C}\otimes\mathbf{D} + \ldots$; and correspondingly, it is natural to define the contraction of T to be contraction$(\mathsf{T}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{C}\cdot\mathbf{D} + \ldots$. Note that this contraction process lowers the rank of the tensor by two, from 2 to 0. Similarly, for a tensor of rank n one can construct a tensor of rank $n-2$ by contraction, but in this case one must specify which slots are to be contracted. For example, if T is a third rank tensor, expressible as $\mathsf{T} = \mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C} + \mathbf{E}\otimes\mathbf{F}\otimes\mathbf{G} + \ldots$, then the contraction of T on its first and third slots is the rank-1 tensor (vector)

$$1\&3\text{contraction}(\mathbf{A}\otimes\mathbf{B}\otimes\mathbf{C} + \mathbf{E}\otimes\mathbf{F}\otimes\mathbf{G} + \ldots) \equiv (\mathbf{A}\cdot\mathbf{C})\mathbf{B} + (\mathbf{E}\cdot\mathbf{G})\mathbf{F} + \ldots \ . \tag{1.6b}$$

All the concepts developed in this section (vectors, tensors, metric tensor, inner product, tensor product, and contraction of a tensor) can be carried over, with no change whatsoever, into *any* vector space[3] that is endowed with a concept of squared length — for example, to the four-dimensional spacetime of special relativity (next chapter)

1.4 Particle Kinetics and Lorentz Force in Geometric Language

In this section we shall illustrate our geometric viewpoint by formulating Newton's laws of motion for particles.

In Newtonian physics, a classical particle moves through Euclidean 3-space as universal time t passes. At time t it is located at some point $\mathbf{x}(t)$ (its *position*). The function $\mathbf{x}(t)$

[3] or, more precisely, any vector space over the real numbers. If the vector space's scalars are complex numbers, as in quantum mechanics, then slight changes are needed.

represents a curve in 3-space, the particle's *trajectory.* The particle's *velocity* $\mathbf{v}(t)$ is the time derivative of its position, its *momentum* $\mathbf{p}(t)$ is the product of its mass m and velocity, its *acceleration* $\mathbf{a}(t)$ is the time derivative of its velocity, and its energy is half its mass times velocity squared:

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt} , \quad \mathbf{p}(t) = m\mathbf{v}(t) , \quad \mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2} , \quad E(t) = \frac{1}{2}m\mathbf{v}^2 . \qquad (1.7a)$$

Since points in 3-space are geometric objects (defined independently of any coordinate system), so also are the trajectory $\mathbf{x}(t)$, the velocity, the momentum, the acceleration and the energy. (Physically, of course, the velocity has an ambiguity; it depends on one's standard of rest.)

Newton's second law of motion states that the particle's momentum can change only if a force $\mathbf{F}$ acts on it, and that its change is given by

$$d\mathbf{p}/dt = m\mathbf{a} = \mathbf{F} . \qquad (1.7b)$$

If the force is produced by an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, then this law of motion takes the familiar Lorentz-force form

$$d\mathbf{p}/dt = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (1.7c)$$

(here we have used the vector cross product, which will not be introduced formally until Sec. 1.7 below). Obviously, these laws of motion are geometric relationships between geometric objects.

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EXERCISES

Exercise 1.1 *Practice: Energy change for charged particle*

Without introducing any coordinates or basis vectors, show that, when a charged particle interacts with electric and magnetic fields, its energy changes at a rate

$$dE/dt = \mathbf{v} \cdot \mathbf{E} . \qquad (1.8)$$

Exercise 1.2 *Practice: Particle moving in a circular orbit*

Consider a particle moving in a circle with uniform speed $v = |\mathbf{v}|$ and uniform magnitude $a = |\mathbf{a}|$ of acceleration. Without introducing any coordinates or basis vectors, show the following:

(a) At any moment of time, let $\mathbf{n} = \mathbf{v}/v$ be the unit vector pointing along the velicity, and let s denote distance that the particle travels in its orbit. By drawing a picture, show that $d\mathbf{n}/ds$ is a unit vector that points to the center of the particle's circular orbit, divided by the radius of the orbit.

(b) Show that the vector (not unit vector) pointing from the particle's location to the center of its orbit is $(v/a)^2\mathbf{a}$.

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1.5 Component Representation of Tensor Algebra

In the Euclidean 3-space of Newtonian physics, there is a unique set of *orthonormal basis vectors* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \equiv \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ associated with any *Cartesian coordinate system* $\{x, y, z\} \equiv \{x^1, x^2, x^3\} \equiv \{x_1, x_2, x_3\}$. [In Cartesian coordinates in Euclidean space, we will usually place indices down, but occasionally we will place them up. It doesn't matter. By definition, in Cartesian coordinates a quantity is the same whether its index is down or up.] The basis vector $\mathbf{e}_j$ points along the x_j coordinate direction, which is orthogonal to all the other coordinate directions, and it has unit length, so

$$\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk} \ . \tag{1.9a}$$

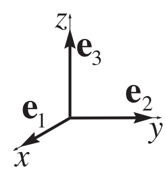

Fig. 1.3: The orthonormal basis vectors $\mathbf{e}_j$ associated with a Euclidean coordinate system in Euclidean 3-space.

Any vector $\mathbf{A}$ in 3-space can be expanded in terms of this basis,

$$\mathbf{A} = A_j \mathbf{e}_j \ . \tag{1.9b}$$

Here and throughout this book, we adopt the Einstein summation convention: repeated indices (in this case j) are to be summed (in this 3-space case over $j = 1, 2, 3$). By virtue of the orthonormality of the basis, the components A_j of $\mathbf{A}$ can be computed as the scalar product

$$A_j = \mathbf{A} \cdot \mathbf{e}_j \ . \tag{1.9c}$$

(The proof of this is straightforward: $\mathbf{A} \cdot \mathbf{e}_j = (A_k \mathbf{e}_k) \cdot \mathbf{e}_j = A_k (\mathbf{e}_k \cdot \mathbf{e}_j) = A_k \delta_{kj} = A_j$.)

Any tensor, say the third-rank tensor $\mathbf{T}(\_,\_,\_)$, can be expanded in terms of tensor products of the basis vectors:

$$\mathbf{T} = T_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \ . \tag{1.9d}$$

The components T_{ijk} of $\mathbf{T}$ can be computed from $\mathbf{T}$ and the basis vectors by the generalization of Eq. (1.9c)

$$T_{ijk} = \mathbf{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) \ . \tag{1.9e}$$

(This equation can be derived using the orthonormality of the basis in the same way as Eq. (1.9c) was derived.) As an important example, the components of the metric are $g_{jk} = \mathbf{g}(\mathbf{e}_j, \mathbf{e}_k) = \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$ [where the first equality is the method (1.9e) of computing tensor components, the second is the definition (1.4b) of the metric, and the third is the orthonormality relation (1.9a)]:

$$g_{jk} = \delta_{jk} \quad \text{in any orthonormal basis in 3-space.} \tag{1.9f}$$

Box 1.2

T2 Vectors and Tensors in Quantum Theory

The laws of quantum theory, like all other laws of Nature, can be expressed as geometric relationships between geometric objects. Most of quantum theory's geometric objects, like those of classical theory, are vectors and tensors:

The quantum state $|\psi\rangle$ of a physical system (e.g. a particle in a harmonic-oscillator potential) is a vector—the analog of a Euclidean-space vector $\mathbf{A}$. There is an inner product, denoted $\langle\phi|\psi\rangle$, between any two states $|\phi\rangle$ and $|\psi\rangle$, analogous to $\mathbf{B}\cdot\mathbf{A}$; but, whereas $\mathbf{B}\cdot\mathbf{A}$ is a real number, $\langle\phi|\psi\rangle$ is a complex number (and we add and subtract quantum states with complex-number coefficients). The Hermitian operators that represent observables (e.g. the Hamiltonian $\hat{H}$ for the particle in the potential) are two-slotted (second-rank), complex-valued functions of vectors; $\langle\phi|\hat{H}|\psi\rangle$ is the complex number that one gets when one inserts ϕ and ψ into the first and second slots of $\hat{H}$. Just as, in Euclidean space, we get a new vector (first-rank tensor) $\mathbf{T}(\_\,, \mathbf{A})$ when we insert the vector $\mathbf{A}$ into the second slot of $\mathbf{T}$, so in quantum theory we get a new vector (physical state) $\hat{H}|\psi\rangle$ (the result of letting $\hat{H}$ "act on" $|\psi\rangle$) when we insert $|\psi\rangle$ into the second slot of $\hat{H}$. In these senses, we can regard $\mathbf{T}$ as a linear map of Euclidean vectors into Euclidean vectors, and $\hat{H}$ as a linear map of states (Hilbert-space vectors) into states.

For the electron in the Hydrogen atom, we can introduce a set of orthonormal basis vectors $\{|1\rangle, |2\rangle, |3\rangle, ...\}$, e.g. the atom's energy eigenstates, with $\langle m|n\rangle = \delta_{mn}$. But by contrast with Newtonian physics, where we only need three basis vectors because our Euclidean space is 3-dimensional, for the particle in a harmonic-oscillator potential we need an infinite number of basis vectors, since the Hilbert space of all states is infinite dimensional. In the particle's quantum-state basis, any observable (e.g. the particle's position $\hat{x}$ or momentum $\hat{p}$) has components computed by inserting the basis vectors into its two slots: $x_{mn} = \langle m|\hat{x}|n\rangle$, and $p_{mn} = \langle m|\hat{p}|n\rangle$. The observable $\hat{x}\hat{p}$ (which maps states into states) has components in this basis $x_{jk}p_{km}$ (a matrix product); and the noncommutation of position and momentum $[\hat{x}, \hat{p}] = i\hbar$ (an important physical law) has components $x_{jk}p_{km} - p_{jk}x_{km} = i\hbar\delta_{mn}$.

In Part VII we shall often use bases that are not orthonormal; in such bases, the metric components will not be δ_{jk}.

The components of a tensor product, e.g. $\mathbf{T}(\_\,,\_\,,\_) \otimes \mathbf{S}(\_\,,\_)$, are easily deduced by inserting the basis vectors into the slots [Eq. (1.9e)]; they are $\mathbf{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)\otimes\mathbf{S}(\mathbf{e}_l, \mathbf{e}_m) = T_{ijk}S_{lm}$ [cf. Eq. (1.5a)]. In words, the components of a tensor product are equal to the ordinary arithmetic product of the components of the individual tensors.

In component notation, the inner product of two vectors and the value of a tensor when vectors are inserted into its slots are given by

$$\mathbf{A}\cdot\mathbf{B} = A_j B_j \ , \quad \mathbf{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{ijk}A_i B_j C_k \ , \tag{1.9g}$$

as one can easily show using previous equations. Finally, the contraction of a tensor [say, the fourth rank tensor $\mathbf{R}(\_\,,\_\,,\_\,,\_)$] on two of its slots [say, the first and third] has components that are easily computed from the tensor's own components:

$$\text{Components of } [1\&3\text{contraction of } \mathbf{R}] = R_{ijik} \tag{1.9h}$$

Note that R_{ijik} is summed on the i index, so it has only two free indices, j and k, and thus is the component of a second rank tensor, as it must be if it is to represent the contraction of a fourth-rank tensor.

1.5.1 Slot-Naming Index Notation

We now pause, in our development of the component version of tensor algebra, to introduce a very important new viewpoint:

Consider the rank-2 tensor $\mathbf{F}(\_,\_)$. We can define a new tensor $\mathbf{G}(\_,\_)$ to be the same as $\mathbf{F}$, but with the slots interchanged; i.e., for any two vectors $\mathbf{A}$ and $\mathbf{B}$ it is true that $\mathbf{G}(\mathbf{A},\mathbf{B}) = \mathbf{F}(\mathbf{B},\mathbf{A})$. We need a simple, compact way to indicate that $\mathbf{F}$ and $\mathbf{G}$ are equal except for an interchange of slots. The best way is to give the slots names, say a and b—i.e., to rewrite $\mathbf{F}(\_,\_)$ as $\mathbf{F}(\_{a},\_{b})$ or more conveniently as F_{ab}; and then to write the relationship between $\mathbf{G}$ and $\mathbf{F}$ as $G_{ab} = F_{ba}$. *"NO!"* some readers might object. This notation is indistinguishable from our notation for components on a particular basis. *"GOOD!"* a more astute reader will exclaim. The relation $G_{ab} = F_{ba}$ in a particular basis is a true statement if and only if "$\mathbf{G} = \mathbf{F}$ with slots interchanged" is true, so why not use the same notation to symbolize both? This, in fact, we shall do. We shall ask our readers to look at any "index equation" such as $G_{ab} = F_{ba}$ like they would look at an Escher drawing: momentarily think of it as a relationship between components of tensors in a specific basis; then do a quick mind-flip and regard it quite differently, as a relationship between geometric, basis-independent tensors with the indices playing the roles of names of slots. This mind-flip approach to tensor algebra will pay substantial dividends.

As an example of the power of this *slot-naming index notation,* consider the contraction of the first and third slots of a third-rank tensor $\mathbf{T}$. In any basis the components of 1&3contraction($\mathbf{T}$) are T_{aba}; cf. Eq. (1.9h). Correspondingly, in slot-naming index notation we denote 1&3contraction($\mathbf{T}$) by the simple expression T_{aba}. We say that the first and third slots are "strangling each other" by the contraction, leaving free only the second slot (named b) and therefore producing a rank-1 tensor (a vector).

1.5.2 Particle Kinetics in Index Notation

As an example of slot-naming index notation, we can rewrite the equations of particle kinetics (1.7) as follows:

$$v_i = \frac{dx_i}{dt}\,, \quad p_i = mv_i\,, a_i = \frac{dv_i}{dt} = \frac{d^2x_i}{dt^2}\,, \quad E = \frac{1}{2}mv_jv_j\,, \frac{dp_i}{dt} = q(E_i + \epsilon_{ijk}v_jB_k)\,. \tag{1.10}$$

(In the last equation ϵ_{ijk} is the so-called Levi-Civita tensor, which is used to produce the cross product; we shall learn about it in Sec. 1.7 below.)

Equations (1.10) can be viewed in either of two ways: (ii) as the basis-independent geometric laws $\mathbf{v} = d\mathbf{x}/dt$, $\mathbf{p} = m\mathbf{v}$, $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{x}/dt^2$, $E = \frac{1}{2}m\mathbf{v}^2$, and $d\mathbf{p}/dt - q(\mathbf{E} +$

$\mathbf{v} \times \mathbf{B}$) written in slot-naming index notation; or as equations for the components of $\mathbf{v}$, $\mathbf{p}$, $\mathbf{a}$, $\mathbf{E}$ and $\mathbf{B}$ in some particular Cartesian coordinate system.

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EXERCISES

Exercise 1.3 *Derivation: Component Manipulation Rules*
Derive the component manipulation rules (1.9g) and (1.9h).

Exercise 1.4 *Example and Practice: Numerics of Component Manipulations*
The third rank tensor $\mathbf{S}(\_,\_,\_)$ and vectors $\mathbf{A}$ and $\mathbf{B}$ have as their only nonzero components $S_{123} = S_{231} = S_{312} = +1$, $A_1 = 3$, $B_1 = 4$, $B_2 = 5$. What are the components of the vector $\mathbf{C} = \mathbf{S}(\mathbf{A}, \mathbf{B}, \_)$, the vector $\mathbf{D} = \mathbf{S}(\mathbf{A}, \_, \mathbf{B})$ and the tensor $\mathbf{W} = \mathbf{A} \otimes \mathbf{B}$?

[*Partial solution:* In component notation, $C_k = S_{ijk} A_i B_j$, where (of course) we sum over the repeated indices i and j. This tells us that $C_1 = S_{231} A_2 B_3$ because S_{231} is the only component of $\mathbf{S}$ whose last index is a 1; and this in turn implies that $C_1 = 0$ since $A_2 = 0$. Similarly, $C_2 = S_{312} A_3 B_1 = 0$ (because $A_3 = 0$). Finally, $C_3 = S_{123} A_1 B_2 = +1 \times 3 \times 5 = 15$. Also, in component notation $W_{ij} = A_i B_j$, so $W_{11} = A_1 \times B_1 = 3 \times 4 = 12$ and $W_{12} = A_1 \times B_2 = 3 \times 5 = 15$.]

Exercise 1.5 *Practice: Meaning of Slot-Naming Index Notation*

(a) The following expressions and equations are written in slot-naming index notation; convert them to geometric, index-free notation: $A_i B_{jk}$; $A_i B_{ji}$, $S_{ijk} = S_{kji}$, $A_i B_i = A_i B_j g_{ij}$.

(b) The following expressions are written in geometric, index-free notation; convert them to slot-naming index notation: $\mathbf{T}(\_,\_,\mathbf{A})$; $\mathbf{T}(\_,\mathbf{S}(\mathbf{B},\_),\_)$.

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1.6 Orthogonal Transformations of Bases

Consider two different Cartesian coordinate systems $\{x, y, z\} \equiv \{x_1, x_2, x_3\}$, and $\{\bar{x}, \bar{y}, \bar{z}\} \equiv \{x_{\bar{1}}, x_{\bar{2}}, x_{\bar{3}}\}$. Denote by $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_{\bar{p}}\}$ the corresponding bases. It must be possible to expand the basis vectors of one basis in terms of those of the other. We shall denote the expansion coefficients by the letter R and shall write

$$\mathbf{e}_i = \mathbf{e}_{\bar{p}} R_{\bar{p}i} , \quad \mathbf{e}_{\bar{p}} = \mathbf{e}_i R_{i\bar{p}} . \tag{1.11}$$

The quantities $R_{\bar{p}i}$ and $R_{i\bar{p}}$ are *not* the components of a tensor; rather, they are the elements of transformation matrices

$$[R_{\bar{p}i}] = \begin{bmatrix} R_{\bar{1}1} & R_{\bar{1}2} & R_{\bar{1}3} \\ R_{\bar{2}1} & R_{\bar{2}2} & R_{\bar{2}3} \\ R_{\bar{3}1} & R_{\bar{3}2} & R_{\bar{3}3} \end{bmatrix} , \quad [R_{i\bar{p}}] = \begin{bmatrix} R_{1\bar{1}} & R_{1\bar{2}} & R_{1\bar{3}} \\ R_{2\bar{1}} & R_{2\bar{2}} & R_{2\bar{3}} \\ R_{3\bar{1}} & R_{3\bar{2}} & R_{3\bar{3}} \end{bmatrix} . \tag{1.12a}$$

(Here and throughout this book we use square brackets to denote matrices.) These two matrices must be the inverse of each other, since one takes us from the barred basis to the unbarred, and the other in the reverse direction, from unbarred to barred:

$$R_{\bar{p}i}R_{i\bar{q}} = \delta_{\bar{p}\bar{q}} , \quad R_{i\bar{p}}R_{\bar{p}j} = \delta_{ij} . \tag{1.12b}$$

The orthonormality requirement for the two bases implies that $\delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = (\mathbf{e}_{\bar{p}}R_{\bar{p}i}) \cdot (\mathbf{e}_{\bar{q}}R_{\bar{q}j}) = R_{\bar{p}i}R_{\bar{q}j}(\mathbf{e}_{\bar{p}} \cdot \mathbf{e}_{\bar{q}}) = R_{\bar{p}i}R_{\bar{q}j}\delta_{\bar{p}\bar{q}} = R_{\bar{p}i}R_{\bar{p}j}$. This says that the transpose of $[R_{\bar{p}i}]$ is its inverse—which we have already denoted by $[R_{i\bar{p}}]$;

$$[R_{i\bar{p}}] \equiv \text{Inverse}\left([R_{\bar{p}i}]\right) = \text{Transpose}\left([R_{\bar{p}i}]\right) . \tag{1.12c}$$

This property implies that the transformation matrix is orthogonal; i.e., the transformation is a reflection or a rotation [see, e.g., Goldstein (1980)]. Thus (as should be obvious and familiar), the bases associated with any two Euclidean coordinate systems are related by a reflection or rotation. Note: Eq. (1.12c) does *not* say that $[R_{i\bar{p}}]$ is a symmetric matrix; in fact, it typically is not.

The fact that a vector $\mathbf{A}$ is a geometric, basis-independent object implies that $\mathbf{A} = A_i\mathbf{e}_i = A_i(\mathbf{e}_{\bar{p}}R_{\bar{p}i}) = (R_{\bar{p}i}A_i)\mathbf{e}_{\bar{p}} = A_{\bar{p}}\mathbf{e}_{\bar{p}}$; i.e.,

$$A_{\bar{p}} = R_{\bar{p}i}A_i , \quad \text{and similarly } A_i = R_{i\bar{p}}A_{\bar{p}} ; \tag{1.13a}$$

and correspondingly for the components of a tensor

$$T_{\bar{p}\bar{q}\bar{r}} = R_{\bar{p}i}R_{\bar{q}j}R_{\bar{r}k}T_{ijk} , \quad T_{ijk} = R_{i\bar{p}}R_{j\bar{q}}R_{k\bar{r}}T_{\bar{p}\bar{q}\bar{r}} . \tag{1.13b}$$

It is instructive to compare the transformation law (1.13a) for the components of a vector with those (1.11) for the bases. To make these laws look natural, we have placed the transformation matrix on the left in the former and on the right in the latter. In Minkowski spacetime, the placement of indices, up or down, will automatically tell us the order.

If we choose the origins of our two coordinate systems to coincide, then the vector $\mathbf{x}$ reaching from the common origin to some point $\mathcal{P}$ whose coordinates are x_j and $x_{\bar{p}}$ has components equal to those coordinates; and as a result, the coordinates themselves obey the same transformation law as any other vector

$$x_{\bar{p}} = R_{\bar{p}i}x_i , \quad x_i = R_{i\bar{p}}x_{\bar{p}} ; \tag{1.13c}$$

The product of two rotation matrices, $[R_{i\bar{p}}R_{\bar{p}\bar{\bar{s}}}]$ is another rotation matrix $[R_{i\bar{\bar{s}}}]$, which transforms the Cartesian bases $\mathbf{e}_{\bar{\bar{s}}}$ to $\mathbf{e}_i$. Under this product rule, the rotation matrices form a mathematical *group*: the *rotation group*, whose "representations" play an important role in quantum theory.

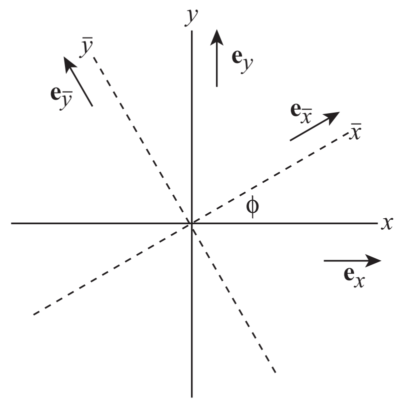

Fig. 1.4: Two Cartesian coordinate systems $\{x, y, z\}$ and $\{\bar{x}, \bar{y}, \bar{z}\}$ and their basis vectors in Euclidean space, rotated by an angle ϕ relative to each other in the x, y plane. The z and $\bar{z}$ axes points out of the paper or screen and are not shown.

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EXERCISES

Exercise 1.6 ***Example and Practice: Rotation in x, y Plane*[4]

Consider two Cartesian coordinate systems rotated with respect to each other in the x, y plane as shown in Fig. 1.4.

(a) Show that the rotation matrix that takes the barred basis vectors to the unbarred basis vectors is

$$[R_{\bar{p}i}] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.14}$$

and show that the inverse of this rotation matrix is, indeed, its transpose, as it must be if this is to represent a rotation.

(b) Verify that the two coordinate systems are related by Eq. (1.13c).

(c) Let A_j be the components of a vector that lies in the x, y plane so $A_z = 0$. The two nonzero components A_x and A_y of this vector can be regarded as describing the two polarizations of an electromagnetic wave propagating in the z direction. Show that $A_{\bar{x}} + iA_{\bar{y}} = (A_x + iA_y)e^{-i\phi}$. One can show (cf. Sec. 26.3.3) that the factor $e^{-i\phi}$ implies that the quantum particle associated with the wave, i.e. the photon, has spin one; i.e., spin angular momentum $\hbar =$(Planck's constant)$/2\pi$.

(d) Let h_{jk} be the components of a symmetric, trace-free tensor that is confined to the x, y plane so $h_{zj} = h_{jz} = 0$ for all j. Then the only nonzero components of this tensor

[4]Exercises marked with double stars are important expansions of the material presented in the text.

are $h_{xx} = -h_{yy}$ and $h_{xy} = h_{yx}$. As we shall see in Sec. 26.3.2, this tensor can be regarded as describing the two polarizations of a gravitational wave propagating in the z direction. Show that $h_{\bar{x}\bar{x}} + ih_{\bar{x}\bar{y}} = (h_{xx} + ih_{xy})e^{-2i\phi}$. The factor $e^{-2i\phi}$ implies that the quantum particle associated with the gravitational wave (the graviton) has spin two (spin angular momentum $2\hbar$); cf. Eq. (26.51) and Sec. 26.3.3.

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1.7 Directional Derivatives, Gradients, Levi-Civita Tensor, Cross Product and Curl

Consider a tensor field $\mathbf{T}(\mathcal{P})$ in Euclidean 3-space and a vector $\mathbf{A}$. We define the *directional derivative* of $\mathbf{T}$ along $\mathbf{A}$ by the obvious limiting procedure

$$\nabla_{\mathbf{A}}\mathbf{T} \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}[\mathbf{T}(\mathbf{x}_{\mathcal{P}} + \epsilon\mathbf{A}) - \mathbf{T}(\mathbf{x}_{\mathcal{P}})] \tag{1.15a}$$

and similarly for the directional derivative of a vector field $\mathbf{B}(\mathcal{P})$ and a scalar field $\psi(\mathcal{P})$. In this definition, the quantity in square brackets is simply the difference between two linear functions of vectors, and we have denoted points, e.g. $\mathcal{P}$, by the vector $\mathbf{x}_{\mathcal{P}}$ that reaches from some arbitrary origin to the point.

It should not be hard to convince oneself that the directional derivative of any tensor field $\mathbf{T}$ is linear in the vector $\mathbf{A}$ along which one differentiates. Correspondingly, if $\mathbf{T}$ has rank n (n slots), then there is another tensor field, denoted $\boldsymbol{\nabla}\mathbf{T}$, with rank $n+1$, such that

$$\nabla_{\mathbf{A}}\mathbf{T} = \boldsymbol{\nabla}\mathbf{T}(\_\,,\,\_\,,\,\_\,,\,\mathbf{A})\ . \tag{1.15b}$$

Here on the right side the first n slots (3 in the case shown) are left empty, and $\mathbf{A}$ is put into the last slot (the "differentiation slot"). The quantity $\boldsymbol{\nabla}\mathbf{T}$ is called the *gradient* of $\mathbf{T}$. In slot-naming index notation, it is conventional to denote this gradient by $T_{abc;d}$, where in general the number of indices preceding the semicolon is the rank of $\mathbf{T}$. Using this notation, the directional derivative of $\mathbf{T}$ along $\mathbf{A}$ reads [cf. Eq. (1.15b)] $T_{abc;j}A_j$.

It is not hard to show that in any Cartesian coordinate system, the components of the gradient are nothing but the partial derivatives of the components of the original tensor,

$$T_{abc;j} = \frac{\partial T_{abc}}{\partial x_j} \equiv T_{abc,j}\ . \tag{1.15c}$$

In a non-Cartesian basis (e.g. the spherical and cylindrical bases often used in electromagnetic theory), the components of the gradient typically are *not* obtained by simple partial differentiation [Eq. (1.15c) fails] because of turning and/or length changes of the basis vectors as we go from one location to another. In Sec. 10.5 we shall learn how to deal with this by using objects called *connection coefficients*. Until then, however, we shall confine ourselves to Cartesian bases (or Lorentz bases in spacetime), so subscript semicolons and subscript commas can be used interchangeably.

Because the gradient and the directional derivatives are defined by the same standard limiting process as one uses when defining elementary derivatives, they obey the standard Leibniz rule for differentiating products:

$$\nabla_{\mathbf{A}}(\mathbf{S}\otimes\mathbf{T}) = (\nabla_{\mathbf{A}}\mathbf{S})\otimes\mathbf{T} + \mathbf{S}\otimes\nabla_{\mathbf{A}}\mathbf{T}\ ,$$
$$\text{i.e., } (S_{ab}T_{cde})_{;j}A_j = (S_{ab;j}A_j)T_{cde} + S_{ab}(T_{cde;j}A_j)\ ; \tag{1.16a}$$

and

$$\nabla_{\mathbf{A}}(f\mathbf{T}) = (\nabla_{\mathbf{A}}f)\mathbf{T} + f\nabla_{\mathbf{A}}\mathbf{T}\ , \quad \text{i.e., } (fT_{abc})_{;j}A_j = (f_{;j}A_j)T_{abc} + fT_{abc;j}A_j\ . \tag{1.16b}$$

In an orthonormal basis these relations should be obvious: They follow from the Leibniz rule for partial derivatives.

Because the components g_{ab} of the metric tensor are constant in any Cartesian coordinate system, Eq. (1.15c) (which is valid in such coordinates) guarantees that $g_{ab;j} = 0$; i.e., the metric has vanishing gradient:

$$\boldsymbol{\nabla}\mathbf{g} = 0\ , \quad \text{i.e., } \ g_{ab;j} = 0\ . \tag{1.17}$$

From the gradient of any vector or tensor we can construct several other important derivatives by contracting on slots: (i) Since the gradient $\boldsymbol{\nabla}\mathbf{A}$ of a vector field $\mathbf{A}$ has two slots, $\boldsymbol{\nabla}\mathbf{A}(\_\ ,\_)$, we can strangle (contract) its slots on each other to obtain a scalar field. That scalar field is the *divergence* of $\mathbf{A}$ and is denoted

$$\boldsymbol{\nabla}\cdot\mathbf{A} \equiv (\text{contraction of } \boldsymbol{\nabla}\mathbf{A}) = A_{a;a}\ . \tag{1.18}$$

(ii) Similarly, if $\mathbf{T}$ is a tensor field of rank three, then $T_{abc;c}$ is its divergence on its third slot, and $T_{abc;b}$ is its divergence on its second slot. (iii) By taking the double gradient and then contracting on the two gradient slots we obtain, from any tensor field $\mathbf{T}$, a new tensor field with the same rank,

$$\nabla^2\mathbf{T} \equiv (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{T}\ , \qquad \text{or, in index notation, } T_{abc;jj}\ . \tag{1.19}$$

Here and henceforth, all indices following a semicolon (or comma) represent gradients (or partial derivatives): $T_{abc;jj} \equiv T_{abc;j;j}$, $T_{abc,jk} \equiv \partial^2 T_{abc}/\partial x_j \partial x_k$. The operator ∇^2 is called the *Laplacian*.

The metric tensor is a fundamental property of the space in which it lives; it embodies the inner product and thence the space's notion of distance or interval and thence the space's geometry. In addition to the metric, there is one (and only one) other fundamental tensor that embodies a piece of Euclidean (or Minkowski) space's geometry: the *Levi-Civita tensor* $\boldsymbol{\epsilon}$.

The Levi-Civita tensor $\boldsymbol{\epsilon}$ has a number of slots equal to the dimensionality of the space in which it lives: 3 slots in the Euclidean 3-space of this chapter, and 4 slots in 4-dimensional spacetime of Chap. 2; and it is antisymmetric in each and every pair of its slots. These properties determine $\boldsymbol{\epsilon}$ uniquely up to a multiplicative constant. That constant is fixed by a compatibility relation between $\boldsymbol{\epsilon}$ and the metric $\mathbf{g}$ plus the concept of "handedness": If $\{\vec{e}_j\}$

is a Cartesian basis (a concept that requires the metric for its definition), and if this basis is right-handed in the usual elementary sense[5] (a property not determined by the metric), then

$$\boldsymbol{\epsilon}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = +1 \ . \tag{1.20a}$$

Equation (1.20a) and the antisymmetry of $\boldsymbol{\epsilon}$ imply that in an orthonormal, right-handed basis, the only nonzero components of $\boldsymbol{\epsilon}$ are

$$\begin{aligned} \epsilon_{123} &= +1 \ , \\ \epsilon_{abc} &= +1 \ \text{ if } a,\ b,\ c \text{ is an even permutation of } 1,\ 2,\ 3 \\ &= -1 \ \text{ if } a,\ b,\ c \text{ is an odd permutation of } 1,\ 2,\ 3 \\ &= 0 \ \text{ if } a,\ b,\ c \text{ are not all different;} \end{aligned} \tag{1.20b}$$

One can show that these components in one right-handed orthonormal frame imply these same components in all other right-handed orthonormal frames by virtue of the fact that the orthogonal transformation matrices have unit determinant; and that in a left-handed Cartesian frame the signs of these components are reversed.

The Levi-Civita tensor is used to define the cross product and the curl:

$$\mathbf{A} \times \mathbf{B} \equiv \boldsymbol{\epsilon}(\_\,, \mathbf{A}, \mathbf{B}) \quad \text{i.e., in slot-naming index notation, } \epsilon_{ijk} A_j B_k \ ; \tag{1.21a}$$

$$\boldsymbol{\nabla} \times \mathbf{A} \equiv (\text{the vector field whose slot-naming index form is } \epsilon_{ijk} A_{k;j}) \ . \tag{1.21b}$$

[Equation (1.21b) is an example of an expression that is complicated if written in index-free notation; it says that $\boldsymbol{\nabla} \times \mathbf{A}$ is the double contraction of the rank-5 tensor $\boldsymbol{\epsilon} \otimes \boldsymbol{\nabla}\mathbf{A}$ on its second and fifth slots, and on its third and fourth slots.]

Although Eqs. (1.21a) and (1.21b) look like complicated ways to deal with concepts that most readers regard as familiar and elementary, they have great power. The power comes from the following property of the Levi-Civita tensor in Euclidean 3-space [readily derivable from its components (1.20b)]:

$$\boxed{\epsilon_{ijm}\epsilon_{klm} = \delta^{ij}_{kl} \equiv \delta^i_k \delta^j_l - \delta^i_l \delta^j_k \ .} \tag{1.22}$$

Here δ^i_k is the Kronecker delta. Examine the 4-index delta function δ^{ij}_{kl} carefully; it says that either the indices above and below each other must be the same ($i = k$ and $j = l$) with a $+$ sign, or the diagonally related indices must be the same ($i = l$ and $j = k$) with a $-$ sign. [We have put the indices ij of δ^{ij}_{kl} up solely to facilitate remembering this rule. Recall (first paragraph of Sec. 1.5) that in Euclidean space and Cartesian coordinates, it does not matter whether indices are up or down.] With the aid of Eq. (1.22) and the index-notation expressions for the cross product and curl, one can quickly and easily derive a wide variety of useful vector identities; see the very important Exercise 1.7.

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[5] Place the thumb of your right hand along the $\mathbf{e}_z = \mathbf{e}_3$ direction and your right hand's fingers along the $\mathbf{e}_x = \mathbf{e}_1$ direction. If, when you bend your fingers, they sweep from the $\mathbf{e}_x = \mathbf{e}_1$ direction to the $\mathbf{e}_y = \mathbf{e}_2$ direction, then the basis is right handed; if they sweep to the $-\mathbf{e}_y = -\mathbf{e}_2$ direction, then the basis is left handed.

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EXERCISES

Exercise 1.7 ***Example and Practice: Vectorial Identities for the Cross Product and Curl*

Here is an example of how to use index notation to derive a vector identity for the double cross product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$: In index notation this quantity is $\epsilon_{ijk}A_j(\epsilon_{klm}B_lC_m)$. By permuting the indices on the second ϵ and then invoking Eq. (1.22), we can write this as $\epsilon_{ijk}\epsilon_{lmk}A_jB_lC_m = \delta^{lm}_{ij}A_jB_lC_m$. By then invoking the meaning (1.22) of the 4-index delta function, we bring this into the form $A_jB_iC_j - A_jB_jC_i$, which is the slot-naming index-notation form of $(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$. Thus, it must be that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$.

Use similar techniques to evaluate the following quantities:

(a) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})$

(b) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})$

(c) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})$

Exercise 1.8 ***Example and Practice: Levi-Civita Tensor in Two Dimensional Euclidean Space*

In Euclidean 2-space, the Levi-Civita tensor is a second-rank, antisymmetric tensor with [by analogy with Eq. (1.20a)]

$$\boldsymbol{\epsilon}(\mathbf{e}_1, \mathbf{e}_2) = +1 \tag{1.23}$$

in a right-handed Cartesian basis.

(a) From this definition, show that the components of $\boldsymbol{\epsilon}$ in a right-handed Cartesian basis are

$$\epsilon_{12} = +1\ , \quad \epsilon_{21} = -1\ , \quad \epsilon_{11} = \epsilon_{22} = 0\ . \tag{1.24}$$

(b) Show that

$$\epsilon_{ik}\epsilon_{jk} = \delta_{ij}\ . \tag{1.25}$$

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1.8 Volumes, Integration and Integral Conservation Laws

The Levi-Civita tensor is the foundation for computing volumes and performing volume integrals in any number of dimensions. In Cartesian coordinates of 2-dimensional Euclidean space (Ex. 1.8 above), the area (i.e. 2-dimensional volume) of a parallelogram whose sides are **A** and **B** is

$$\text{2-Volume} = \epsilon_{ab} A_a B_b = A_1 B_2 - A_2 B_1 = \det \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} , \tag{1.26}$$

a relation that should be familiar from elementary geometry. Equally familiar should be the expression for the 3-dimensional volume of a parallelopiped with legs **A**, **B**, and **C**:

$$\text{3-Volume} = \epsilon_{ijk} A_i B_j C_k = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \det \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} . \tag{1.27}$$

Recall that this volume has a sign: it is positive if $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is a right handed set of vectors and negative if left-handed. Equations (1.26) and (1.27) are foundations from which one can derive the usual formulae $dA = dx\,dy$ and $dV = dx\,dy\,dz$ for the area and volume of elementary surface and volume elements with Cartesian side lengths dx, dy and dz (Ex. 1.9.

In Euclidean 3-space, we define the vectorial surface area of a 2-dimensional parallelogram with legs **A** and **B** to be

$$\boldsymbol{\Sigma} = \mathbf{A} \times \mathbf{B} = \boldsymbol{\epsilon}(\_, \mathbf{A}, \mathbf{B}) . \tag{1.28}$$

This vectorial surface area has a magnitude equal to the area of the parallelogram and a direction perpendicular to it. Such vectorial surface areas are the foundation for surface integrals in 3-dimensional space, and for the familiar *Gauss theorem*

$$\int_{\mathcal{V}_3} (\boldsymbol{\nabla} \cdot \mathbf{A}) dV = \int_{\partial \mathcal{V}_3} \mathbf{A} \cdot d\boldsymbol{\Sigma} \tag{1.29a}$$

(where $\mathcal{V}_3$ is a compact 3-dimensional region and $\partial\mathcal{V}_3$ is its closed two-dimensional boundary) and *Stokes theorem*

$$\int_{\mathcal{V}_2} \boldsymbol{\nabla} \times \mathbf{A} \cdot d\boldsymbol{\Sigma} = \int_{\partial \mathcal{V}_2} \mathbf{A} \cdot d\mathbf{l} \tag{1.29b}$$

(where $\mathcal{V}_2$ is a compact 2-dimensional region, $\partial\mathcal{V}_2$ is the 1-dimensional closed curve that bounds it, and the last integral is a line integral around that curve).

Notice that in Euclidean 3-space, the vectorial surface area $\boldsymbol{\epsilon}(\_, \mathbf{A}, \mathbf{B})$ can be thought of as an object that is waiting for us to insert a third leg **C** so as to compute a volume $\boldsymbol{\epsilon}(\mathbf{C}, \mathbf{A}, \mathbf{B})$—the volume of the parallelopiped with legs **C**, **A**, and **B**.

This mathematics is illustrated by the integral and differential conservation laws for electric charge and for particles: The total charge and the total number of particles inside a three dimensional region of space $\mathcal{V}_3$ are $\int_{\mathcal{V}_3} \rho_e dV$ and $\int_{\mathcal{V}_3} n dV$, where ρ_e is the charge density and n the number density of particles. The rates that charge and particles flow out of $\mathcal{V}_3$ are the integrals of the current density $\mathbf{j}$ and the particle flux vector $\mathbf{S}$ over its boundary $\partial\mathcal{V}_3$. Therefore, the laws of charge conservation and particle conservation say

$$\frac{d}{dt}\int_{\mathcal{V}_3} \rho_e dV + \int_{\partial\mathcal{V}_3} \mathbf{j}\cdot d\boldsymbol{\Sigma} = 0 \, , \qquad \frac{d}{dt}\int_{\mathcal{V}_3} n dV + \int_{\partial\mathcal{V}_3} \mathbf{S}\cdot d\boldsymbol{\Sigma} = 0 \, . \tag{1.30}$$

Pull the time derivative inside each volume integral (where it becomes a partial derivative), and apply Gauss's law to each surface integral; the results are $\int_{\mathcal{V}_3}(\partial\rho_e/\partial t + \boldsymbol{\nabla}\cdot\mathbf{j})dV = 0$ and similarly for particles. The only way these equations can be true for all choices of $\mathcal{V}_3$ is by the integrands vanishing:

$$\partial\rho_e/\partial t + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \, , \quad \partial n/\partial t + \boldsymbol{\nabla}\cdot\mathbf{S} = 0 \, . \tag{1.31}$$

These are the differential conservation laws for charge and for particles. They have a standard, universal form: the time derivative of the density of a quantity plus the divergence of its flux vanishes.

Note that the integral conservation laws (1.30) and the differential conservation laws (1.31) required no coordinate system or basis for their description; and no coordinates or basis were used in deriving the differential laws from the integral laws. This is an example of the fundamental principle that *the Newtonian physical laws are all expressible as geometric relationships between geometric objects.*

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EXERCISES

Exercise 1.9 *Derivation and Practice: Volume Elements in Cartesian Coordinates*

Use Eqs. (1.26) and (1.27) to derive the usual formulae $dA = dxdy$ and $dV = dxdydz$ for the 2-dimensional and 3-dimensional integration elements in Cartesian coordinates. Hint: Use as the edges of the integration volumes $dx\,\mathbf{e}_x$, $dy\,\mathbf{e}_y$, and $dz\,\mathbf{e}_z$.

Exercise 1.10 *Example and Practice: Integral of a Vector Field Over a Sphere*

Integrate the vector field $\mathbf{A} = z\mathbf{e}_z$ over a sphere with radius a with center at the origin of the Cartesian coordinate system. Hints:

(a) Introduce spherical polar coordinates on the sphere, and construct the vectorial integration element $d\boldsymbol{\Sigma}$ from the two legs $ad\theta\,\mathbf{e}_{\hat\theta}$ and $a\sin\theta d\phi\,\mathbf{e}_{\hat\phi}$. Here $\mathbf{e}_{\hat\theta}$ and $\mathbf{e}_{\hat\phi}$ are unit-length vectors along the θ and ϕ directions. Explain the factors $ad\theta$ and $a\sin\theta d\phi$ in the definitions of the legs. Show that

$$d\boldsymbol{\Sigma} = \boldsymbol{\epsilon}(\_\,, \mathbf{e}_{\hat\theta}, \mathbf{e}_{\hat\phi})a^2\sin\theta d\theta d\phi \, . \tag{1.32}$$

(b) Using $z = a\cos\theta$ and $\mathbf{e}_z = \cos\theta\mathbf{e}_{\hat r} - \sin\theta\mathbf{e}_{\hat\phi}$ on the sphere (where $\mathbf{e}_{\hat r}$ is the unit vector pointing in the radial direction), show that $\mathbf{A}\cdot d\boldsymbol{\Sigma} = a\cos^2\theta\,\boldsymbol{\epsilon}(\mathbf{e}_{\hat r}, \mathbf{e}_{\hat\theta}, \mathbf{e}_{\hat\phi})\,a^2\sin\theta d\theta d\phi$.

(c) Explain why $\boldsymbol{\epsilon}(\mathbf{e}_{\hat r}, \mathbf{e}_{\hat\theta}, \mathbf{e}_{\hat\phi}) = 1$.

(d) Perform the integral $\int \mathbf{A} \cdot d\boldsymbol{\Sigma}$ over the sphere's surface to obtain your final answer $(4\pi/3)a^3$. This, of course, is the volume of the sphere. Explain pictorially why this had to be the answer.

Exercise 1.11 *Example: Faraday's Law of Induction*

One of Maxwell's equations says that $\boldsymbol{\nabla} \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. This is a geometric relationship between geometric objects; it requires no coordinates or basis for its statement. By integrating this equation over a two-dimensional surface $\mathcal{V}_2$ with boundary curve $\partial\mathcal{V}_2$ and applying Stokes' theorem, derive Faraday's law of induction — again, a geometric relationship between geometric objects.

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1.9 The Stress Tensor and Momentum Conservation

Press your hands together in the x–y plane and feel the force that one hand exerts on the other across a tiny area A — say, one square millimeter of your hands' palms (Fig. 1.5). That force, of course, is a vector $\mathbf{F}$. It has a normal component (along the x direction). It also has a tangential component: if you try to slide your hands past each other, you feel a component of force along their surface, a "shear" force in the y and z directions. Not only is the force $\mathbf{F}$ vectorial; so is the 2-surface across which it acts, $\boldsymbol{\Sigma} = A\,\mathbf{e}_x$. (Here $\mathbf{e}_x$ is the unit vector orthogonal to the tiny area A, and we have chosen the negative side of the surface to be the $-x$ side and the positive side to be $+x$. With this choice, the force $\mathbf{F}$ is that which the negative hand, on the $-x$ side, exerts on the positive hand.)

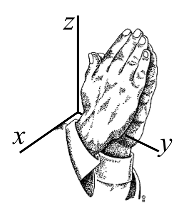

Fig. 1.5: Hands, pressed together, exert a stress on each other.

Now, it should be obvious that the force $\mathbf{F}$ is a linear function of our chosen surface $\boldsymbol{\Sigma}$. Therefore, there must be a tensor, the *stress tensor*, that reports the force to us when we insert the surface into its second slot:

$$\mathbf{F}(\_) = \mathbf{T}(\_, \boldsymbol{\Sigma}) \;, \quad \text{i.e.,}\; F_i = T_{ij}\Sigma_j \;. \tag{1.33}$$

Newton's law of action and reaction tells us that the force that the positive hand exerts on the negative hand must be equal and opposite to that that which the negative hand exerts on the positive. This shows up trivially in Eq. (1.33): By changing the sign of $\boldsymbol{\Sigma}$, one reverses which hand is regarded as negative and which positive; and since $\mathbf{T}$ is linear in $\mathbf{\Sigma}$, one also reverses the sign of the force.

The definition (1.33) of the stress tensor gives rise to the the following physical meaning of its components:

$$T_{jk} = \begin{pmatrix} j\text{-component of force per unit area} \\ \text{across a surface perpendicular to } \vec{e}_k \end{pmatrix}$$
$$= \begin{pmatrix} j\text{-component of momentum that crosses a unit} \\ \text{area which is perpendicular to } \vec{e}_k\text{, per unit time,} \\ \text{with the crossing being from } -x^k \text{ to } +x^k \end{pmatrix} . \tag{1.34}$$

The stresses inside a table with a heavy weight on it are described by the stress tensor $\mathbf{T}$, as are the stresses in a flowing fluid or plasma, in the electromagnetic field, and in any other physical medium. Accordingly, we shall use the stress tensor as an important mathematical tool in our study of force balance in kinetic theory (Chap. 2), elasticity theory (Part III), fluid mechanics (Part IV), and plasma physics (Part V).

It is not obvious from its definition, but the stress tensor $\mathbf{T}$ is always symmetric in its two slots. To see this, consider a small cube with side L in any medium (or field) (Fig. 1.6). The medium outside the cube exerts forces, and thence also torques, on the cube's faces. The z-component of the torque is produced by the shear forces on the front and back faces and on the left and right. As shown in the figure, the shear forces on the front and back faces have magnitudes $T_{xy}L^2$ and point in opposite directions, so they exert identical torques on the cube, $N_z = T_{xy}L^2(L/2)$ (where $L/2$ is the distance of each face from the cube's center). Similarly, the shear forces on the left and right faces have magnitudes $T_{yx}L^2$ and point in opposite directions, thereby exerting identical torques on the cube, $N_z = -T_{yx}L^2(L/2)$. Adding the torques from all four faces and equating them to the rate of change of angular momentum, $\frac{1}{12}\rho L^5 d\Omega_z/dt$ (where ρ is the mass density, $\frac{1}{12}\rho L^5$ is the cube's moment of inertia, and Ω_z is the z component of its angular velocity), we obtain $(T_{yx} - T_{xy})L^3 = \frac{1}{12}\rho L^5 d\Omega_z/dt$. Now, let the cube's edge length become arbitrarily small, $L \to 0$. If $T_{yx} - T_{xy}$ does not vanish, then the cube will be set into rotation with an infinitely large angular acceleration, $d\Omega_z/dt \propto 1/L^2 \to \infty$ — an obviously unphysical behavior. Therefore $T_{yx} = T_{xy}$, and similarly for all other components; *the stress tensor is always symmetric under interchange of its two slots.*

Two examples will make the stress tensor more concrete:

Perfect fluid: Inside a perfect fluid there is an isotropic pressure P, so $T_{xx} = T_{yy} = T_{zz} = P$ (the normal forces per unit area across surfaces in the yz plane, the zx plane and the xy plane are all equal to P). The fluid cannot support any shear forces, so the off-diagonal components of $\mathbf{T}$ vanish. We can summarize this by $T_{ij} = P\delta_{ij}$ or equivalently, since δ_{ij} are the components of the Euclidean metric, $T_{ij} = Pg_{ij}$. The frame-independent version of this is

$$\mathbf{T} = P\mathbf{g} \quad \text{or, in slot-naming index notation } T_{ij} = P_{ij} \; . \tag{1.35}$$

Note that, *as always*, the formula in slot-naming index notation looks identically the same as the formula $T_{ij} = P\delta_{ij}$ for the components in our chosen Cartesian coordinate system.

To check this result, consider a 2-surface $\mathbf{\Sigma} = A\mathbf{n}$ with area A oriented perpendicular to some arbitrary unit vector $\mathbf{n}$. The vectorial force that the fluid exerts across $\mathbf{\Sigma}$ is, in index

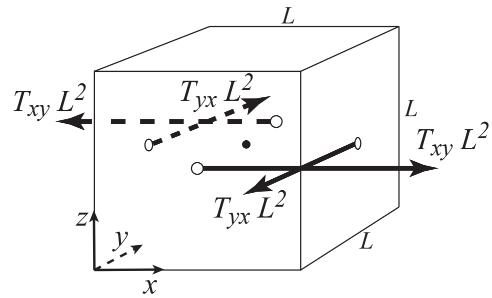

Fig. 1.6: The shear forces exerted on the left, right, front and back faces of a vanishingly small cube. The resulting torque about the z direction will set the cube into rotation with an arbitrarily large angular acceleration unless the stress tensor is symmetric.

notation, $F_j = T_{jk}\Sigma_k = P g_{jk} A n_k = P A n_j$; i.e. it is a normal force with magnitude equal to the fluid pressure P times the surface area A. This is what it should be.

Electromagnetic field: See Ex. 1.13 below.

The stress tensor plays a central role in the *Newtonian law of momentum conservation*: Recall the physical interpretation of T_{jk} as the j-component of momentum that crosses a unit area perpendicular to $\mathbf{e}_k$ per unit time [Eq. (1.34)]. Apply this definition to the little cube in Fig. 1.6. The momentum that flows *into* the cube in unit time across the front face (at $y = 0$) is $T_{jy}L^2$, and across the back face (at $y = L$) is $-T_{jy}L^2$; and their sum is $-T_{jy,y}L^3$. Adding to this the contributions from the side faces and the top and bottom faces, we find for the rate of change of total momentum inside the cube $(-T_{jx,x} - T_{jy,y} - T_{jz,z})L^3 = -T_{jk,k}L^3$. Since the cube's volume is L^3, this says that

$$\partial(\text{momentum density})/dt + \boldsymbol{\nabla} \cdot \mathbf{T} = 0 \; . \tag{1.36}$$

This has the standard form for any local conservation law: the time derivative of the density of some quantity (here momentum), plus the divergence of the flux of that quantity (here momentum flux is the stress tensor), is zero. We shall make extensive use of this Newtonian *local law of momentum conservation* in Part III (elasticity theory), Part IV (fluid mechanics) and Part V (plasma physics).

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EXERCISES

Exercise 1.12 *****Example: Equations of Motion for a Perfect Fluid***

(a) Consider a perfect fluid that moves with a velocity $\mathbf{v}$ that varies in time and space. Explain why the fluid's momentum density is $\rho\mathbf{v}$, and explain why its momentum flux (stress tensor) is

$$\mathbf{T} = P\mathbf{g} + \rho\mathbf{v} \otimes \mathbf{v} \; , \quad \text{or, in slot-naming index notation } T_{ij} = P\delta_{ij} + \rho v_i v_j \; . \tag{1.37a}$$

(b) Explain why the law of mass conservation for this fluid is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0. \tag{1.37b}$$

(c) Explain why the derivative operator

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \tag{1.37c}$$

describes the rate of change as measured by somebody who moves locally with the fluid, i.e. with velocity $\mathbf{v}$. Sometimes this is called the fluid's *advective time derivative.*

(d) Show that the fluid's law of mass conservation (1.37b) can be rewritten as

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\boldsymbol{\nabla} \cdot \mathbf{v} \;, \tag{1.37d}$$

which says that the divergence of the fluid's velocity field is minus the fractional rate of change of its density, as measured in the fluid's local rest frame.

(e) Show that the law of momentum conservation (1.36) for the fluid can be written as

$$\rho \frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla} P \tag{1.37e}$$

This is called the fluid's *Euler equation.* Explain why this Euler equation is Newton's second law of motion, "F=ma", written on a per unit volume basis.

In Part V of this book, we shall use Eqs. (1.37) to study the dynamical behaviors of fluids. For many applications, the Euler equation will need to be augmented by the force per unit volume exerted by the fluid's internal viscosity.

Exercise 1.13 ***Problem: Electromagnetic Stress Tensor*

(a) An electric field $\mathbf{E}$ exerts (in Gaussian cgs units) a pressure $\mathbf{E}^2/8\pi$ orthogonal to itself and a tension of this same magnitude along itself. Similarly, a magnetic field $\mathbf{B}$ exerts a pressure $\mathbf{B}^2/8\pi$ orthogonal to itself and a tension of this same magnitude along itself. Verify that the following stress tensor embodies these stresses:

$$\mathbf{T} = \frac{1}{8\pi}\left[(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{g} - 2(\mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B})\right] \;. \tag{1.38}$$

(b) Consider an electromagnetic field interacting with a material that has a charge density ρ_e and a current density $\mathbf{j}$. Compute the divergence of the electromagnetic stress tensor (1.38) and evaluate the derivatives using Maxwell's equation. Show that the result is the negative of the force density that the electromagnetic field exerts on the material. Use momentum conservation to explain why this had to be so.

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1.10 Geometrized Units and Relativistic Particles for Newtonian Readers

Readers who are skipping the relativistic parts of this book will need to know two important pieces of relativity: (i) geometrized units, and (ii) the (relativistic) energy and momentum of a moving particle.

Geometrized Units: The speed of light is independent of one's reference frame (i.e. independent of how fast one moves. This is a fundamental tenet of special relativity, and in the era before 1983, when the meter and the second were defined independently, it was tested and confirmed experimentally with very high precisionḂy 1983, this constancy had been become so universally accepted that it was used to redefine the meter (which is hard to measure precisely) in terms of the second (which is much easier to measure with modern technology[6]): The meter is now related to the second in such a way that the speed of light is precisely $c = 299,792,458$ m s^{-1} ; i.e., one meter is the distance traveled by light in $(1/299,792,458)$ seconds. Because of this constancy of the light speed, it is permissible when studying special relativity to set c to unity. Doing so is equivalent to the relationship

$$c = 2.99792458 \times 10^{10} \mathrm{cm\,s}^{-1} = 1 \tag{1.39a}$$

between seconds and centimeters; i.e., equivalent to

$$1 \text{ second} = 2.99792458 \times 10^{10} \text{ cm} . \tag{1.39b}$$

We shall refer to units in which $c = 1$ as *geometrized units*, and we shall adopt them throughout this book, when dealing with relativistic physics, since they make equations look much simpler. Occasionally it will be useful to restore the factors of c to an equation, thereby converting it to ordinary (SI or Gaussian-cgs) units. This restoration is achieved easily using dimensional considerations. For example, the equivalence of mass m and energy $\mathcal{E}$ is written in geometrized units as $\mathcal{E} = m$. In cgs units $\mathcal{E}$ has dimensions ergs = gram cm^2 sec^{-2}, while m has dimensions of grams, so to make $\mathcal{E} = m$ dimensionally correct we must multiply the right side by a power of c that has dimensions cm^2/sec^2, i.e. by c^2; thereby we obtain $\mathcal{E} = mc^2$.

Energy and momentum of a moving particle. A particle with rest mass m, moving with velocity $\mathbf{v} = d\mathbf{x}/dt$ and speed $v = |\mathbf{v}|$, has a relativistic energy $\mathcal{E}$ (including its rest-mass), relativistic energy E (excluding its rest mass) and relativistic momentum $\mathbf{p}$ given by

$$\mathcal{E} = \frac{m}{\sqrt{1-v^2}} \equiv \frac{m}{\sqrt{1-v^2/c^2}} \equiv E + m \;, \quad \mathbf{p} = \mathcal{E}\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1-v^2}} \;. \tag{1.40}$$

[6] The second is defined as the duration of 9,192,631,770 periods of the radiation produced by a certain hyperfine transition in the ground state of a ^{133}Cs atom that is at rest in empty space. Today (2008) all fundamental physical units except mass units (e.g. the kilogram) are defined similarly in terms of fundamental constants of nature.

In the low-velocity, Newtonian limit, the energy E with rest mass removed and the momentum $\mathbf{p}$ and take their familiar, Newtonian forms:

$$\text{When } v \ll c \equiv 1, \quad E \to \frac{1}{2}mv^2 \quad \text{and } \mathbf{p} \to m\mathbf{v} \; . \tag{1.41}$$

A particle with zero rest mass (a photon or a graviton[7]) always moves with the speed of light $v = c = 1$, and like other particles it has momentum $\mathbf{p} = \mathcal{E}\mathbf{v}$, so the magnitude of its momentum is equal to its energy: $|\mathbf{p}| = \mathcal{E}v = \mathcal{E}$.

When particles interact (e.g. in chemical reactions, nuclear reactions, and elementary-particle collisions) the sum of the particle energies $\mathcal{E}$ is conserved, as is the sum of the particle momenta $\mathbf{p}$.

For further details and explanations, see Chap. 2.

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EXERCISES

Exercise 1.14 *Practice: Geometrized Units*
Convert the following equations from the geometrized units in which they are written to cgs/Gaussian units:

(a) The "Planck time" t_P expressed in terms of Newton's gravitation constant G and Planck's constant $\hbar$, $t_P = \sqrt{G\hbar}$. What is the numerical value of t_P in seconds? in meters?

(b) The energy $E = 2m$ obtained from the annihilation of an electron and a positron, each with rest mass m.

(c) The Lorentz force law $m d\mathbf{v}/dt = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

(d) The expression $\mathbf{p} = \hbar\omega\mathbf{n}$ for the momentum $\mathbf{p}$ of a photon in terms of its angular frequency ω and direction $\mathbf{n}$ of propagation.

How tall are you, in seconds? How old are you, in centimeters?

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Bibliographic Note

The geometric viewpoint on the laws of physics, which we present and advocate in this chapter, is not common (but it should be because of its great power). For example, the vast majority of mechanics and electrodynamics textbooks with which we are familiar [e.g.

[7] We do not know for sure that photons and gravitons are massless, but the laws of physics as currently understood require them to be massless and there are tight experimental limits on their rest masses.

Box 1.3
Important Concepts in Chapter 1

- Geometric Principle for laws of physics: last paragraph of Sec. 1.2
- Geometric objects: points, scalars, vectors, tensors, length: Secs. 1.2 and 1.3
- Operations on geometric objects: Inner product, Eq. (1.4a); tensor product (outer product), Eq. (1.5); contraction, Eqs. (1.6)
- Component representation of tensor algebra Sec. 1.5
- Slot-naming index notation, Sec. 1.5.1
- Orthogonal transformations, Sec. 1.6
- Differentiation of tensors, Sec. 1.7
- Levi-Civita tensor, Eqs (1.20) in 3 dimensions; Ex. 1.8 in 2 dimensions
  - Vector cross product defined using Levi-Civita tensor, Eqs. (1.20)
  - Contraction of Levi-Civita tensor with itself, Eq. (1.22)
- Integration of vectors; Gauss and Stokes theorems, Sec. 1.8
- Conservation laws:
  - going back and forth between integral and differential conservation laws: Eqs. (1.30), (1.31)
  - general form of a differential conservation law, Eq. (1.31) and sentences following it.
  - momentum conservation, Eq. (1.36)
- Stress tensor, Sec. 1.9
  - for a moving perfect fluid, Eq. (1.37a)
  - for electric and magnetic fields, Eq. (1.38)
- Geometrized units, Sec. 1.10
- Energy and momentum of a relativistic, moving particle, Eqs. (1.40), (1.41)

Goldstein (1980), Griffiths (1999), and Jackson (1999)] define a tensor as a matrix-like entity whose components transform under rotations in the manner described by Eq. (1.13b). This is a complicated definition that hides the great simplicity of a tensor as nothing more than a linear function of vectors, and hides the possibility to think about tensors geometrically, without the aid of any coordinate system or basis.

The geometric viewpoint comes to the physics community from mathematicians, largely by way of relativity theory; by now, most relativity textbooks espouse it. See the Bibliography to Chap. 2. Fortunately, this viewpoint is gradually seeping into the physics curriculum. [For example, John Safko, a relativity theorist, has introduced it into portions of the most recent version of Goldstein's classic textbook, Goldstein, Poole and Safko (2002).] We hope that this chapter will accelerate that seepage.

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