Contents

Part II

OPTICS

Optics

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Prior to the opening up of the electromagnetic spectrum and the development of quantum mechanics, the study of optics was only concerned with visible light.

Reflection and refraction of light were first described by the Greeks and further studied by medieval scholastics like Roger Bacon, who explained the rainbow and used refraction in the design of crude magnifying lenses and spectacles. However, it was not until the seventeenth century that there arose a strong commercial interest in developing the telescope and the compound microscope.

Naturally, the discovery of Snell's law in 1621 and the observation of diffractive phenomena stimulated serious speculation about the physical nature of light. The corpuscular and wave theories were propounded by Newton in YYYY and Huygens in YYYY, respectively. The corpuscular theory initially held sway, but the studies of interference by Young in YYYY and the derivation of a wave equation for electromagnetic disturbances by Maxwell in YYYY seemed to settle the matter in favor of the undulatory theory, only for the debate to be resurrected in YYYY with the discovery of the photoelectric effect. After quantum mechanics was developed in the 1920's, the dispute was abandoned, the wave and particle descriptions of light became "complementary", and Hamilton's optics-inspired formulation of classical mechanics was modified to produce the Schrödinger equation.

Many physics students are all too familiar with this potted history and may consequently regard optics as an ancient precursor to modern physics that has been completely subsumed by quantum mechanics. However, this is not the case. Optics has developed dramatically and independently from quantum mechanics in recent decades, and is now a major branch of classical physics. It is no longer concerned primarily with light. The principles of optics are routinely applied to all types of wave propagation: from all parts of the electromagnetic spectrum, to quantum mechanical waves, e.g. of electrons and neutrinos, to waves in elastic solids (Part IV of this book), fluids (Part V), plasmas (Part VI) and the geometry of space-time (Part VII). There is a commonality, for instance, to seismology, oceanography and radio physics that allows ideas to be freely transported between these different disciplines. Even in the study of visible light, there have been major developments: the invention of the laser has led to the modern theory of coherence and has begotten the new field of nonlinear optics.

An even greater revolution has occured in optical technology. From the credit card and white light hologram to the laser scanner at a supermarket checkout, from laser printers

to CD's, DVD's and BD's, from radio telescopes capable of nanoradian angular resolution to Fabry-Perot systems that detect displacements smaller than the size of an elementary particle, we are surrounded by sophisticated optical devices in our everyday and scientific lives. Many of these devices turn out to be clever and direct applications of the fundamental optical principles that we shall discuss.

The treatment of optics in this text differs from that found in traditional texts in that we shall assume familiarity with basic classical and quantum mechanics and, consequently, fluency in the language of Fourier transforms. This inversion of the historical development reflects contemporary priorities and allows us to emphasize those aspects of the subject that involve fresh concepts and modern applications.

In Chap. 7, we shall discuss optical (wave-propagation) phenomena in the *geometric optics* approximation. This approximation is accurate whenever the wavelength and the wave period are short compared with the lengthscales and timescales on which the wave amplitude and the waves' environment vary. We shall show how a wave equation can be solved approximately in such a way that optical rays become the classical trajectories of particles, e.g. photons, and how, in general, ray systems develop singularities or caustics where the geometric optics approximation breaks down and we must revert to the wave description.

In Chap. 8 we will develop the theory of *diffraction* that arises when the geometric optics approximation fails and the waves' energy spreads in a non-particle-like way. We shall analyze diffraction in two limiting regimes, called *Fresnel* and *Fraunhofer*, after the physicists who discovered them, in which the wavefronts are approximately planar or spherical, respectively. Insofar as we are working with a linear theory of wave propagation, we shall make heavy use of Fourier methods and shall show how elementary applications of Fourier transforms can be used to design powerful optics instruments.

Most elementary diffractive phenomena involve the superposition of an infinite number of waves. However, in many optical applications, only a small number of waves from a common source are combined. This is known as *interference* and is the subject of Chap. 9. In this chapter we will also introduce the notion of coherence, which is a quantitative measure of the distributions of the combining waves and their capacity to interfere constructively.

The final chapter on optics, Chap. 10, is concerned with *nonlinear phenomena* that arise when waves, propagating through a medium, become sufficiently strong to couple to each other. These nonlinear phenomena can occur for all types of waves (we shall meet them for fluid waves in Part V and plasma waves in Part VI). For light (the focus of Chap. 10), they have become especially important; the nonlinear effects that arise when laser light is shone through certain crystals are having a strong impact on technology and on fundamental scientific research. We shall explore several examples.

Chapter 7

Geometric Optics

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Box 7.1
Reader's Guide

- This chapter does not depend substantially on any previous chapter.

- Secs. 7.1–7.4 of this chapter are foundations for the remaining Optics chapters: 8, 9, and 10.

- The discussion of caustics in Sec. 7.5 is a foundation for Sec. 8.6 on diffraction at a caustic.

- Secs. 7.2 and 7.3 (plane, monochromatic waves and wavepackets in a homogeneous, time-independent medium, the dispersion relation, and the geometric optics equations) will be used extensively in subsequent Parts of this book, including

    - Chap. 12 for elastodynamic waves
    - Chap. 16 for waves in fluids
    - Sec. 18.7, and Chaps. 20–20, for waves in plasmas
    - Chap. 26 for gravitational waves.

7.1 Overview

Geometric optics, the study of "rays," is the oldest approach to optics. It is an accurate description of wave propagation whenever the wavelengths and periods of the waves are far

smaller than the lengthscales and timescales on which the wave amplitude and the medium supporting the waves vary.

After reviewing wave propagation in a homogeneous medium (Sec. 7.2), we shall begin our study of geometric optics in Sec. 7.3. There we shall derive the geometric-optic propagation equations with the aid of the *eikonal* approximation, and we shall elucidate the connection to Hamilton-Jacobi theory, which we will assume that the reader has already encountered. This connection will be made more explicit by demonstrating that a classical, geometric-optics wave can be interpreted as a flux of quanta. In Sec. 7.4 we shall specialize the geometric optics formalism to any situation where a bundle of nearly parallel rays is being guided and manipulated by some sort of apparatus. This is called the *paraxial approximation*, and we shall illustrate it using the problem of magnetically focusing a beam of charged particles and shall show how matrix methods can be used to describe the particle (i.e. ray) trajectories. In Sec. 7.5, we shall discuss the formation of images in geometric optics, illustrating our treatment with gravitational lenses. We shall pay special attention to the behavior of images at caustics, and its relationship to *catastrophe theory*. Finally, in Sec. 7.6, we shall turn from scalar waves to the vector waves of electromagnetic radiation. We shall deduce the geometric-optics propagation law for the waves' polarization vector and shall explore the classical version of the *geometric phase*.

7.2 Waves in a Homogeneous Medium

7.2.1 Monochromatic, Plane Waves

Consider a monochromatic plane wave propagating through a homogeneous medium. Independently of the physical nature of the wave, it can be described mathematically by

$$\psi = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \equiv Ae^{i\varphi} , \tag{7.1}$$

where ψ is any oscillatory physical quantity associated with the wave, for example, the y-component of the magnetic field associated with an electromagnetic wave. If, as is usually the case, the physical quantity is real (not complex), then we must take the real part of Eq. (7.1). In Eq. (7.1), A is the wave's *complex amplitude*, $\varphi = \mathbf{k} \cdot \mathbf{x} - \omega t$ is the wave's *phase*, t and $\mathbf{x}$ are time and location in space, $\omega = 2\pi f$ is the wave's *angular frequency*, and $\mathbf{k}$ is its *wave vector* (with $k \equiv |\mathbf{k}|$ its *wave number*, $\lambda = 2\pi/k$ its *wavelength*, $\lambda = \lambda/2\pi$ its reduced wavelength and $\hat{\mathbf{k}} \equiv \mathbf{k}/k$ its *propagation direction*). Surfaces of constant phase are orthogonal to the propagation direction $\hat{\mathbf{k}}$ and move with the *phase velocity*

$$\mathbf{V}_{\mathrm{ph}} \equiv \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\varphi} = -\frac{(\partial\varphi/\partial t)_{\mathbf{x}}}{(\partial\varphi/\partial\mathbf{x})_t} = \frac{\omega}{k}\hat{\mathbf{k}} ; \tag{7.2a}$$

cf. Fig. 7.1. Lest there be confusion, Eq. (7.2a) is short-hand notation for the Cartesian-component equation

$$V_{\mathrm{ph}\,j} \equiv \left(\frac{\partial x_j}{\partial t}\right)_{\varphi} = -\frac{(\partial\varphi/\partial t)_{\mathbf{x}}}{(\partial\varphi/\partial x_j)_t} = \frac{\omega}{k}\hat{\mathbf{k}}_j . \tag{7.2b}$$
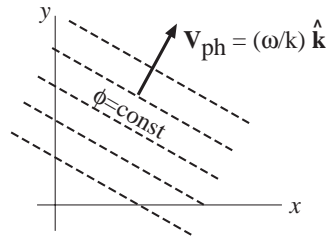
Fig. 7.1: A monochromatic plane wave in a homogeneous medium.

The frequency ω is determined by the wave vector $\mathbf{k}$ in a manner that depends on the wave's physical nature; the functional relationship

$$\boxed{\omega = \Omega(\mathbf{k})} \tag{7.3}$$

is called the wave's *dispersion relation* because (as we shall see in Ex. 7.2) it governs the dispersion (spreading) of a wave packet that is constructed by superposing plane waves.

Some examples of plane waves that we shall study in this book are: (i) Electromagnetic waves propagating through a dielectric medium with index of refraction $\mathfrak{n}$ (this chapter), for which ψ could be any Cartesian component of the electric or magnetic field or vector potential and the dispersion relation is

$$\omega = \Omega(\mathbf{k}) = Ck \equiv C|\mathbf{k}| \ , \tag{7.4}$$

with $C = c/\mathfrak{n}$ the propagation speed and c the speed of light in vacuum. (ii) Sound waves propagating through a solid (Sec. 12.2.3) or fluid (liquid or vapor; Secs. 16.5 and 7.3.1), for which ψ could be the pressure or density perturbation produced by the sound wave (or it could be a potential whose gradient is the velocity perturbation), and the dispersion relation is the same as for electromagnetic waves, Eq. (7.4), but with C now the sound speed. (iii) Waves on the surface of a deep body of water (depth $\gg \lambda$; Sec. 16.2), for which ψ could be the height of the water above equilibrium, and the dispersion relation is [Eq. (16.9)]:

$$\omega = \Omega(\mathbf{k}) = \sqrt{gk} = \sqrt{g|\mathbf{k}|} \tag{7.5}$$

with g the acceleration of gravity. (iv) Flexural waves on a stiff beam or rod (Sec. 12.3.4), for which ψ could be the transverse displacement of the beam from equilibrium and the dispersion relation is

$$\omega = \Omega(\mathbf{k}) = \sqrt{\frac{D}{\Lambda}k^2} = \sqrt{\frac{D}{\Lambda}\mathbf{k} \cdot \mathbf{k}} \ , \tag{7.6}$$

with Λ the rod's mass per unit length and D its "flexural rigidity" [Eq. (11.82)]. (v) Alfvén waves (bending oscillations of plasma-laden magnetic field lines in a magnetized, nonrelativistic plasma; Sec. 18.7.2), for which ψ could be the transverse displacement of the field and plasma, and the dispersion relation is [Eq. (18.76)]

$$\omega = \Omega(\mathbf{k}) = \mathbf{a} \cdot \mathbf{k}, \tag{7.7}$$

with $\mathbf{a} = \mathbf{B}/\sqrt{\mu_o\rho}$, $[= \mathbf{B}/\sqrt{4\pi\rho}]$ [1] the *Alfvén speed*, $\mathbf{B}$ the (homogeneous) magnetic field, μ_o the magnetic permitivity of the vacuum, and ρ the plasma mass density.

In general, one can derive the dispersion relation $\omega = \Omega(\mathbf{k})$ by inserting the plane-wave ansatz (7.1) into the dynamical equations that govern one's physical system [e.g. Maxwell's equations, or the equations of elastodynamics (Chap. 12), or the equations for a magnetized plasma (Part VI) or ...]. We shall do so time and again in this book.

7.2.2 Wave Packets

Waves in the real world are not precisely monochromatic and planar. Instead, they occupy wave packets that are somewhat localized in space and time. Such wave packets can be constructed as superpositions of plane waves:

$$\psi(\mathbf{x}, t) = \int \mathbf{A}(\mathbf{k})e^{i\alpha(\mathbf{k})}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}d^3k \text{ , where } \mathbf{A}(\mathbf{k}) \text{ is concentrated around some } \mathbf{k} = \mathbf{k}_o.$$

$$(7.8a)$$

Here $\mathbf{A}$ and α (both real) are the modulus and phase of the complex amplitude $\mathbf{A}e^{i\alpha}$, and the integration element is $d^3k \equiv d\mathcal{V}_k \equiv dk_x dk_y dk_z$ in terms of components of $\mathbf{k}$ along Cartesian axes x, y, z. In the integral (7.8a), the contributions from adjacent $\mathbf{k}$'s will tend to cancel each other except in that region of space and time where the oscillatory phase factor changes little with changing $\mathbf{k}$, when $\mathbf{k}$ is near $\mathbf{k}_o$. This is the spacetime region in which the wave packet is concentrated, and its center is where $\boldsymbol{\nabla}_{\mathbf{k}}(\text{phasefactor}) = 0$:

$$\left(\frac{\partial\alpha}{\partial k_j} + \frac{\partial}{\partial k_j}(\mathbf{k}\cdot\mathbf{x} - \omega t)\right)_{\mathbf{k}=\mathbf{k}_o} = 0 \ . \tag{7.8b}$$

Evaluating the derivative with the aid of the wave's dispersion relation $\omega = \Omega(\mathbf{k})$, we obtain for the location of the wave packet's center

$$x_j - \left(\frac{\partial\Omega}{\partial k_j}\right)_{\mathbf{k}=\mathbf{k}_o} t = -\left(\frac{\partial\alpha}{\partial k_j}\right)_{\mathbf{k}=\mathbf{k}_o} = \text{const} \ . \tag{7.8c}$$

This tells us that the wave packet moves with the *group velocity*

$$\boxed{\mathbf{V}_g = \nabla_{\mathbf{k}}\Omega \ , \quad \text{i.e.} \quad \mathbf{V}_{gj} = \left(\frac{\partial\Omega}{\partial k_j}\right)_{\mathbf{k}=\mathbf{k}_o}} \ . \tag{7.9}$$

When, as for electromagnetic waves in a dielectric medium or sound waves in a solid or fluid, the dispersion relation has the simple form (7.4), $\omega = \Omega(\mathbf{k}) = Ck$ with $k \equiv |\mathbf{k}|$, then the group and phase velocities are the same

$$\mathbf{V}_g = \mathbf{V}_{\text{ph}} = C\hat{\mathbf{k}} \ , \tag{7.10}$$

and the waves are said to be *dispersionless*. If the dispersion relation has any other form, then the group and phase velocities are different, and the wave is said to exhibit *dispersion*;

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[1]Gaussian unit equivalents will be given with square brackets.

(a) Deep water wavepacket   **(b)** Flexural wavepacket on a beam   **(c)** Alfven wavepacket
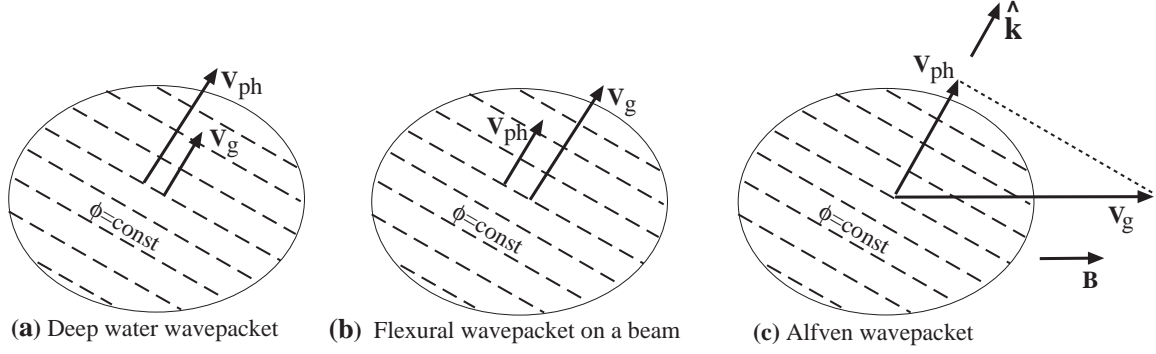
Fig. 7.2: (a) A wave packet of waves on a deep body of water. The packet is localized in the spatial region bounded by the thin ellipse. The packet's (ellipse's) center moves with the group velocity $\mathbf{V}_g$. The ellipse will grow slowly in size due to wave-packet dispersion (spreading; Ex. 7.2). The surfaces of constant phase (the wave's oscillations) move twice as fast as the ellipse and in the same direction, $\mathbf{V}_{\mathrm{ph}} = 2\mathbf{V}_g$ [Eq. (7.11)]. This means that the wave's oscillations arise at the back of the packet and move forward through the packet, disappearing at the front. The wavelength of these oscillations is $\lambda = 2\pi/k_o$, where $k_o = |\mathbf{k}_o|$ is the wavenumber about which the wave packet is concentrated [Eq. (7.8a) and associated discussion]. (b) A flexural wave packet on a beam, for which $\mathbf{V}_{\mathrm{ph}} = \frac{1}{2}\mathbf{V}_g$ [Eq. (7.12)] so the wave's oscillations arise at the packet's front and, traveling more slowly than the packet, disappear at its back. (c) An Alfvén wave packet. Its center moves with a group velocity $\mathbf{V}_g$ that points along the direction of the background magnetic field [Eq. (7.13)], and its surfaces of constant phase (the wave's oscillations) move with a phase velocity $\mathbf{V}_{\mathrm{ph}}$ that can be in any direction $\hat{\mathbf{k}}$. The phase speed is the projection of the group velocity onto the phase propagation direction, $|\mathbf{V}_{\mathrm{ph}}| = \mathbf{V}_g \cdot \hat{\mathbf{k}}$ [Eq. (7.13)], which implies that the wave's oscillations remain fixed inside the packet as the packet moves; their pattern inside the ellipse does not change.

cf. Ex. 7.2. Examples are (see above): (iii) Waves on a deep body of water [dispersion relation (7.5); Fig. 7.2a] for which

$$\mathbf{V}_g = \frac{1}{2}\mathbf{V}_{\mathrm{ph}} = \frac{1}{2}\sqrt{\frac{g}{k}}\,\hat{\mathbf{k}}\,. \tag{7.11}$$

(iv) Flexural waves on a rod or beam [dispersion relation (7.6); Fig. 7.2b] for which

$$\mathbf{V}_g = 2\mathbf{V}_{\mathrm{ph}} = 2\sqrt{\frac{D}{\Lambda}}\,k\hat{\mathbf{k}}\,. \tag{7.12}$$

(v) Alfvén waves in a magnetized plasma [dispersion relation (7.7); Fig. 7.2c] for which

$$\mathbf{V}_g = \mathbf{a}\,, \quad \mathbf{V}_{\mathrm{ph}} = (\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}\,. \tag{7.13}$$

Notice that, depending on the dispersion relation, the group speed $|\mathbf{V}_g|$ can be less than or greater than the phase speed, and if the homogeneous medium is anisotropic (e.g., for a magnetized plasma), the group velocity can point in a different direction than the phase velocity.

It should be obvious, physically, that the energy contained in a wave packet must remain always with the packet and cannot move into the region outside the packet where the wave amplitude vanishes. Correspondingly, *the wave packet's energy must propagate with the group velocity* $\mathbf{V}_g$ *and not with the phase velocity* $\mathbf{V}_{\mathrm{ph}}$. Similarly, when one examines the wave packet from a quantum mechanical viewpoint, *its quanta must move with the group velocity* $\mathbf{V}_g$. Since we have required that the wave packet have its wave vectors concentrated around $\mathbf{k}_o$, the energy and momentum of each of the packet's quanta are given by the standard quantum mechanical relations

$$\boxed{\mathcal{E} = \hbar\Omega(\mathbf{k}_o) \text{ and } \mathbf{p} = \hbar\mathbf{k}_o .} \tag{7.14}$$

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EXERCISES

Exercise 7.1 *Practice: Group and Phase Velocities*

Derive the group and phase velocities (7.10)–(7.13) from the dispersion relations (7.4)–(7.7).

Exercise 7.2 *Example: Gaussian Wave Packet and its Dispersion*

Consider a one-dimensional wave packet, $\psi(x,t) = \int \mathtt{A}(k)e^{i\alpha(k)}e^{i(kx-\omega t)}dk$ with dispersion relation $\omega = \Omega(k)$. For concreteness, let $\mathtt{A}(k)$ be a narrow Gaussian peaked around k_o: $\mathtt{A} \propto \exp[-\kappa^2/2(\Delta k)^2]$, where $\kappa = k - k_o$.

(a) Expand α as $\alpha(k) = \alpha_o - x_o\kappa$ with x_o a constant, and assume for simplicity that higher order terms are negligible. Similarly expand $\omega \equiv \Omega(k)$ to quadratic order, and explain why the coefficients are related to the group velocity V_g at $k = k_o$ by $\Omega = \omega_o + V_g\kappa + (dV_g/dk)\kappa^2/2$.

(b) Show that the wave packet is given by

$$\psi \propto \exp[i(\alpha_o + k_o x - \omega_o t)] \int_{-\infty}^{+\infty} \exp[i\kappa(x - x_o - V_g t)]\exp\left[-\frac{\kappa^2}{2}\left(\frac{1}{(\Delta k)^2} + i\frac{dV_g}{dk}t\right)\right] d\kappa .$$
$$\tag{7.15a}$$

The term in front of the integral describes the phase evolution of the waves inside the packet; cf. Fig. 7.2.

(c) Evaluate the integral analytically (with the help of Mathematica or Maple, if you wish). Show, from your answer, that the modulus of ψ is given by

$$\boxed{|\psi| \propto \exp\left[-\frac{(x - x_o - V_g t)^2}{2L^2}\right] , \quad \text{where } L = \frac{1}{2\Delta k}\sqrt{1 + \left(\frac{dV_g}{dk}(\Delta k)^2 t\right)^2}} \tag{7.15b}$$

is the packet's half width.

(d) Discuss the relationship of this result, at time $t = 0$, to the uncertainty principle for the localization of the packet's quanta.

(e) Equation (7.15b) shows that the wave packet spreads (i.e. disperses) due to its containing a range of group velocities. How long does it take for the packet to enlarge by a factor 2? For what range of initial half widths can a water wave on the ocean spread by less than a factor 2 while traveling from Hawaii to California?

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7.3 Waves in an Inhomogeneous, Time-Varying Medium: The Eikonal Approximation and Geometric Optics

Suppose that the medium in which the waves propagate is spatially inhomogeneous and varies with time. If the lengthscale $\mathcal{L}$ and timescale $\mathcal{T}$ for substantial variations are long compared to the waves' reduced wavelength and period,

$$\mathcal{L} \gg \lambda = 1/k , \quad \mathcal{T} \gg 1/\omega , \tag{7.16}$$

then the waves can be regarded locally as planar and monochromatic. The medium's inhomogeneities and time variations may produce variations in the wave vector $\mathbf{k}$ and frequency ω, but those variations should be substantial only on scales $\gtrsim \mathcal{L} \gg 1/k$ and $\gtrsim \mathcal{T} \gg 1/\omega$. This intuitively obvious fact can be proved rigorously using a two-lengthscale expansion, i.e. an expansion of the wave equation in powers of $\lambda/\mathcal{L} = 1/k\mathcal{L}$ and $1/\omega\mathcal{T}$. Such an expansion, in this context of wave propagation, is called the *eikonal approximation* [2] or the *geometric optics approximation*. When the waves are those of elementary quantum mechanics, it is called the *WKB approximation*. The eikonal approximation converts the laws of wave propagation into a remarkably simple form in which the waves' amplitude is transported along trajectories in spacetime called *rays*. In the language of quantum mechanics, these rays are the world lines of the wave's quanta (photons for light, phonons for sound, plasmons for Alfvén waves, gravitons for gravitational waves), and the law by which the wave amplitude is transported along the rays is one which conserves quanta. These ray-based propagation laws are called the laws of *geometric optics*.

In this section we shall develop and study the eikonal approximation and its resulting laws of geometric optics. We shall begin in Sec. 7.3.1 with a full development of the eikonal approximation and its geometric-optics consequences for a prototypical dispersion-free wave equation that represents, for example, sound waves in a weakly inhomogeneous fluid. In Sec. 7.3.3 we shall extend our analysis to cover all other types of waves. In Sec. 7.3.4 and a number of exercises we shall explore examples of geometric-optics waves, and in Sec. 7.3.5 we shall discuss conditions under which the eikonal approximation breaks down, and some non-geometric-optics phenomena that result from the breakdown. Finally, in Sec. 7.3.6 we shall return to nondispersive light and sound waves, and deduce Fermat's Principle and explore some of its consequences.

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[2] After the Greek word $\epsilon\iota\kappa\omega\nu$ meaning image.

7.3.1 Geometric Optics for a Prototypical Wave Equation

Our prototypical wave equation is

$$\frac{\partial}{\partial t}\left(W\frac{\partial\psi}{\partial t}\right) - \boldsymbol{\nabla}\cdot(WC^2\boldsymbol{\nabla}\psi) = 0 \; . \tag{7.17}$$

Here $\psi(\mathbf{x},t)$ is the quantity that oscillates (the *wave field*), $C(\mathbf{x},t)$ will turn out to be the wave's slowly varying *propagation speed*, and $W(\mathbf{x},t)$ is a slowly varying *weighting function* that depends on the properties of the medium through which the wave propagates. As we shall see, W has no influence on the wave's dispersion relation or on its geometric-optics rays, but does influence the law of transport for the waves' amplitude.

The wave equation (7.17) describes sound waves propagating through a static, inhomogeneous fluid (Ex. 16.12), in which case ψ is the wave's pressure perturbation δP, $C(\mathbf{x}) = \sqrt{(\partial P/\partial\rho)_s}$ is the adiabatic sound speed, and the weighting function is $W(\mathbf{x}) = \rho/C^2$, with ρ the fluid's unperturbed density. This wave equation also describes waves on the surface of a lake or pond or the ocean, in the limit that the slowly varying depth of the undisturbed water $h_o(\mathbf{x})$ is small compared to the wavelength (shallow-water waves; e.g. tsunamis); see Ex. 16.2. In this case $W = 1$ and $C = \sqrt{gh_o}$ with g the acceleration of gravity. In both cases, sound-waves in a fluid and shallow-water waves, if we turn on a slow time dependence in C and W, then additional terms enter the wave equation (7.17). For pedagogical simplicity, we leave those terms out but do allow W and C to be slowly varying in time, as well as in space: $W = W(\mathbf{x},t)$ and $C = C(\mathbf{x},t)$.

Associated with the wave equation (7.17) are an energy density $U(\mathbf{x},t)$ and energy flux $\mathbf{F}(\mathbf{x},t)$ given by

$$U = W\left[\frac{1}{2}\dot{\psi}^2 + \frac{1}{2}C^2(\boldsymbol{\nabla}\psi)^2\right] \; , \quad \mathbf{F} = -WC^2\dot{\psi}\boldsymbol{\nabla}\psi \; ; \tag{7.18}$$

see Ex. 7.4. Here and below the dot denotes a time derivative, $\dot{\psi} \equiv \psi_{,t} \equiv \partial\psi/\partial t$. It is straightforward to verify that, *if C and W are independent of time t*, then the scalar wave equation (7.17) guarantees that the U and $\mathbf{F}$ of Eq. (7.18) satisfy the law of energy conservation

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{F} = 0 \; ; \tag{7.19}$$

cf. Ex. 7.4.

We now specialize to a weakly inhomogeneous and slowly time-varying fluid and to *nearly plane waves*, and we seek a solution of the wave equation (7.17) that *locally* has approximately the plane-wave form $\psi \simeq Ae^{i\mathbf{k}\cdot\mathbf{x}-\omega t}$. Motivated by this plane-wave form, (i) we express the waves as the product of a real amplitude $A(\mathbf{x},t)$ that varies slowly on the length and time scales $\mathcal{L}$ and $\mathcal{T}$, and the exponential of a complex phase $\varphi(\mathbf{x},t)$ that varies rapidly on the timescale $1/\omega$ and lengthscale $\lambdabar$:

$$\psi(\mathbf{x},t) = A(\mathbf{x},t)e^{i\varphi(\mathbf{x},t)} \; ; \tag{7.20}$$

and (ii) we *define* the wave vector (field) and angular frequency (field) by

$$\mathbf{k}(\mathbf{x},t) \equiv \boldsymbol{\nabla}\varphi \; , \quad \omega(\mathbf{x},t) \equiv -\partial\varphi/\partial t \; . \tag{7.21}$$

Box 7.2

Bookkeeping Parameter in Two-Lengthscale Expansions

When developing a two-lengthscale expansion, it is sometimes helpful to introduce a "bookkeeping" parameter σ and rewrite the anszatz (7.20) in a fleshed-out form

$$\psi = (A + \sigma B + \ldots)e^{i\varphi/\sigma} .. \tag{1}$$

The numerical value of σ is unity so it can be dropped when the analysis is finished. We use σ to tell us how various terms scale when λ is reduced at fixed $\mathcal{L}$ and $\mathcal{R}$. A has no attached σ and so scales as λ^0, B is multiplied by σ and so scales proportional to λ, and φ is multiplied by σ^{-1} and so scales as λ^{-1}. When one uses these σ's in the evaluation of the wave equation, the first term on the second line of Eq. (7.22) gets multiplied by σ^{-2}, the second term by σ^{-1}, and the omitted terms by σ^0. These factors of σ help us in quickly grouping together all terms that scale in a similar manner, and identifying which of the groupings is leading order, and which subleading, in the two-lengthscale expansion. In Eq. (7.22) the omitted σ^0 terms are the first ones in which B appears; they produce a propagation law for B, which can be regarded as a *post-geometric-optics correction.*

In addition to our two-lengthscale requirement $\mathcal{L} \gg 1/k$ and $\mathcal{T} \gg 1/\omega$, we also require that A, $\mathbf{k}$ and ω vary slowly, i.e., vary on lengthscales $\mathcal{R}$ and timescales $\mathcal{T}'$ long compared to $\lambda = 1/k$ and $1/\omega$.[3] This requirement guarantees that the waves are locally planar ($\varphi \simeq \mathbf{k} \cdot x - \omega t + \text{constant}$).

We now insert the Eikonal-approximated wave field (7.20) into the wave equation (7.17), perform the differentiations with the aid of Eqs. (7.21), and collect terms in a manner dictated by a two-lengthscale expansion (see Box 7.2):

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left(W\frac{\partial \psi}{\partial t}\right) - \boldsymbol{\nabla} \cdot (WC^2\boldsymbol{\nabla}\psi) \\ &= \left(-\omega^2 + C^2 k^2\right)W\psi + \left[-2(\omega\dot{A} + C^2 k_j A_{,j})W - (W\omega)_{,t}A - (WC^2 k_j)_{,j}A\right]e^{i\varphi} + \quad (7.22) \end{aligned}$$

The first term on the second line, $(-\omega^2 + C^2 k^2)W\psi$ scales as λ^{-2} when we make the reduced wavelength λ shorter and shorter while holding the macroscopic lengthscales $\mathcal{L}$ and $\mathcal{R}$ fixed; the second term (in square brackets) scales as λ^{-1}; and the omitted terms scale as λ^0. This is what we mean by "collecting terms in a manner dictated by a two-lengthscale expansion". Because of their different scaling, the first and second terms must vanish separately; they cannot possibly cancel each other.

The vanishing of the first term in the eikonal-approximated wave equation (7.22) says that the waves' frequency field $\omega(\mathbf{x}, t) \equiv -\partial\varphi/\partial t$ and wave-vector field $\mathbf{k} \equiv \boldsymbol{\nabla}\varphi$ satisfy the

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[3]Note: these variations can arise both (i) from the influence of the medium's inhomogeneity (which puts limits $\mathcal{R} \lesssim \mathcal{L}$ and $\mathcal{T}' \lesssim \mathcal{T}$ on the wave's variations, and also (ii) from the chosen form of the wave. For example, the wave might be traveling outward from a source and so have nearly spherical phase fronts with radii of curvature $r \simeq$ (distance from source); then $\mathcal{R} = \min(r, \mathcal{L})$.

dispersion relation

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t) \equiv C(\mathbf{x}, t)k \, , \tag{7.23}$$

where (as throughout this chapter) $k \equiv |\mathbf{k}|$. Notice that, as promised, this dispersion relation is independent of the weighting function W in the wave equation. Notice further that *this dispersion relation is identical to that for a precisely plane wave in a homogeneous medium, Eq. (7.4), except that the propagation speed C is now a slowly varying function of space and time.* This will always be so:

One can always deduce the geometric-optics dispersion relation by (i) considering a precisely plane, monochromatic wave in a precisely homogeneous, time-independent medium, obtaining $\omega = \Omega(\mathbf{k})$ in a functional form that involves the medium's properties (e.g. density); and then (ii) allowing the properties to be slowly varying functions of $\mathbf{x}$ and t [in this case, $C(\mathbf{x}, t)$]. The resulting dispersion relation then acquires its $\mathbf{x}$ and t dependence from the properties of the medium. See Sec. 7.3.3.

The vanishing of the second term in the eikonal-approximated wave equation (7.22) says that the waves' real amplitude A is transported with the group velocity $\mathbf{V}_g = C\hat{\mathbf{k}}$ in the following manner:

$$\frac{dA}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{V}_g \cdot \boldsymbol{\nabla} \right) A = -\frac{1}{2W\omega} \left[\frac{\partial(W\omega)}{\partial t} + \boldsymbol{\nabla} \cdot (WC^2\mathbf{k}) \right] A \, . \tag{7.24}$$

This propagation law, by contrast with the dispersion relation, does depend on the weighting function W. We shall return to this propagation law shortly and shall understand more deeply its dependence on W, but first we must investigate in detail the directions in spacetime along which A is transported.

The time derivative d/dt appearing in the propagation law (7.24) is similar to the derivative with respect to proper time along a world line in special relativity, $d/d\tau = u^0 \partial/\partial t + \mathbf{u} \cdot \boldsymbol{\nabla}$ (with u^α the world line's 4-velocity). This analogy tells us that the waves' amplitude A is being propagated along some sort of world lines. Those world lines (called the waves' *rays*), in fact, are governed by Hamilton's equations of particle mechanics with the dispersion relation $\Omega(\mathbf{x}, t, \mathbf{k})$ playing the role of the Hamiltonian and $\mathbf{k}$ playing the role of momentum:

$$\boxed{\frac{dx_j}{dt} = \left(\frac{\partial\Omega}{\partial k_j} \right)_{\mathbf{x},t} \equiv V_{gj}} \, , \quad \boxed{\frac{dk_j}{dt} = -\left(\frac{\partial\Omega}{\partial x_j} \right)_{\mathbf{k},t}} \, , \quad \boxed{\frac{d\omega}{dt} = \left(\frac{\partial\Omega}{\partial t} \right)_{\mathbf{x},\mathbf{k}}} \, . \tag{7.25}$$

The first of these Hamilton equations is just our definition of the group velocity, with which [according to Eq. (7.24)] the amplitude is transported. The second tells us how the wave vector $\mathbf{k}$ changes along a ray, and together with our knowledge of $C(\mathbf{x}, t)$, it tells us how the group velocity $\mathbf{V}_g = C\hat{\mathbf{k}}$ changes along a ray, and thence tells us the ray itself. The third tells us how the waves' frequency changes along a ray.

To deduce the second and third of these Hamilton equations, we begin by inserting the definitions $\omega = -\partial\varphi/\partial t$ and $\mathbf{k} = \boldsymbol{\nabla}\varphi$ [Eqs. (7.21)] into the dispersion relation $\omega = \Omega(\mathbf{x}, t; \mathbf{k})$ for an arbitrary wave, thereby obtaining

$$\boxed{\frac{\partial\varphi}{\partial t} + \Omega(\mathbf{x}, t; \boldsymbol{\nabla}\varphi) = 0} \, . \tag{7.26a}$$

This equation is known in optics as the *eikonal equation*. It is formally the same as the Hamilton-Jacobi equation of classical mechanics[4] if we identify Ω with the Hamiltonian and φ with Hamilton's principal function; cf. Ex. 7.9. This suggests that, to derive the second and third of Eqs. (7.25), we can follow the same procedure as is used to derive Hamilton's equations of motion: We take the gradient of Eq. (7.26a) to obtain

$$\frac{\partial^2 \varphi}{\partial t \partial x_j} + \frac{\partial \Omega}{\partial k_l}\frac{\partial^2 \varphi}{\partial x_l \partial x_j} + \frac{\partial \Omega}{\partial x_j} = 0 \ , \tag{7.26b}$$

where the partial derivatives of Ω are with respect to its arguments $(\mathbf{x}, t; \mathbf{k})$; we then use $\partial\varphi/\partial x_j = k_j$ and $\partial\Omega/\partial k_l = V_{gl}$ to write this as $dk_j/dt = -\partial\Omega/\partial x_j$. This is the second of Hamilton's equations (7.25), and it tells us how the wave vector changes along a ray. The third Hamilton equation, $d\omega/dt = \partial\Omega/\partial t$ [Eq. (7.25)] is obtained by taking the time derivative of the eikonal equation (7.26a).

Not only is the waves' amplitude A propagated along the rays, so is their phase:

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial t} + \mathbf{V}_g \cdot \boldsymbol{\nabla}\varphi = -\omega + \mathbf{V}_g \cdot \mathbf{k} \ . \tag{7.27}$$

Since our sound waves in an inhomogeneous fluid have $\omega = Ck$ and $\mathbf{V}_g = C\hat{\mathbf{k}}$, this vanishes. Therefore, *for the special case of sound waves the phase is constant along each ray*

$$\boxed{d\varphi/dt = 0} \tag{7.28}$$

The same will be true for any type of wave that has the dispersion-free dispersion relation $\Omega = Ck$, for example, light propagating in a dielectric medium.

7.3.2 Connection of Geometric Optics to Quantum Theory

Although the waves $\psi = Ae^{i\varphi}$ are classical and our analysis is classical, their propagation laws in the eikonal approximation can be described most nicely in quantum mechanical language.[5] Quantum mechanics insists that, associated with any wave in the geometric optics regime, there are real quanta: the wave's quantum mechanical particles. If the wave is electromagnetic, the quanta are photons; if it is gravitational, they are gravitons; if it is sound, they are phonons; if it is a plasma wave (e.g. Alfvén), they are plasmons. When we multiply the wave's $\mathbf{k}$ and ω by Planck's constant, we obtain the particles' momentum and energy,

$$\boxed{\mathbf{p} = \hbar\mathbf{k} \ , \quad \mathcal{E} = \hbar\omega \ .} \tag{7.29}$$

Although the originators of the 19th century theory of classical waves were unaware of these quanta, once quantum mechanics had been formulated, the quanta became a powerful conceptual tool for thinking about classical waves:

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[4]See, for example, Goldstein (1980).

[5]This is intimately related to the fact that quantum mechanics underlies classical mechanics; the classical world is an approximation to the quantum world.

In particular, we can regard the rays as the world lines of the quanta, and by multiplying the dispersion relation by $\hbar$ we can obtain the Hamiltonian for the quanta's world lines

$$\boxed{H(\mathbf{x}, t; \mathbf{p}) = \hbar\Omega(\mathbf{x}, t; \mathbf{k} = \mathbf{p}/\hbar) \,.}$$ (7.30)

Hamilton's equations (7.25) for the rays then become, immediately, Hamilton's equations for the quanta, $dx_j/dt = \partial H/\partial p_j$, $dp_j/dt = -\partial H/\partial x_j$, $d\mathcal{E}/dt = \partial H/\partial t$.

Return, now, to the propagation law (7.24) for the waves' amplitude. It is enlightening to explore the consequences of this propagation law for the waves' energy. By inserting the ansatz $\psi = \Re(Ae^{i\varphi}) = A\cos(\varphi)$ into Eqs. (7.18) for the energy density U and energy flux $\mathbf{F}$, and averaging over a wavelength and wave period so $\overline{\cos^2\varphi} = \overline{\sin^2\varphi} = 1/2$, we find that

$$U = \frac{1}{2}WC^2k^2A^2 = \frac{1}{2}W\omega^2A^2 \,, \quad \mathbf{F} = U(C\hat{\mathbf{k}}) = U\mathbf{V}_g \,.$$ (7.31)

Inserting these into the energy conservation expression and using the propagation law (7.24) for A, we obtain

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F} = U\frac{\partial \ln C}{\partial t} \,.$$ (7.32)

Thus, as the propagation speed C slowly changes at fixed location in space due to a slow change in the medium's properties, the medium slowly pumps energy into the waves or removes it from them.

This slow energy change can be understood more deeply using quantum concepts. The number density and number flux of quanta are

$$\boxed{n = \frac{U}{\hbar\omega} \,, \quad \mathbf{S} = \frac{\mathbf{F}}{\hbar\omega} = n\mathbf{V}_g \,.}$$ (7.33a)

By combining these with the the energy (non)conservation equation (7.32), we obtain

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = n\left[\frac{\partial \ln C}{\partial t} - \frac{d\ln\omega}{dt}\right] \,.$$

The third Hamilton equation tells us that $d\omega/dt = \partial\Omega/\partial t = \partial(Ck)/\partial t = k\partial C/\partial t$, whence $d\ln\omega/dt = \partial\ln C/\partial t$, which, when inserted into the above equation, implies that *the quanta are conserved:*

$$\boxed{\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = 0 \,.}$$ (7.33b)

Since $\mathbf{F} = n\mathbf{V}_g$ and $d/dt = \partial/\partial t + \mathbf{V}_g \cdot \boldsymbol{\nabla}$, we can rewrite this conservation law as a propagation law for the number density of quanta:

$$\boxed{\frac{dn}{dt} + n\boldsymbol{\nabla} \cdot \mathbf{V}_g = 0 \,.}$$ (7.33c)

The propagation law for the waves' amplitude, Eq. (7.24), can now be understood much more deeply: *The amplitude propagation law is nothing but the law of conservation of quanta*

in a slowly varying medium, rewritten in terms of the amplitude. This is true quite generally, for any kind of wave (Sec. 7.3.3); and the quickest route to the amplitude propagation law is often to express the wave's energy density U in terms of the amplitude and then invoke conservation of quanta, Eqs. (7.33).

In Ex. 7.3 we shall show that the conservation law (7.33c) is equivalent to

$$\frac{d(nC\mathcal{A})}{dt} = 0 , \quad \text{i.e.,} \quad nC\mathcal{A} \text{ is a constant along each ray.} \tag{7.34a}$$

Here $\mathcal{A}$ is the cross sectional area of a bundle of rays surrounding the ray along which the wave is propagating. Equivalently, by virtue of Eqs. (7.33a) and (7.31) for the number density of quanta in terms of the wave amplitude A,

$$\boxed{\frac{d}{dt} A\sqrt{CW\omega\mathcal{A}} = 0 \quad \text{i.e.,} \quad nCW\omega\mathcal{A} \text{ is a constant along each ray.}} \tag{7.34b}$$

<center>***************************</center>

<center>**EXERCISES**</center>

Exercise 7.3 ** *Derivation and Example: Amplitude Propagation for Dispersionless Waves Expressed as Constancy of Something Along a Ray*

(a) In connection with Eq. (7.33c), explain why $\mathbf{\nabla} \cdot \mathbf{V}_g = d\ln\mathcal{V}/dt$, where $\mathcal{V}$ is the tiny volume occupied by a collection of the wave's quanta.

(b) Choose for the collection of quanta those that occupy a cross sectional area $\mathcal{A}$ orthogonal to a chosen ray, and a longitudinal length Δs along the ray, so $\mathcal{V} = \mathcal{A}\Delta s$. Show that $d\ln\Delta s/dt = d\ln C/dt$ and correspondingly, $d\ln\mathcal{V}/dt = d\ln(C\mathcal{A})/dt$.

(c) Thence, show that the conservation law (7.33c) is equivalent to the constancy of $nC\mathcal{A}$ along a ray, Eq. (7.34a).

(d) From this, derive the constancy of $A\sqrt{CW\omega\mathcal{A}}$ along a ray (where A is the wave's amplitude), Eq. (7.34b).

Exercise 7.4 *** $\boxed{T2}$ *Example: Energy Density and Flux, and Adiabatic Invariant, for a Dispersionless Wave*

(a) Show that the scalar wave equation (7.17) for sound waves in a fluid follows from the variational principle

$$\delta \int \mathcal{L} dt d^3x = 0 , \tag{7.35a}$$

where $\mathcal{L}$ is the Lagrangian density

$$\mathcal{L} = W\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}C^2(\mathbf{\nabla}\psi)^2\right] . \tag{7.35b}$$

(not to be confused with the lengthscale $\mathcal{L}$ of inhomogeneities in the medium).

(b) For any scalar-field Lagrangian $\mathcal{L}(\dot{\psi}, \boldsymbol{\nabla}\psi, \mathbf{x}, t)$, there is a *canonical*, relativistic proce-
dure for constructing a stress-energy tensor:

$$T_\mu{}^\nu = -\frac{\partial \mathcal{L}}{\partial \psi_{,\nu}}\psi_{,\mu} + \delta_\mu{}^\nu \mathcal{L} \ . \tag{7.35c}$$

Show that, if $\mathcal{L}$ has no explicit time dependence (e.g., for the Lagrangian (7.35b)
if $C = C(\mathbf{x})$ and $W = W(\mathbf{x})$ do not depend on time t), then the field's energy is
conserved, $T^{0\nu}{}_{,\nu} = 0$. A similar calculation shows that if the Lagrangian has no
explicit space dependence (e.g., if C and W are independent of x), then the field's
momentum is conserved, $T^{j\nu}{}_{,\nu} = 0$. Here and throughout this chapter we use Cartesian
spatial coordinates, so spatial partial derivatives (denoted by commas) are the same
as covariant derivatives.

(c) Show that expression (7.35c) for the field's energy density $U = T^{00} = -T_0{}^0$ and its
energy flux $F_i = T^{0i} = -T_0{}^0$ agree with Eqs. (7.18).

(d) Now, regard the wave amplitude ψ as a generalized (field) coordinate. Use the La-
grangian $L = \int \mathcal{L}d^3x$ to define a field momentum Π conjugate to this ψ, and then
compute a *wave action*

$$J \equiv \int_0^{2\pi/\omega} \int \Pi(\partial\psi/\partial t)d^3x\,dt \ , \tag{7.35d}$$

which is the continuum analog of Eq. (7.44). The temporal integral is over one wave
period. Show that this J is proportional to the wave energy divided by the frequency
and thence to the number of quanta in the wave. [*Comment:* It is shown in standard
texts on classical mechanics that, for approximately periodic oscillations, the particle
action (7.44), with the integral limited to one period of oscillation of q, is an *adiabatic
invariant*. By the extension of that proof to continuum physics, the wave action (7.35d)
is also an adiabatic invariant. This means that the wave action (and thence also the
number of quanta in the waves) is conserved when the medium [in our case the index
of refraction $\mathfrak{n}(\mathbf{x})$] changes very slowly in time—a result asserted in the text, and a
result that also follows from quantum mechanics. We shall study the particle version
(7.44) of this adiabatic invariant in detail when we analyze charged particle motion in
a magnetic field in Chap. 19.]

Exercise 7.5 *Problem: Propagation of Sound Waves in a Wind*

Consider sound waves propagating in an atmosphere with a horizontal wind. Assume
that the sound speed C, as measured in the air's local rest frame, is constant. Let the
wind velocity $\mathbf{u} = u_x\mathbf{e}_x$ increase linearly with height z above the ground according to
$u_x = Sz$, where S is the constant shearing rate. Just consider rays in the $x - z$ plane.

(a) Give an expression for the dispersion relation $\omega = \Omega(\mathbf{x}, t; \mathbf{k})$. [Hint: in the local rest
frame of the air, Ω should have its standard sound-wave form.]

(b) Show that k_x is constant along a ray path and then demonstrate that sound waves will not propagate when

$$\left| \frac{\omega}{k_x} - u_x(z) \right| < c . \tag{7.36}$$

(c) Consider sound rays generated on the ground which make an angle θ to the horizontal initially. Derive the equations describing the rays and use them to sketch the rays distinguishing values of θ both less than and greater than $\pi/2$. (You might like to perform this exercise numerically.)

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7.3.3   Geometric Optics for a General Wave

With the simple case of non-dispersive sound waves (previous two subsections) as our model, we now study an arbitrary kind of wave in a weakly inhomogeneous and slowly time varying medium — e.g. any of the examples in Sec. 7.2.1: light waves in a dielectric medium, deep water waves, flexural waves on a stiff beam, or Alfvén waves. Whatever may be the wave, we seek a solution to its wave equation using the eikonal approximation $\psi = Ae^{i\varphi}$ with slowly varying amplitude A and rapidly varying phase φ. Depending on the nature of the wave, ψ and A might be a scalar (e.g. sound waves), a vector (e.g. light waves), or a tensor (e.g. gravitational waves).

When we insert the ansatz $\psi = Ae^{i\varphi}$ into the wave equation and collect terms in the manner dictated by our two lengthscale expansion [as in Eq. (7.22) and Box 7.2], the leading order term will arise from letting every temporal or spatial derivative act on the $e^{i\varphi}$. This is precisely where the derivatives would operate in the case of a plane wave in a homogeneous medium, and here as there the result of each differentiation is $\partial e^{i\varphi}/\partial t = -\omega e^{i\varphi}$ or $\partial e^{i\varphi}/\partial x_j = k_j e^{i\varphi}$. Correspondingly, *the leading order terms in the wave equation here will be identical to those in the homogeneous plane wave case: they will be the dispersion relation* multiplied by something times the wave,

$$[-\omega^2 + \Omega^2(\mathbf{x}, t; \mathbf{k})] \times (\text{something}) Ae^{i\varphi} = 0 , \tag{7.37}$$

with the spatial and temporal dependence in Ω^2 entering through the medium's properties. This guarantees that (as we claimed in Sec. 7.3.1) *the dispersion relation can be obtained by analyzing a plane, monochromatic wave in a homogeneous, time-independent medium and then letting the medium's properties, in the dispersion relation, vary slowly with $\mathbf{x}$ and t.*

Each next-order ("subleading") term in the wave equation will entail just one of the wave operator's derivatives acting on a slowly-varying quantity (A or a medium property or ω or $\mathbf{k}$), and all the other derivatives acting on $e^{i\varphi}$. The subleading terms that interest us, for the moment, are those where the one derivative acts on A thereby propagating it. The subleading terms, therefore, can be deduced from the leading-order terms (7.37) by replacing just one $\omega Ae^{i\varphi} = -A(e^{i\varphi})_{,t}$ by $-A_{,t}e^{i\varphi}$, and replacing just one $k_j Ae^{i\varphi} = A(e^{i\varphi})_{,j}$ by $A_{,j}e^{i\varphi}$.

A little thought then reveals that the equation for the vanishing of the subleading terms must take the form [deducible from the leading terms (7.37)]

$$-2i\omega\frac{\partial A}{\partial t} - 2i\Omega(\mathbf{k}, \mathbf{x}, t)\frac{\partial\Omega(\mathbf{k}, \mathbf{x}, t)}{\partial k_j}\frac{\partial A}{\partial x_j} = \text{terms proportional to } A . \qquad (7.38)$$

Using the dispersion relation $\omega = \Omega(\mathbf{x}, t; \mathbf{k})$ and the group velocity (first Hamilton equation) $\partial\Omega/\partial k_j = V_{g\,j}$, we bring this into the "propagate A along a ray" form

$$\frac{dA}{dt} \equiv \frac{\partial A}{\partial t} + \mathbf{V}_g \cdot \boldsymbol{\nabla} A = \text{terms proportional to } A . \qquad (7.39)$$

Let us return to the leading order terms (7.37) in the wave equation, i.e. to the dispersion relation $\omega = \Omega(\mathbf{x}, t; k)$. For our general wave, as for the dispersionless sound wave of the previous two sections, the argument embodied in Eqs. (7.26) shows that *the rays are determined by Hamilton's equations* (7.25),

$$\boxed{\frac{dx_j}{dt} = \left(\frac{\partial\Omega}{\partial k_j}\right)_{\mathbf{x},t} \equiv V_{g\,j}} , \quad \boxed{\frac{dk_j}{dt} = -\left(\frac{\partial\Omega}{\partial x_j}\right)_{\mathbf{k},t}} , \quad \boxed{\frac{d\omega}{dt} = \left(\frac{\partial\Omega}{\partial t}\right)_{\mathbf{x},\mathbf{k}}} , \qquad (7.40)$$

but using the general wave's dispersion relation $\Omega(\mathbf{k}, \mathbf{x}, t)$ rather than the light-and-sound dispersion relation $\Omega = Ck$. *These Hamilton equations include propagation laws for* $\omega = -\partial\varphi/\partial t$ *and* $k_j = \partial\varphi/\partial x_j$, *from which we can deduce the propagation law (7.27) for* φ *along the rays*

$$\boxed{\frac{d\varphi}{dt} = -\omega + \mathbf{V}_g \cdot \mathbf{k}} . \qquad (7.41)$$

For waves with dispersion, by contrast with sound in a fluid and other waves that have $\Omega = Ck$, φ *will not be constant along a ray.*

For our general wave, as for sound, *the Hamilton equations for the rays can be reinterpreted as Hamilton's equations for the world lines of the waves' quanta* [Eq. (7.30) and associated discussion]. And *for our general wave, as for sound, the medium's slow variations are incapable of creating or destroying wave quanta.* [This is a general feature of quantum theory; creation and destruction of quanta require imposed oscillations at the high frequency and short wavelength of the waves themselves, or at some multiple or submultiple of them (in the case of nonlinear creation and annihilation processes; Chap. 10).] Correspondingly, if one knows the relationship between the waves' energy density U and their amplitude A, and thence the relationship between the waves' quantum number density $n = U/\hbar\omega$ and A, then *from the quantum conservation law*

$$\boxed{\partial n/\partial t + \nabla \cdot (n\mathbf{V}_g) = 0} \qquad (7.42)$$

one can deduce the propagation law for A — *and the result must be the same propagation law as one obtains from the subleading terms in the eikonal approximation.*

Because light waves in a dielectric medium have the same dispersion relation $\Omega = Ck$ as sound, Hamilton's equations will take the same form and have the same consequences as for sound. However, light is a vectorial phenomenon, and the propagation law for its vectorial amplitude differs from that for the scalar amplitude of a sound wave; see Sec. YYYYYY.

7.3.4 Examples of Geometric-Optics Wave Propagation

Spherical scalar waves.

As a simple example of these geometric-optics propagation laws, consider a scalar wave propagating radially outward from the origin at the speed of light in flat spacetime. Setting the speed of light to unity, the dispersion relation is Eq. (7.4) with $C = 1$: $\Omega = k$. It is straightforward (Ex 7.6) to integrate Hamilton's equations and learn that the rays have the simple form $\{r = t + \text{constant}, \theta = \text{constant}, \phi = \text{constant}, \mathbf{k} = \omega \mathbf{e}_r\}$ in spherical polar coordinates, with $\mathbf{e}_r$ the unit radial vector. Since the dispersion relation is the same as for light and sound, we know that the waves' phase φ must be conserved along a ray, i.e. it must be a function of $t - r, \theta, \phi$. In order that the waves propagate radially, it is essential that $\mathbf{k} = \boldsymbol{\nabla}\varphi$ point very nearly radially; this implies that φ must be a rapidly varying function of $t - r$ and a slowly varying function of (θ, ϕ). The law of conservation of quanta in this case reduces to the propagation law $d(rA)/dt = 0$ (Ex. 7.6) so rA is also a constant along the ray; we shall call it $\mathcal{B}$. Putting this all together, we conclude that

$$\psi = \frac{\mathcal{B}(t - r, \theta, \phi)}{r} e^{i\varphi(t - r, \theta, \phi)} \ , \tag{7.43}$$

where the phase is rapidly varying in $t - r$ and slowly varying in the angles, and the amplitude is slowly varying in $t - r$ and the angles.

Flexural waves.

As another example of the geometric-optics propagation laws, consider flexural waves on a spacecraft's tapering antenna. The dispersion relation is $\Omega = k^2\sqrt{D/\Lambda}$ [Eq. (7.6)] with $D/\Lambda \propto d^2$, where d is the antenna diameter (cf. Chaps. 11 and 12). Since Ω is independent of t, as the waves propagate from the spacecraft to the antenna's tip, their frequency ω is conserved [third of Eqs. (7.40)], which implies by the dispersion relation that $k = (D/\Lambda)^{-1/4}\omega^{1/2} \propto d^{-1/2}$, whence the wavelength decreases as $d^{1/2}$. The group velocity is $V_g = 2(D/\Lambda)^{1/4}\omega^{1/2} \propto d^{1/2}$. Since the energy per quantum $\hbar\omega$ is constant, particle conservation implies that the waves' energy must be conserved, which in this one-dimensional problem, means that the energy flux must be constant along the antenna. On physical grounds the constant energy flux must be proportional to $A^2 V_g$, which means that the amplitude A must increase $\propto d^{-1/4}$ as the flexural waves approach the antenna's end. A qualitatively similar phenomenon is seen in the "cracking" of a bullwhip.

Light through lens, and Alfvén waves

Fig. 7.3 sketches two other examples: light propagating through a lens, and Alfvén waves propagating in the magnetosphere of a planet. In Sec. 7.3.6 and the exercises we shall explore a variety of other applications, but first we shall describe how the geometric-optics propagation laws can fail (Sec. 7.3.5).

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EXERCISES

Exercise 7.6 *Derivation and Practice: Quasi-Spherical Solution to Vacuum Scalar Wave Equation*
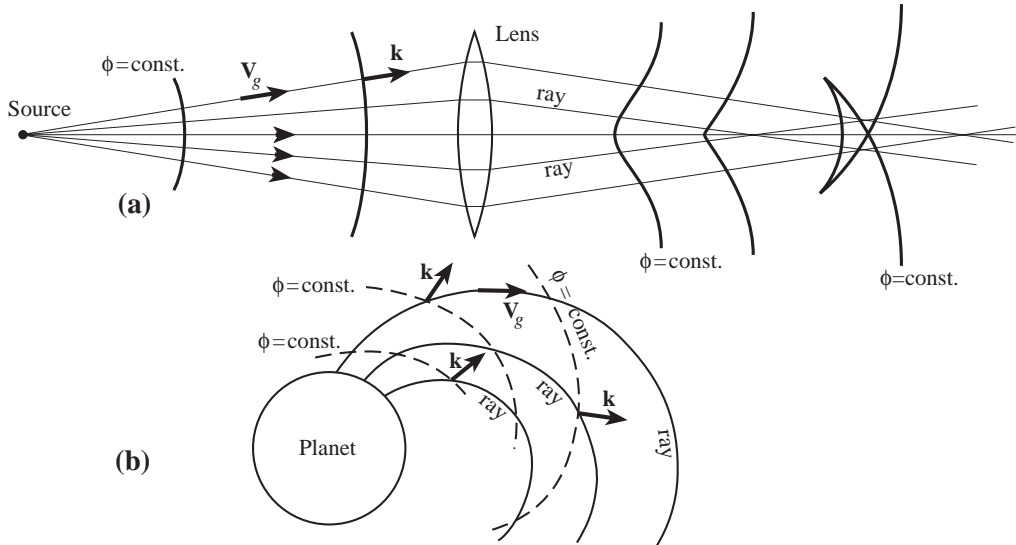
Fig. 7.3: (a) The rays and the surfaces of constant phase φ at a fixed time for light passing through a converging lens [dispersion relation $\Omega = ck/n(\mathbf{x})$]. In this case the rays (which always point along $\mathbf{V}_g$) are parallel to the wave vector $\mathbf{k} = \boldsymbol{\nabla}\varphi$ and thus also parallel to the phase velocity $\mathbf{V}_{\mathrm{ph}}$, and the waves propagate along the rays with a speed $V_g = V_{\mathrm{ph}} = c/\mathfrak{n}$ that is independent of wavelength. The strange self-intersecting shape of the last phase front is due to caustics; see Sec. 7.5.2. (b) The rays and surfaces of constant phase for Alfvén waves in the magnetosphere of a planet [dispersion relation $\Omega = \mathbf{a}(\mathbf{x}) \cdot \mathbf{k}$]. In this case because $\mathbf{V}_g = \mathbf{a} \equiv \mathbf{B}/\sqrt{4\pi\rho}$, the rays are parallel to the magnetic field lines and not parallel to the wave vector, and the waves propagate along the field lines with speeds $V_g = B/\sqrt{4\pi\rho}$ that are independent of wavelength; cf. Fig. 7.2b. As a consequence, if some electric discharge excites Alfvén waves on the planetary surface, then they will be observable by a spacecraft when it passes magnetic field lines on which the discharge occurred. As the waves propagate, because $\mathbf{B}$ and ρ are time independent and thence $\partial\Omega/\partial t = 0$, the frequency and energy of each quantum is conserved, and conservation of quanta implies conservation of wave energy. Because the Alfvén speed generally diminishes with distance from the planet, conservation of wave energy typically requires the waves' energy density and amplitude to increase as they climb upward.

Derive the quasi-spherical solution (7.43) of the vacuum scalar wave equation $-\partial^2\psi/\partial t^2 + \nabla^2\psi = 0$ from the geometric optics laws by the procedure sketched in the text.

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7.3.5 Relation to Wave Packets; Breakdown of the Eikonal Approximation and Geometric Optics

The form $\psi = Ae^{i\varphi}$ of the waves in the eikonal approximation is remarkably general. At some initial moment of time, A and φ can have any form whatsoever, so long as the two-lengthscale constraints are satisfied [A, $\omega \equiv -\partial\varphi/\partial t$, $\mathbf{k} \equiv \boldsymbol{\nabla}\varphi$, and dispersion relation $\Omega(\mathbf{k}; \mathbf{x}, t)$ all vary on lengthscales long compared to $\lambda = 1/k$ and timescales long compared

to $1/\omega$]. For example, ψ could be as nearly planar as is allowed by the inhomogeneities of the dispersion relation. At the other extreme, ψ could be a moderately narrow wave packet, confined initially to a small region of space (though not too small; its size must be large compared to its mean reduced wavelength). In either case, the evolution will be governed by the above propagation laws.

Of course, the eikonal approximation *is* an approximation. Its propagation laws make errors, though when the two-lengthscale constraints are well satisfied, the errors will be small for sufficiently short propagation times. Wave packets provide an important example. *Dispersion (different group velocities for different wave vectors) causes wave packets to spread (widen; disperse) as they propagate; see Ex. 7.2. This spreading is not included in the geometric optics propagation laws;* it is a fundamentally wave-based phenomenon and is lost when one goes to the particle-motion regime. In the limit that the wave packet becomes very large compared to its wavelength or that the packet propagates for only a short time, the spreading is small (Ex. 7.2). This is the geometric-optics regime, and geometric optics ignores the spreading.

Many other wave phenomena are missed by geometric optics. Examples are *diffraction*, e.g. at a geometric-optics caustic (Chap. 8; Sec. 7.5.2), *nonlinear wave-wave coupling* (Chaps. 10, 16 and 22), and *parametric amplification* of waves by rapid time variations of the medium (Chap. 10)—which shows up in quantum mechanics as *particle production* (i.e., a breakdown of the law of conservation of quanta). In Chap. 27, we shall study such particle production in inflationary models of the early universe.

7.3.6   Fermat's Principle

The Hamilton equations of optics allow us to solve for the paths of rays in media that vary both spatially and temporally. When the medium is time independent, the rays $\mathbf{x}(t)$ can be computed from a variational principle named after Fermat. This *Fermat's principle* is the optical analogue of Maupertuis' principle of least action in classical mechanics.[6] In classical mechanics, this principle states that, when a particle moves from one point to another through a time-independent potential (so its energy, the Hamiltonian, is conserved), then the path $\mathbf{q}(t)$ that it follows is one that extremizes the action

$$J = \int \mathbf{p} \cdot d\mathbf{q} \,, \tag{7.44}$$

(where $\mathbf{q}$, $\mathbf{p}$ are the particle's generalized coordinates and momentum), subject to the constraint that the paths have a fixed starting point, a fixed endpoint, and constant energy. The proof, which can be found in any text on analytical dynamics, carries over directly to optics when we replace the Hamiltonian by Ω, $\mathbf{q}$ by $\mathbf{x}$, and $\mathbf{p}$ by $\mathbf{k}$. The resulting Fermat principle, stated with some care, has the following form:

Consider waves whose Hamiltonian $\Omega(\mathbf{k}, \mathbf{x})$ is independent of time. Choose an initial location $\mathbf{x}_{\text{initial}}$ and a final location $\mathbf{x}_{\text{final}}$ in space, and ask what are the rays $\mathbf{x}(t)$ that connect these two points. The rays (usually only one) are those paths that satisfy the variational

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[6]Goldstein (1980).

principle

$$\delta \int \mathbf{k} \cdot d\mathbf{x} = 0 \ . \tag{7.45}$$

In this variational principle, $\mathbf{k}$ *must be expressed in terms of the trial path* $\mathbf{x}(t)$ *using Hamilton's equation* $dx^j/dt = -\partial\Omega/\partial k_j$; *the rate that the trial path is traversed (i.e., the magnitude of the group velocity) must be adjusted so as to keep* Ω *constant along the trial path* (which means that the total time taken to go from $\mathbf{x}_{\text{initial}}$ to $\mathbf{x}_{\text{final}}$ can differ from one trial path to another); and, of course, the trial paths must all begin at $\mathbf{x}_{\text{initial}}$ and end at $\mathbf{x}_{\text{final}}$.

Notice that, once a ray has been identified via this action principle, it has $\mathbf{k} = \boldsymbol{\nabla}\varphi$, and therefore the extremal value of the action $\int \mathbf{k} \cdot d\mathbf{x}$ along the ray is equal to the waves' phase difference $\Delta\varphi$ between $\mathbf{x}_{\text{initial}}$ and $\mathbf{x}_{\text{final}}$. Correspondingly, for any trial path we can think of the action as a phase difference along that path and *we can think of the action principle as one of extremal phase difference* $\Delta\varphi$. This can be reexpressed in a form closely related to **Feynman's path-integral formulation of quantum mechanics:** We can regard *all* the trial paths as being followed with equal probability; for each path we are to construct a probability amplitude $e^{i\Delta\varphi}$; and we must then add together these amplitudes

$$\sum_{\text{all paths}} e^{i\Delta\varphi} \tag{7.46}$$

to get the net complex amplitude for quanta associated with the waves to travel from $\mathbf{x}_{\text{initial}}$ to $\mathbf{x}_{\text{final}}$. The contributions from almost all neighboring paths will interfere destructively. The only exceptions are those paths whose neighbors have the same values of $\Delta\varphi$, to first order in the path difference. These are the paths that extremize the action (7.45); i.e., they are the wave's rays, the actual paths of the quanta.

Specialization to $\Omega = C(\mathbf{x})k$

Fermat's principle takes on an especially simple form when not only is the Hamiltonian $\Omega(\mathbf{k}, \mathbf{x})$ time independent, but it also has the simple dispersion-free form $\Omega = C(\mathbf{x})k$ — a form valid for propagation of light through a time-independent dielectric, and sound waves through a time-independent, inhomogeneous fluid, and electromagnetic or gravitational waves through a time-independent, Newtonian gravitational field. In this case, the Hamiltonian dictates that for each trial path, $\mathbf{k}$ is parallel to $d\mathbf{x}$, and therefore $\mathbf{k} \cdot d\mathbf{x} = kds$, where s is distance along the path. Using the dispersion relation $k = \Omega/C$ and noting that Hamilton's equation $dx^j/dt = \partial\Omega/\partial k_j$ implies $ds/dt = C$ for the rate of traversal of the trial path, we see that $\mathbf{k} \cdot d\mathbf{x} = kds = \Omega dt$. Since the trial paths are constrained to have Ω constant, *Fermat's principle (7.45) becomes a principle of extremal time: The rays between* $\mathbf{x}_{\text{initial}}$ *and* $\mathbf{x}_{\text{final}}$ *are those paths along which*

$$\int dt = \int \frac{ds}{C(\mathbf{x})} = \int \frac{\mathfrak{n}(\mathbf{x})}{c} ds \tag{7.47}$$

is extremal. In the last expression we have adopted the convention used for light in a dielectric medium, that $C(\mathbf{x}) = c/\mathfrak{n}(\mathbf{x})$, where c is the speed of light in vacuum and $\mathfrak{n}$ is the medium's
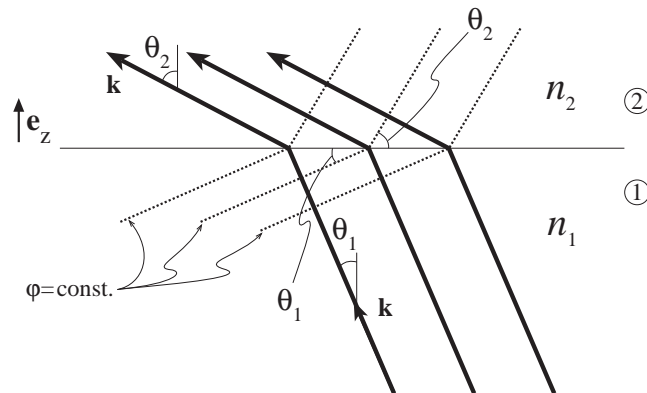
Fig. 7.4: Illustration of Snell's law of refraction at the interface between two media where the refractive indices are $\mathfrak{n}_1$ and $\mathfrak{n}_2$ (assumed less than $\mathfrak{n}_1$). As the wavefronts must be continuous across the interface, and the wavelengths are inversely proportional to the refractive index, we have from simple trigonometry that $\mathfrak{n}_1 \sin\theta_1 = \mathfrak{n}_2 \sin\theta_2$.

index of refraction. Since c is constant, *the rays are paths of extremal optical path length* $\int \mathfrak{n}(\mathbf{x})ds$.

We can use Fermat's principle to demonstrate that, if the medium contains no opaque objects, then there will always be at least one ray connecting any two points. This is because there is a lower bound on the optical path between any two points given by $\mathfrak{n}_{\min}L$, where $\mathfrak{n}_{\min}$ is the lowest value of the refractive index anywhere in the medium and L is the distance between the two points. This means that for some path the optical path length must be a minimum, and that path is then a ray connecting the two points.

From the principle of extremal time, we can derive the Euler-Lagrange differential equation for the ray. For ease of derivation, we write the action principle in the form

$$\delta \int \mathfrak{n}(\mathbf{x})\sqrt{\frac{d\mathbf{x}}{d\mathbf{s}} \cdot \frac{d\mathbf{x}}{d\mathbf{s}}} \, ds, \tag{7.48}$$

where the quantity in the square root is identically one. Performing a variation in the usual manner then gives

$$\boxed{\frac{d}{ds}\left(\mathfrak{n}\frac{d\mathbf{x}}{ds}\right) = \boldsymbol{\nabla}\mathfrak{n} \,, \quad \text{i.e.} \quad \frac{d}{ds}\left(\frac{1}{C}\frac{d\mathbf{x}}{ds}\right) = \boldsymbol{\nabla}\left(\frac{1}{C}\right).} \tag{7.49}$$

This is equivalent to Hamilton's equations for the ray, as one can readily verify using the Hamiltonian $\Omega = kc/\mathfrak{n}$ (Ex. 7.7).

Equation (7.49) is a second-order differential equation requiring two boundary conditions to define a solution. We can either choose these to be the location of the start of the ray and its starting direction, or the start and end of the ray. A simple case arises *when the medium is stratified*, i.e. when $\mathfrak{n} = \mathfrak{n}(z)$, where (x, y, z) are Cartesian coordinates. Projecting Eq. (7.49) perpendicular to $\mathbf{e}_z$, we discover that $\mathfrak{n}dy/ds$ and $\mathfrak{n}dx/ds$ are constant, which implies

$$\boxed{\mathfrak{n}\sin\theta = \text{constant}} \tag{7.50}$$

where θ is the angle between the ray and $\mathbf{e}_z$. This is Snell's law of refraction. Snell's law is just a mathematical statement that the rays are normal to surfaces (wavefronts) on which the eikonal φ is constant (cf. Fig. 7.4).

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<div align="center">**EXERCISES**</div>

Exercise 7.7 *Derivation: Hamilton's Equations for Dispersionless Waves; Fermat's Principle*

Show that Hamilton's equations for the standard dispersionless dispersion relation (7.4) imply the same ray equation (7.49) as we derived using Fermat's principle.

Exercise 7.8 *Example: Self-Focusing Optical Fibers*

Optical fibers in which the refractive index varies with radius are commonly used to transport optical signals. Provided that the diameter of the fiber is many wavelengths, we can use geometric optics. Let the refractive index be

$$\mathfrak{n} = \mathfrak{n}_0(1 - \alpha^2 r^2)^{1/2} \tag{7.51a}$$

where n_0 and α are constants and r is radial distance from the fiber's axis.

(a) Consider a ray that leaves the axis of the fiber along a direction that makes a small angle θ to the axis. Solve the ray transport equation (7.49) to show that the radius of the ray is given by

$$r = \frac{\sin\theta}{\alpha}\left|\sin\left(\frac{\alpha z}{\cos\theta}\right)\right| \tag{7.51b}$$

where z measures distance along the fiber.

(b) Next consider the propagation time T for a light pulse propagating along a long length L of fiber. Show that

$$T = \frac{\mathfrak{n}_0 L}{c}[1 + O(\theta^4)] \tag{7.51c}$$

and comment on the implications of this result for the use of fiber optics for communication.

Exercise 7.9 *** *Example: Geometric Optics for the Schrödinger Equation*
Consider the non-relativistic Schrödinger equation for a particle moving in a time-dependent, 3-dimensional potential well.

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2 + V(\mathbf{x}, t)\right]\psi . \tag{7.52}$$

(a) Seek a geometric optics solution to this equation with the form $\psi = Ae^{iS/\hbar}$, where A and V are assumed to vary on a lengthscale $\mathcal{L}$ and timescale $\mathcal{T}$ long compared to those, $1/k$ and $1/\omega$, on which S varies. Show that the leading order terms in the two-lengthscale expansion of the Schrödinger equation give the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla} S)^2 + V = 0 . \tag{7.53a}$$

Our notation $\varphi \equiv S/\hbar$ for the phase φ of the wave function ψ is motivated by the fact that the geometric-optics limit of quantum mechanics is classical mechanics, and the function $S = \hbar\varphi$ becomes, in that limit, "Hamilton's principal function," which obeys the Hamilton-Jacobi equation.[7]

(b) From this equation derive the equation of motion for the rays (which of course is identical to the equation of motion for a wave packet and therefore is also the equation of motion for a classical particle):

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{m} , \quad \frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla} V , \tag{7.53b}$$

where $\mathbf{p} = \boldsymbol{\nabla} S$.

(c) Derive the propagation equation for the wave amplitude A and show that it implies

$$\frac{d|A|^2}{dt} + |A|^2 \frac{\boldsymbol{\nabla} \cdot \mathbf{p}}{m} = 0 \tag{7.53c}$$

Interpret this equation quantum mechanically.

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7.4 Paraxial Optics

It is quite common in optics to be concerned with a bundle of rays that are almost parallel. This implies that the angle that the rays make with some reference ray can be treated as small—an approximation that underlies the first order theory of simple optical instruments like the telescope and the microscope. This approximation is called *paraxial optics*, and it permits one to linearize the geometric optics equations and use matrix methods to trace their rays.

We shall develop the paraxial optics formalism for waves whose dispersion relation has the simple, time-independent, nondispersive form $\Omega = kc/\mathfrak{n}(\mathbf{x})$. This applies to light in a dielectric medium — the usual application. As we shall see below, it also applies to charged particles in a storage ring (Sec. 7.4.2) and to light being lensed by a weak gravitational field (Sec. 7.5).

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[7]See, e.g., Chap. 10 of Goldstein (1980).

We start by linearizing the ray propagation equation (7.49). Let z measure distance along a reference ray. Let the two dimensional vector $\mathbf{x}(z)$ be the transverse displacement of some other ray from this reference ray, and denote by $(x, y) = (x_1, x_2)$ the Cartesian components of $\mathbf{x}$, with the transverse Cartesian basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ transported parallely along the reference ray.[8] Under paraxial conditions, $|\mathbf{x}|$ is small compared to the z-lengthscales of the propagation. Now, let us Taylor expand the refractive index, $\mathfrak{n}(\mathbf{x}, z)$

$$\mathfrak{n}(\mathbf{x}, z) = \mathfrak{n}(0, z) + x_i \mathfrak{n}_{,i}(0, z) + \frac{1}{2} x_i x_j \mathfrak{n}_{,ij}(0, z) + \dots , \tag{7.54a}$$

where the subscript commas denote partial derivatives with respect to the transverse coordinates, $\mathfrak{n}_{,i} \equiv \partial \mathfrak{n} / \partial x_i$. The linearized form of Eq. (7.49) is then given by

$$\frac{d}{dz}\left(\mathfrak{n}(0, z)\frac{dx_i}{dz}\right) = \mathfrak{n}_{,i}(0, z) + x_j \mathfrak{n}_{,ij}(0, z) . \tag{7.54b}$$

We are usually concerned with aligned optical systems in which there is a particular choice of reference ray called the *optic axis*, for which the term $\mathfrak{n}_{,i}(0, z)$ on the right hand side of Eq. (7.54b) vanishes. If we choose the reference ray to be the optic axis, then Eq. (7.54b) is a linear, homogeneous, second-order equation for $\mathbf{x}(z)$,

$$\boxed{(d/dz)(\mathfrak{n}\, dx_i/dz) = x_j \mathfrak{n}_{,ij}} \tag{7.55}$$

Here $\mathfrak{n}$ and $\mathfrak{n}_{,ij}$ are evaluated on the reference ray. It is helpful to regard z as "time" and think of Eq. (7.55) as an equation for the two dimensional motion of a particle (the ray) in a quadratic potential well. We can solve Eq. (7.55) given starting values $\mathbf{x}(z')$, $\dot{\mathbf{x}}(z')$ where the dot denotes differentiation with respect to z, and z' is the starting location. The solution at some point z is linearly related to the starting values. We can capitalize on this linearity by treating $\{\mathbf{x}(z), \dot{\mathbf{x}}(z)\}$ as a 4 dimensional vector $V_i(z)$, with

$$\boxed{V_1 = x, \quad V_2 = \dot{x}, \quad V_3 = y, \quad V_4 = \dot{y},} \tag{7.56}$$

and embodying the linear transformation from location z' to location z in a *transfer matrix* $J_{ab}(z, z')$:

$$\boxed{V_a(z) = J_{ab}(z, z') \cdot V_b(z').} \tag{7.57}$$

The transfer matrix contains full information about the change of position and direction of all rays that propagate from z' to z. As is always the case for linear systems, the transfer matrix for propagation over a large interval, from z' to z, can be written as the product of the matrices for two subintervals, from z' to z'' and from z'' to z:

$$\boxed{J_{ac}(z, z') = J_{ab}(z, z'') J_{bc}(z'', z').} \tag{7.58}$$

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[8]By parallel transport, we mean this: the basis vector $\mathbf{e}_x$ is carried a short distance along the reference ray, keeping it parallel to itself. Then, if the reference ray has bent a bit, $\mathbf{e}_x$ is projected into the new plane that is transverse to the ray.

7.4.1 Axisymmetric, Paraxial Systems

If the index of refraction is everywhere axisymmetric, so $\mathfrak{n} = \mathfrak{n}(\sqrt{x^2 + y^2}, z)$, then there is *no coupling between the motions of rays along the x and y directions, and the equations of motion along x are identical to those along y.* In other words, $J_{11} = J_{33}$, $J_{12} = J_{34}$, $J_{21} = J_{43}$, and $J_{22} = J_{44}$ are the only nonzero components of the transfer matrix. This reduces the dimensionality of the propagation problem from 4 dimensions to 2: V_a can be regarded as either $\{x(z), \dot{x}(z)\}$ or $\{y(z), \dot{y}(z)\}$, and in both cases the 2×2 transfer matrix J_{ab} is the same.

Let us illustrate the paraxial formalism by deriving the transfer matrices of a few simple, axisymmetric optical elements. In our derivations it is helpful conceptually to focus on rays that move in the x-z plane, i.e. that have $y = \dot{y} = 0$. We shall write the 2-dimensional V_i as a column vector

$$V_a = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{7.59}$$

The simplest case is a straight section of length d extending from z' to $z = z' + d$. The components of V will change according to

$$\begin{aligned} x &= x' + \dot{x}'d \\ \dot{x} &= \dot{x}' \end{aligned} \tag{7.60}$$

so

$$J_{ab} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \text{ for straight section of length } d, \tag{7.61}$$

where $x' = x(z')$ etc. Next, consider a thin lens with focal length f. The usual convention in optics is to *give f a positive sign when the lens is converging and a negative sign when diverging.* A thin lens gives a deflection to the ray that is linearly proportional to its displacement from the optic axis, but does not change its transverse location. Correspondingly, the transfer matrix in crossing the lens (ignoring its thickness) is:

$$J_{ab} = \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix} \text{ for thin lens with focal length } f. \tag{7.62}$$

Similarly, a spherical mirror with radius of curvature R (again *adopting a positive sign for a converging mirror and a negative sign for a diverging mirror*) has a transfer matrix

$$J_{ab} = \begin{pmatrix} 1 & 0 \\ -2R^{-1} & 1 \end{pmatrix} \text{ for spherical mirror with radius of curvature } R. \tag{7.63}$$

As a simple illustration let us consider rays that leave a point source which is located a distance u in front of a converging lens of focal length f and solve for the ray positions a distance v behind the lens (Fig. 7.5). The total transfer matrix is the product of the transfer
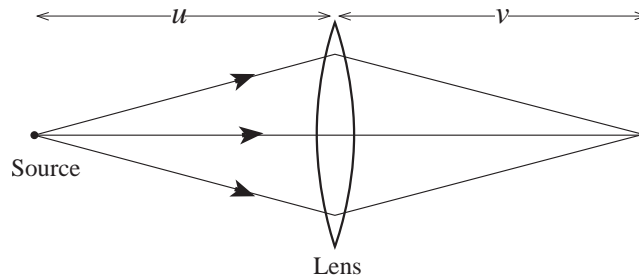
Fig. 7.5: Simple converging lens used to illustrate the use of transfer matrices. The total transfer matrix is formed by taking the product of the straight section transfer matrix with the lens matrix and another straight section matrix.

matrix for a straight section, Eq. (7.61) with the product of the lens transfer matrix and a second straight-section transfer matrix:

$$J_{ab} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - vf^{-1} & u + v - uvf^{-1} \\ -f^{-1} & 1 - uf^{-1} \end{pmatrix} \tag{7.64}$$

When the 1-2 element (upper right entry) of this transfer matrix vanishes, the position of the ray after traversing the optical system is independent of the starting direction. In other words, *rays from the point source form a point image; when this happens, the planes containing the source and the image are said be conjugate.* The condition for this to occur is

$$\boxed{\frac{1}{u} + \frac{1}{v} = \frac{1}{f}} . \tag{7.65}$$

This is the standard thin lens equation. *The linear magnification of the image* is given by $M = J_{11} = 1 - v/f$, i.e.

$$\boxed{M = -\frac{v}{u}} , \tag{7.66}$$

where the negative sign means that the image is inverted. Note that, if a ray is reversed in direction, it remains a ray, but with the source and image planes interchanged; u and v are exchanged, Eq. (7.65) is unaffected, and the magnification (7.66) is inverted: $M \to 1/M$.

7.4.2   Converging Magnetic Lens

Since geometric optics is the same as particle dynamics, matrix equations can be used to describe paraxial motions of electrons and ions in a storage ring. (Note, however, that the Hamiltonian for such particles is dispersive, since the Hamiltonian does not depend linearly on the particle momentum, and so for our simple matrix formalism to be valid, we must confine attention to a *mono-energetic beam.*)

The simplest practical lens for charged particles is a quadrupolar magnet. Quadrupolar magnetic fields are used to guide particles around storage rings. If we orient our axes appropriately, the magnet's magnetic field can be expressed in the form

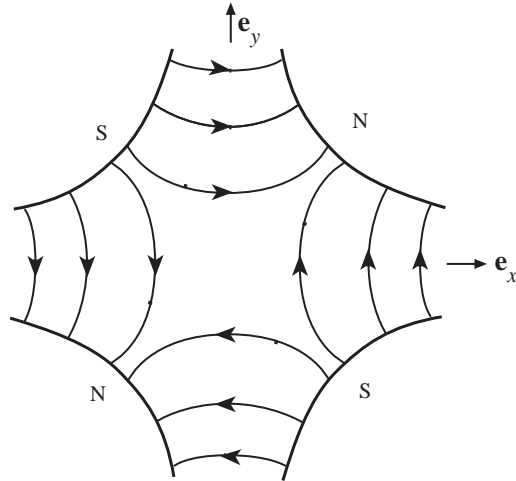$$\mathbf{B} = \frac{B_0}{r_0}(y\mathbf{e}_x + x\mathbf{e}_y) \tag{7.67}$$

Fig. 7.6: Quadrupolar Magnetic Lens. The magnetic field lines lie in a plane perpendicular to the optic axis. Positively charged particles moving along $\mathbf{e}_z$ are converged when $y = 0$ and diverged when $x = 0$.

[Fig. 7.6]. Particles traversing this magnetic field will be subjected to a Lorentz force which will curve their trajectories. In the paraxial approximation, a particle's coordinates will satisfy the two differential equations

$$\ddot{x} = -\frac{x}{\lambda^2}\,, \quad \ddot{y} = \frac{y}{\lambda^2}\,, \tag{7.68a}$$

where the dots (as above) mean $d/dz = v^{-1}d/dt$ and

$$\lambda = \left(\frac{pr_0}{qB_0}\right)^{1/2} \tag{7.68b}$$

with q the particle's charge (assumed positive) and p its momentum. The motions in the x and y directions are decoupled. It is convenient in this case to work with two 2-dimensional vectors, $\{V_{x1}, V_{x2}\} \equiv \{x, \dot{x}\}$ and $\{V_{y1}, V_{y2}\} = \{y, \dot{y}\}$. From the elementary solutions to the equations of motion (7.68a), we infer that the transfer matrices from the magnet's entrance to its exit are $J_{x\,ab}, J_{y\,ab}$, where

$$J_{x\,ab} = \begin{pmatrix} \cos\phi & \lambda\sin\phi \\ -\lambda^{-1}\sin\phi & \cos\phi \end{pmatrix} \tag{7.69a}$$

$$J_{y\,ab} = \begin{pmatrix} \cosh\phi & \lambda\sinh\phi \\ \lambda^{-1}\sinh\phi & \cosh\phi \end{pmatrix} \tag{7.69b}$$

and

$$\phi = L/\lambda \tag{7.69c}$$

with L the distance from entrance to exit.

The matrices $J_{x\,ab}, J_{y\,ab}$ can be decomposed as follows

$$J_{x\,ab} = \begin{pmatrix} 1 & \lambda\tan\phi/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\sin\phi/\lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda\tan\phi/2 \\ 0 & 1 \end{pmatrix} \tag{7.69d}$$

$$J_{y\,ab} = \begin{pmatrix} 1 & \lambda\tanh\phi/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \sinh\phi/\lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & \lambda\tanh\phi/2 \\ 0 & 1 \end{pmatrix} \tag{7.69e}$$

Comparing with Eqs. (7.61), (7.62), we see that *the action of a single magnet is equivalent to the action of a straight section, followed by a thin lens, followed by another straight section.* Unfortunately, if the lens is focusing in the x direction, it must be de-focusing in the y direction and *vice versa.* However, *we can construct a lens that is focusing along both directions by combining two magnets that have opposite polarity but the same focusing strength $\phi = L/\lambda$:*

Consider the motion in the x direction first. Let $f_+ = \lambda/\sin\phi$ be the equivalent focal length of the first converging lens and $f_- = -\lambda/\sinh\phi$ that of the second diverging lens. If we separate the magnets by a distance s, this must be added to the two effective lengths of the two magnets to give an equivalent separation, $d = \lambda\tan(\phi/2) + s + \lambda\tanh(\phi/2)$ for the two equivalent thin lenses. The combined transfer matrix for the two thin lenses separated by this distance d is then

$$\begin{pmatrix} 1 & 0 \\ -f_-^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -f_+^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 - df_+^{-1} & d \\ -f_*^{-1} & 1 - df_-^{-1} \end{pmatrix} \tag{7.70a}$$

where

$$\frac{1}{f_*} = \frac{1}{f_-} + \frac{1}{f_+} - \frac{d}{f_-f_+} = \frac{\sin\phi}{\lambda} - \frac{\sinh\phi}{\lambda} + \frac{d\sin\phi\sinh\phi}{\lambda^2} . \tag{7.70b}$$

If we assume that $\phi \ll 1$ and $s \ll L$, then we can expand as a Taylor series in ϕ to obtain

$$f_* \simeq \frac{3\lambda}{2\phi^3} = \frac{3\lambda^4}{2L^3} . \tag{7.71}$$

The effective focal length f_* of the combined magnets is positive and so the lens has a net focusing effect. From the symmetry of Eq. (7.70b) under interchange of f_+ and f_-, it should be clear that f_* is independent of the order in which the magnets are encountered. Therefore, if we were to repeat the calculation for the motion in the y direction we would get the same focusing effect. (The diagonal elements of the transfer matrix are interchanged but as they are both close to unity, this is a fairly small difference.)

The combination of two quadrupole lenses of opposite polarity can therefore imitate the action of a converging lens. Combinations of magnets like this are used to collimate particle beams in storage rings and particle accelerators.

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<p align="center">**EXERCISES**</p>

Exercise 7.10 *Problem: Matrix Optics for a Simple Refracting Telescope*

Consider a simple refracting telescope (Fig. 7.7) that comprises two converging lenses. This telescope takes parallel rays of light from distant stars, which make an angle $\theta \ll 1$ with the optic axis, and converts them into parallel rays making an angle $M\theta$. Here M is the magnification with M negative, $|M| \gg 1$ and $|M\theta| \ll 1$

.

(a) Use matrix methods to investigate how the output rays depend on the separation of the two lenses and hence find the condition that the output rays are parallel when the input rays are parallel.

(b) How does the magnification M depend on the ratio of the focal lengths of the two lenses?

Exercise 7.11 *Example: Rays bouncing between two mirrors*

Consider two spherical mirrors each with radius of curvature R, separated by distance d so as to form an "optical cavity," as shown in Fig. 7.8. A laser beam bounces back and forth between the two mirrors. The center of the beam travels along a geometric-optics ray.

(a) Show, using matrix methods, that the central ray hits one of the mirrors (either one) at successive locations $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \ldots$ (where $\mathbf{x} \equiv (x, y)$ is a 2 dimensional vector in the plane perpendicular to the optic axis), which satisfy the difference equation

$$\mathbf{x}_{k+2} - 2b\mathbf{x}_{k+1} + \mathbf{x}_k = 0$$

where

$$b = 1 - \frac{4d}{R} + \frac{2d^2}{R^2}.$$

Explain why this is a difference-equation analogue of the simple-harmonic-oscillator equation.

(b) Show that this difference equation has the general solution

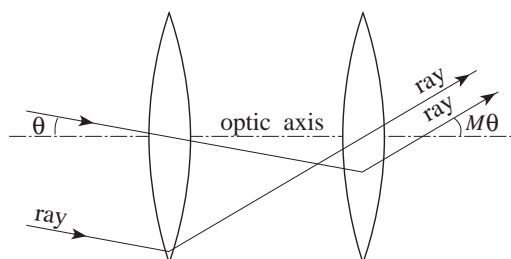$$\mathbf{x}_k = \mathbf{A} \cos(k \cos^{-1} b) + \mathbf{B} \sin(k \cos^{-1} b) .$$

Fig. 7.7: Simple refracting telescope. By convention $\theta > 0$ and $M\theta < 0$, so the image is inverted.

Obviously, $\mathbf{A}$ is the transverse position $\mathbf{x}_0$ of the ray at its 0'th bounce. The ray's 0'th position $\mathbf{x}_0$ and its 0'th direction of motion $\dot{\mathbf{x}}_0$ together determine $\mathbf{B}$.

(c) Show that if $0 \leq d \leq 2R$, the mirror system is "stable". In other words, all rays oscillate about the optic axis. Similarly, show that if $d > 2R$, the mirror system is unstable and the rays diverge from the optic axis.

(d) For an appropriate choice of initial conditions $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$, the laser beam's successive spots on the mirror lie on a circle centered on the optic axis. When operated in this manner, the cavity is called a *Harriet delay line*. How must d/R be chosen so that the spots have an angular step size θ? (There are two possible choices.)

<div align="center">****************************</div>

7.5   T2 Caustics and Catastrophes—Gravitational Lenses

Albert Einstein's general relativity theory (Part VII of this book) predicts that light rays should be deflected by the gravitational field of the Sun. Newton's law of gravity combined with his corpuscular theory of light also predict such a deflection, but through an angle half as great as relativity predicts. A famous measurement, during a 1919 solar eclipse, confirmed the relativistic prediction, thereby making Einstein world famous.

The deflection of light by gravitational fields allows a cosmologically distant galaxy to behave like a crude lense and, in particular, to produce multiple images of a more distant quasar. Many examples of this phenomenon have been observed. The optics of these gravitational lenses provides an excellent illustration of the use of Fermat's principle and also the properties of caustics.[9]

7.5.1   T2 Formation of Multiple Images

The action of a gravitational lens can only be understood properly using general relativity. However, when the gravitational field is weak, there exists an equivalent Newtonian model
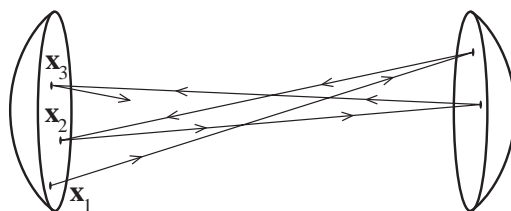
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[9]Schneider, Ehlers & Falco (1999).

Fig. 7.8: An optical cavity formed by two mirrors, and a light beam bouncing back and forth inside it.
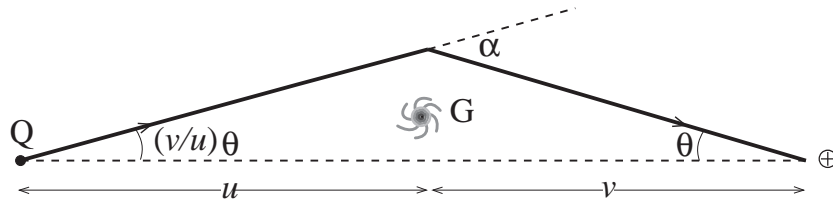
Fig. 7.9: Geometry for a gravitational lens. Light from a distant quasar, Q treated as a point source, passes by a galaxy G and is deflected through an angle α on its way to earth $\oplus$. The galaxy is a distance u from the quasar and v from earth.

which is adequate for our purposes. In this model, curved spacetime behaves as if it were spatially flat and endowed with a refractive index given by

$$\mathfrak{n} = 1 - \frac{2\Phi}{c^2} \tag{7.72}$$

where Φ is the Newtonian gravitational potential, normalized to vanish far from the source of the gravitational field and chosen to have a negative sign (so, e.g., the field at a distance r from a point mass M is $\Phi = -GM/r$). Time is treated in the Newtonian manner in this model. We will justify this index-of-refraction model in Part VII.

Consider, first, a ray which passes by a point mass M with an impact parameter b. The ray trajectory is determined by solving the paraxial ray equation (7.55), $d/dz(\mathfrak{n}d\mathbf{x}/dz) = (\mathbf{x} \cdot \boldsymbol{\nabla})(\boldsymbol{\nabla}\mathfrak{n})$, where $\mathbf{x}(z)$ is the ray's transverse position relative to an optic axis that passes through the point mass, and z is distance along the optic axis.

The ray will be bent through a deflection angle α, cf. Fig. 7.9. An equivalent way of expressing the motion is to say that the photons moving with speed c are subject to a Newtonian gravitational force and are accelerated kinematically with twice the Newtonian acceleration. The problem is therefore just that of computing the deflection of a charged particle passing by an oppositely charged particle. The deflection, under the impulse approximation, is given by

$$\alpha = \frac{4GM}{bc^2} = \frac{-4\Phi(r = b)}{c^2} \ , \tag{7.73}$$

where b is the ray's impact parameter. For a ray from a distant star, passing close to the limb of the sun, this deflection is $\alpha = 1.75$ arc seconds.

Now, let us replace the sun with a cosmologically distant galaxy and the star with a more distant quasar. Let the distance from the galaxy to Earth be v and from the galaxy to the quasar be u (Fig. 7.9).[10]

It is instructive to make an order of magnitude estimate of $\Phi \sim -GM/b$, where M is the mass interior to the impact parameter, b. It is convenient to relate the deflection to

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[10]There is a complication in the fact that the universe is expanding and that it contains large scale gravitational fields. However, as we discuss in Chap. 27, the universe is spatially flat and so the relation between angles and lengths is the same as in the solar system. We can also imagine performing this observation after the universe has expanded to its present age everywhere and stopping the expansion so as to measure the distances u, v. If we use this definition of distance, we can ignore cosmological effects in understanding the optics.

the mean square velocity σ^2 of the constituent stars (measured in one dimension). (This quantity can be measured spectroscopically.) We find that $\Phi \sim -3\sigma^2/2$. Therefore, an order of magnitude estimate of the angle of deflection is $\alpha \sim 6\sigma^2/c^2$. If we do a more careful calculation for a simple model of a galaxy in which the mass density varies inversely with the square of the radius, then we obtain

$$\alpha \sim \frac{4\pi\sigma^2}{c^2} \tag{7.74}$$

For typical galaxies, $\sigma \sim 300$ km s^{-1} and $\alpha \sim 1 - 2$ arc sec. The paraxial approximation therefore is fully justified. Now, the distances u, v are roughly ten billion light years $\sim 10^{26}$m and so the transverse displacement of the ray due to the galaxy is $\sim v\alpha/2 \sim 3 \times 10^{20}$m, which is well within the galaxy. This means that light from a quasar lying behind the galaxy can pass by either side of the galaxy. We should then see at least two distinct images of the quasar separated by an angular distance $\sim \alpha/2$.

The imaging is illustrated in Fig. 7.9. First trace a ray backward from the observer, in the absence of the intervening galaxy, to the quasar. We call this the reference ray. Next, reinstate the galaxy and consider a *virtual ray* that propagates at an angle $\boldsymbol{\theta}$ (a 2-dimensional vector on the sky) to the reference ray in a straight line from the earth to the galaxy where it is deflected toward the quasar. (A virtual ray is a path that will become a real ray if it satisfies Fermat's principle.) The optical phase for light propagating along this virtual ray will exceed that along the reference ray by an amount $\Delta\varphi$ called the *phase delay*. There are two contributions to $\Delta\varphi$: First, the geometrical length of the path is longer than the reference ray by an amount $(u + v)v\theta^2/2u$ (cf. Fig. 7.9), and thus the travel time is longer by an amount $(u + v)v\theta^2/2uc$. Second, the light is delayed as it passes through the potential well by a time $\int(\mathfrak{n} - 1)ds/c = -2\int \Phi ds/c^3$, where ds is an element of length along the path. We can express this second delay as $2\Phi_2/c^3$. Here

$$\boxed{\Phi_2 = \int \Phi ds} \tag{7.75}$$

is the *two-dimensional (2D) Newtonian potential* and can be computed from the 2D Poisson equation

$$\boxed{\nabla^2\Phi_2 = 4\pi G\Sigma , \quad \text{where } \Sigma = \int \rho ds} \tag{7.76a}$$

is the surface density of mass in the galaxy integrated along the line of sight.

Therefore, the phase delay $\Delta\varphi$ is given by

$$\Delta\varphi = \omega \left(\frac{(u + v)v}{2uc}\theta^2 - \frac{2\Phi_2(\boldsymbol{\theta})}{c^3} \right) . \tag{7.76b}$$

We can now invoke Fermat's principle. Of all possible virtual rays, parametrized by the angular coordinate $\boldsymbol{\theta}$, the only ones that correspond to real rays are those for which the phase difference is stationary, i.e. those for which

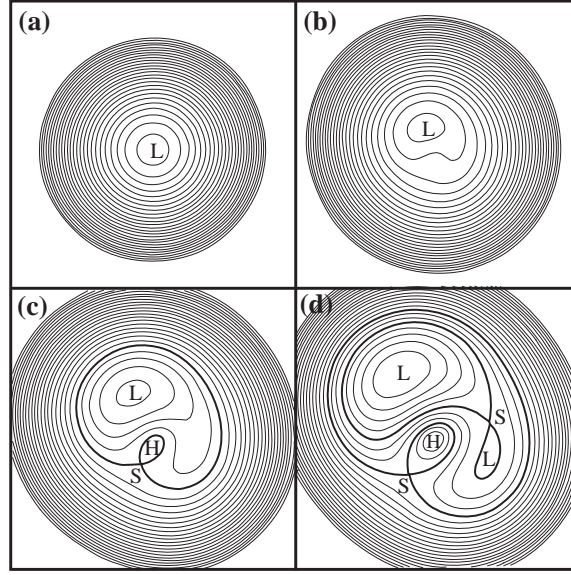$$\frac{\partial\Delta\varphi}{\partial\theta_j} = 0 , \tag{7.76c}$$

Fig. 7.10: Contour plots of the phase delay $\Delta\varphi(\boldsymbol{\theta})$ for four different gravitational lenses. a) In the absence of a lens $\Phi_2 = 0$, the phase delay (7.76b) has a single minimum corresponding to a single undeflected image. b) When a small galaxy with a shallow potential Φ_2 is interposed, it pushes the phase delay $\Delta\varphi$ up in its vicinity [Eq. (7.76b) with negative Φ_2], so the minimum and hence the image are deflected slightly away from the galaxy's center. c) When a galaxy with a deeper potential well is present, the delay surface will be raised so much near the galaxy's center that additional stationary points will be created and two more images will be produced. d) If the potential well deepens even more, five images can be produced. In all four plots the local extrema of $\Delta\varphi$ are denoted L for a low point (local minimum), H for a high point (local maximum) and S for saddle point.

where θ_j (with $j = x, y$) are the Cartesian components of $\boldsymbol{\theta}$. Differentiating Eq. (7.76b) we obtain *a 2D vector equation for the angular location of the images as viewed from Earth:*

$$\boxed{\theta_j = \frac{2u}{(u+v)vc^2}\frac{\partial\Phi_2}{\partial\theta_j}.}$$

(7.76d)

Note that $\partial\Phi_2/\partial\theta_j$ is a function of θ_j, so if $\Phi_2(\theta_j)$ is known, this becomes a (usually) nonlinear equation to be solved for the vector θ_j. Referring to Fig. 7.9, and using simple geometry, we can identify the *deflection angle for the image's ray:*

$$\boldsymbol{\alpha}_j = \frac{2}{vc^2}\frac{\partial\Phi_2}{\partial\theta_j}$$

(7.76e)

We can understand quite a lot about the properties of the images by inspecting a *contour plot of the phase delay function* $\Delta\varphi(\boldsymbol{\theta})$ (Fig. 7.10). When the galaxy is very light or quite distant from the line of sight, then there is a single minimum in the phase delay. However, a massive galaxy along the line of sight to the quasar can create two or even four additional stationary points and therefore three or five images. Note that with a transparent galaxy,

the additional images are created in pairs. Note, in addition, that the stationary points are not necessarily minima. They can be minima labeled by L in the figure, maxima labeled by H, or saddle points labeled by S. This is inconsistent with Fermat's original statement of his principle ("minimum phase delay"), but there are images at all the stationary points nevertheless.

Now *suppose that the quasar is displaced by a small angle $\delta\boldsymbol{\theta}'$ as seen from Earth in the absence of the lens.* This is equivalent to moving the lens by a small angle $-\delta\boldsymbol{\theta}'$ as seen from Earth. Equation (7.76d) says that *the image will be displaced by a small angle $\delta\boldsymbol{\theta}$* satisfying the equation

$$\delta\theta_i - \delta\theta'_i = \frac{2u}{(u+v)vc^2}\frac{\partial^2\Phi_2}{\partial\theta_i\partial\theta_j}\delta\theta_j \ . \tag{7.77a}$$

By combining with Eq. (7.76b), we can rewrite this as

$$\delta\theta'_i = H_{ij}\delta\theta_j \ , \tag{7.77b}$$

where the matrix $[H_{ij}]$ is

$$\boxed{H_{ij} = \left(\frac{uc/\omega}{(u+v)v}\right)\frac{\partial^2\Delta\varphi}{\partial\theta_i\partial\theta_j} = \delta_{ij} - \frac{2u}{(u+v)vc^2}\Phi_{2,ij} \ .} \tag{7.77c}$$

Now consider a small solid angle of source $d\Omega' = d\theta'_1 d\theta'_2$. Its image in the presence of the lens will have solid angle $d\Omega = d\theta_1 d\theta_2$. Because the specific intensity of the light, $I_\nu = dE/dtdAd\Omega d\nu$ is conserved along a ray (unaffected by the lensing), the flux of light received from the source is $I_\nu d\Omega'$ in the absence of the lens and $I_\nu d\Omega$ in its presence. The ratio of these fluxes is the *magnification $M = d\Omega/d\Omega'$*. From Eq. (7.77b) we see that the magnification is just the determinant of the inverse of the matrix H_{ij}:

$$\boxed{M = \frac{d\Omega}{d\Omega'} = \frac{1}{\det[H_{ij}]} \ .} \tag{7.78}$$

The curvature of the phase delay surface [embodied in $\det[\partial^2\Delta\varphi/\partial\theta_i\partial\theta_j]$ which appears in Eq.7.77c)] is therefore a quantitative measure of the magnification. Small curvature implies large magnification of the images and *vice versa*. Furthermore images associated with saddle points in the phase delay surface have opposite parity to the source. Those associated with maxima and minima have the same parity as the source, although the maxima are rotated on the sky by 180°. These effects have been seen in observed gravitational lenses.

There is an additional, immediate contact to the observations and this is that *the phase delay function at the stationary points is equal to ω times the extra time it takes a signal to arrive along that ray.* In order of magnitude, the time delay difference will be $\sim v\alpha^2/8c \sim$ 1 month. Now, many quasars are intrinsically variable, and if we monitor the variation in two or more images, then we should be able to measure the time delay between the two images. This, in turn, allows us to measure the distance to the quasar and, consequently, provides a measurement of the size of the universe.
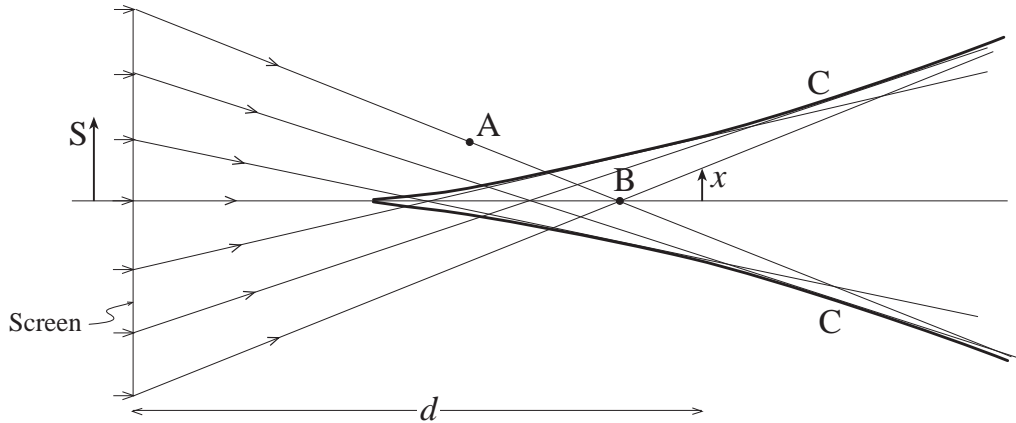
Fig. 7.11: The formation of caustics by a phase changing screen which is constrained to be circularly symmetric. Light from a distant source is refracted at the screen. The envelope of the refracted rays forms a caustic surface C. An observer at a point A, outside C, will see a single image of the distant source, whereas one at point B, inside C, will see three images. If the observer at B moves toward the caustic, then she will see two of the images approach each other, merge and then vanish. If the source has a finite angular size, the observed angular size of the two merging images will increase prior to their vanishing, and the energy flux from the two images will also increase. In this example, the caustic terminates in a cusp point which becomes structurally unstable if we remove the constraint that the phase screen be axisymmetric. THIS DIAGRAM MUST BE REDRAWN. IT COULD BE IMPROVED IF WE SHOW THAT THERE ARE THREE RAYS PASSING THROUGH OFF AXIS POINTS WITH C AND MOVE B TO ONE OF THEM. HOWEVER IT GOT RATHER COMPLICATED.

7.5.2 ⬚T2⬚ Catastrophe Optics — Formation of Caustics

Many simple optical instruments are carefully made so as to form point images from point sources. However, naturally occuring optical systems, and indeed precision optical instruments, when examined in detail, bring light to a focus on a 2D surface in 3D space, called a *caustic*.[11] Caustics are quite familiar and can be formed when sunlight is refracted or reflected by the choppy water on the surface of a swimming pool. The bright lines one sees on the pool's bottom are intersections of 2D caustics with the pool's 2D surface Another good example is the cusped curve (called a nephroid) formed by light from a distant source reflected off the cylindrical walls of a mug of coffee onto the surface of the coffee. What is surprising is that caustics formed under quite general conditions can be classified into a rather small number of types.

   As a simple, concrete example that illustrates the theory of caustics, let us consider the problem of the refraction of light by an axisymmetric, converging lens, for example a gravitational lens (cf. Fig. 7.11). Consider a set of rays from a distant source with impact parameter s at the lens. Let these rays pass through a point of observation a distance d from the lens with radial coordinate $x = s - \theta d \ll d$. As we have just shown the true rays will be those for which the total phase $\varphi(s,x)$ is stationary with respect to variations of s.
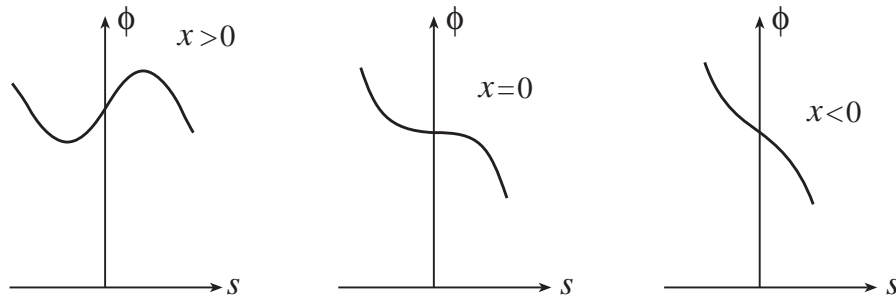
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[11]See, for example, Berry & Upstill (1980).

Fig. 7.12: Optical phase φ for three different observer locations measured by a one dimensional coordinate x. (The second coordinate is suppressed.) The true rays are refracted at the values of s corresponding to the maxima and minima of the phase. There are two such rays, when the observer lies inside the caustic surface, $(x > 0)$ and now rays when she lies outside $(x < 0)$.

Now, if x is small enough and d is large enough, then there will typically be three rays that pass through any point of observation. In the case of a gravitational lens, the astronomer would see three images of the source. However, when x is large and the astronomer is well away from the optic axis, there will only be one ray and one image. There is therefore an axisymmetric surface, called a *caustic* where the number of images changes from one to three.

Now let us consider the behavior of the phase φ as we cross this caustic. From Fig. 7.11, it is clear that the two disappearing images approach one another and then vanish. Algebraically, this means that, by changing the parameter (often called a *control* variable) x, the variation of $\varphi(s, x)$ with s (often called a *state* variable) changes locally through a set of curves like those in Fig. 7.12, where a maximum and a minimum smoothly merge through a point of inflexion and then vanish. It is clear that close enough to the caustic, $\varphi(s, x)$, for given s, has the form of a cubic. By changing the origins of s and x, this cubic can be written in the form of a Taylor series for which the leading terms are:

$$\varphi(s; d, x) = \frac{1}{3}as^3 - bxs + \dots \tag{7.79}$$

where the factor $1/3$ is just a convention and we have dropped a constant. Note that, by changing coordinates, we have removed the quadratic terms. Now, for any given lens we can compute the coefficients a, b accurately through a careful Taylor expansion about the caustic. However, their precise form does not interest us here as we are only concerned with scaling laws.

Now, invoking Fermat's Principle and differentiating Eq. (7.79) with respect to s, we see that there are two true rays and two images for $x > 0$, (passing through $s = \pm(bx/a)^{1/2}$), and no images for $x < 0$. $x = 0$ marks the location of the caustic at this distance behind the lens. We can now compute the magnification of the images as the caustic is approached. This is given by

$$M \propto \frac{ds}{dx} = \frac{1}{2}\left(\frac{b}{ax}\right)^{1/2} . \tag{7.80}$$

Notice that *the magnification, and thus also the total flux in each image, scales inversely*

with the square root of the distance from the caustic. This does not depend on the optical details (i.e. on the coefficients in our power series expansions). It therefore is equally true for reflection at a spherical mirror, or refraction by a gravitational lens, or refraction by the rippled surface of the water in a swimming pool. This is just one example of several scaling laws which apply to caustics.

The theory of optical caustics is a special case of a more general formalism called *catastrophe theory*, and caustics are examples of *catastrophes*. In this theory, it is shown that there are only a few types of catastrophe and they have many generic properties. The key mathematical requirement is that the behavior of the solution should be structurally stable. That is to say, if we make small changes in the physical conditions, the scaling laws etc. are robust.

The catastrophe that we have just considered is the most elementary example and is called the *fold*. A gravitational lens example is shown in shown in Fig. 7.13. The next simplest catastrophe, known as the *cusp*, is the curve where two fold surfaces meet. (The point cusp displayed in Fig. 7.12, is actually structurally unstable as a consequence of the assumed strict axisymmetry. However if we regard s, x as just 1D Cartesian coordinates, then Fig. 7.12 provides a true representation of the geometry.) In total there are seven elementary catastrophes. Catastrophe theory has many interesting applications in optics, dynamics, and other branches of physics.

Let us conclude with an important remark. If we have a point source, the magnification will diverge to infinity as the caustic is approached, according to Eq. 7.80. Two factors prevent the magnification from becoming truly infinite. The first is that a point source is only an idealization, and if we allow the source to have finite size, different parts will produce caustics at slightly different locations. The second is that geometric optics, on which our analysis was based, pretends that the wavelength of light is vanishingly small. In actuality, the wavelength is always nonzero, and near a caustic its finiteness leads to diffraction effects, which limit the magnification to a finite value. Diffraction is the subject of the next chapter.

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EXERCISES

Exercise 7.12 *Example: Stellar gravitational lens*

Consider a star with mass M that is located a distance v from us and acts as a gravitational lens to produce multiple images of a second star that is a further distance u behind the lens.

(a) Verify Eq. (7.73) under the assumption that the deflection angle is small.

(b) Use this equation to show that when the second star lies on the continuation of the Earth-lens line, it will produce a thin-ring image at the observer of angular radius given by

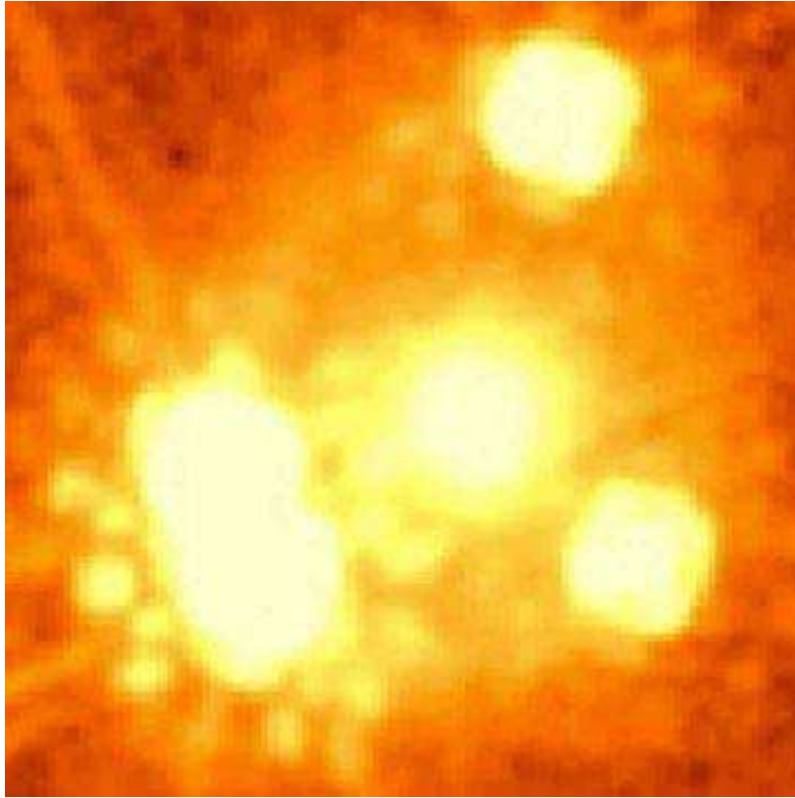$$\theta_E = \left(\frac{4GM}{Dc^2}\right)^{1/2}, \tag{7.81}$$

Fig. 7.13: Gravitational lens in which a distant quasar, Q1115+080, is quadruply imaged by an intervening galaxy. (There is also a fifth, unseen image.) Two of the images are observed to be much brighter and closer than the other two because the quasar is located quite close to a fold caustic surface. The images are observed to vary in the order predicted by modeling the gravitational potential of the galaxy and this variation can be used to estimate the size of the universe.

and evaluate the distance D in terms of the distances u, v. (This ring is known as the *Einstein ring* and the radius as the *Einstein radius*.)

(c) Show that when the source is displaced from this line, there will be just two images, one lying within the Einstein ring, the other lying outside. Find their locations.

(d) Denote the ratio of the fluxes in these two images by R. Show that the angular radii of the two images can be expressed in the form $\theta_\pm = \pm\theta_E R^{\pm 1/4}$ and explain how to measure the Einstein radius observationally.

Incidentally, Einstein effectively performed this calculation, in one of his unpublished notebooks, prior to understanding general relativity.

Exercise 7.13 *Challenge: Catastrophe Optics of an Elliptical Lens*

Consider an elliptical gravitational lens where the potential at the lens plane varies as

$$\Phi_2(\boldsymbol{\theta}) = (1 + A\theta_1^2 + 2B\theta_1\theta_2 + C\theta_2^2)^q; \qquad 0 < q < 1/2.$$

Determine the generic form of the caustic surfaces and the types of catastrophe encountered. Note that it is in the spirit of catastrophe theory *not* to compute exact expressions but to determine scaling laws and to understand the qualitative behavior of the images.

<center>****************************</center>

7.6 Polarization

In our geometric optics analyses thus far, we have either dealt with a scalar wave (e.g., a sound wave) or simply supposed that individual components of vector or tensor waves can be treated as scalars. For most purposes, this is indeed the case and we shall continue to use this simplification in the following chapters. However, there are some important wave properties that are unique to vector (or tensor) waves. Most of these come under the heading of *polarization* effects. In Part VII we shall study polarization effects for (tensorial) gravitational waves. Here and in several other chapters we shall examine them for electromagnetic waves.

7.6.1 Polarization Vector and its Geometric-Optics Propagation Law

An electromagnetic wave *in vacuo* has its electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ perpendicular to its propagation direction $\hat{\mathbf{k}}$ and perpendicular to each other. In a medium, $\mathbf{E}$ and $\mathbf{B}$ may or may not remain perpendicular to $\hat{\mathbf{k}}$, depending on the medium's properties. For example, an Alfvén wave has its vibrating magnetic field *perpendicular* to the background magnetic field, which can make an arbitrary angle with respect to $\hat{\mathbf{k}}$. By contrast, in the simplest case of an *isotropic* dielectric medium, where the dispersion relation has our standard dispersion-free form $\Omega = (c/\mathfrak{n})k$, the group and phase velocities are parallel to $\hat{\mathbf{k}}$, and $\mathbf{E}$ and $\mathbf{B}$ turn out to be perpendicular to $\hat{\mathbf{k}}$ and to each other—as in vacuum. In this section, we shall confine attention to this simple situation, and to linearly polarized waves, for which $\mathbf{E}$ oscillates linearly back and forth along a polarization direction $\hat{\mathbf{f}}$ that is perpendicular to $\hat{\mathbf{k}}$:

$$\mathbf{E} = A e^{i\varphi}\,\hat{\mathbf{f}}\,, \quad \hat{\mathbf{f}} \cdot \hat{\mathbf{k}} \equiv \hat{\mathbf{f}} \cdot \boldsymbol{\nabla}\varphi = 0\,. \tag{7.82}$$

In the eikonal approximation, $Ae^{i\varphi} \equiv \psi$ satisfies the geometric-optics propagation laws of Sec. 7.3, and the polarization vector $\hat{\mathbf{f}}$, like the amplitude A, will propagate along the rays. The propagation law for $\hat{\mathbf{f}}$ can be derived by applying the eikonal approximation to Maxwell's equations, but it is easier to infer that law by simple physical reasoning: (i) If the ray is straight, then the medium, being isotropic, is unable to distinguish a slow right-handed rotation of $\hat{\mathbf{f}}$ from a slow left-handed rotation, so there will be no rotation at all: $\hat{\mathbf{f}}$ will continue always to point in the same direction, i.e. $\hat{\mathbf{f}}$ will be kept parallel to itself during transport along the ray. (ii) If the ray bends, so $d\hat{\mathbf{k}}/ds \neq 0$ (where s is distance along the ray), then $\hat{\mathbf{f}}$ will have to change as well, so as always to remain perpendicular to $\hat{\mathbf{k}}$. The
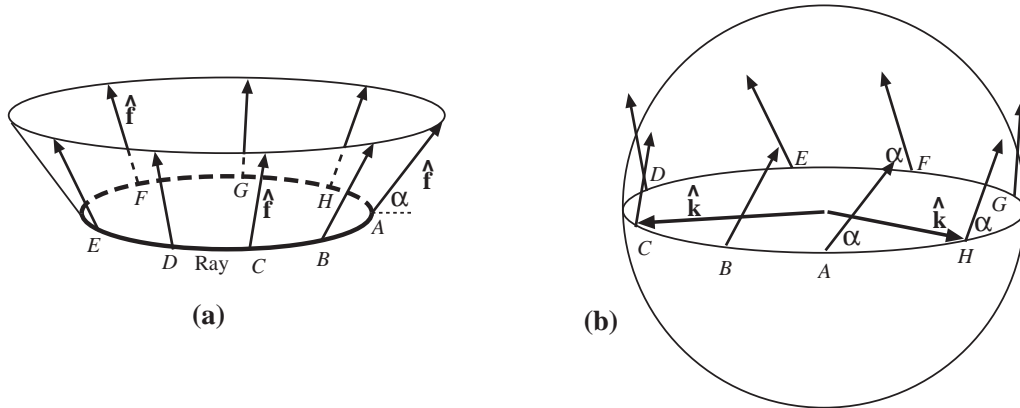
Fig. 7.14: (a) The ray along the optic axis of a circular loop of optical fiber, and the polarization vector $\hat{\mathbf{f}}$ that is transported along the ray by the geometric-optics transport law $d\hat{\mathbf{f}}/ds = -\hat{\mathbf{k}}(\hat{\mathbf{f}} \cdot d\hat{\mathbf{k}}/ds)$. (b) The polarization vector $\hat{\mathbf{f}}$ drawn on the unit sphere. The vector from the center of the sphere to each of the points A, B, ..., is the ray's propagation direction $\hat{\mathbf{k}}$, and the polarization vector (which is orthogonal to $\hat{\mathbf{k}}$ and thus tangent to the sphere) is identical to that in the physical space of the ray [drawing (a)].

direction of $\hat{\mathbf{f}}$'s change must be $\hat{\mathbf{k}}$, since the medium, being isotropic, cannot provide any other preferred direction for the change. The magnitude of the change is determined by the requirement that $\hat{\mathbf{f}} \cdot \hat{\mathbf{k}}$ remain zero all along the ray and that $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$. This immediately implies that the propagation law for $\hat{\mathbf{f}}$ is

$$\boxed{\frac{d\hat{\mathbf{f}}}{ds} = -\hat{\mathbf{k}}\left(\hat{\mathbf{f}} \cdot \frac{d\hat{\mathbf{k}}}{ds}\right)}. \tag{7.83}$$

We say that the vector $\hat{\mathbf{f}}$ is *parallel-transported* along $\hat{\mathbf{k}}$. Here "parallel transport" means: (i) Carry $\hat{\mathbf{f}}$ a short distance along the trajectory keeping it parallel to itself in 3-dimensional space. This will cause $\hat{\mathbf{f}}$ to no longer be perpendicular to $\hat{\mathbf{k}}$. (ii) Project $\hat{\mathbf{f}}$ perpendicular to $\hat{\mathbf{k}}$ by adding onto it the appropriate multiple of $\hat{\mathbf{k}}$. (The techniques of differential geometry for curved surfaces, which we shall develop in Part VII when studying general relativity, give powerful mathematical tools for analyzing this parallel transport.)

7.6.2   T2 Geometric Phase

We shall use the polarization propagation law (7.83) to illustrate a quite general phenomenon known as the *Geometric* phase.[12]

Consider linearly polarized, monochromatic light beam that propagates in an optical fiber. The fiber's optic axis is the principal ray along which the light propagates. We can imagine bending the fiber into any desired shape, and thereby controlling the shape of the ray. The ray's shape in turn will control the propagation of the polarization via Eq. (7.83).
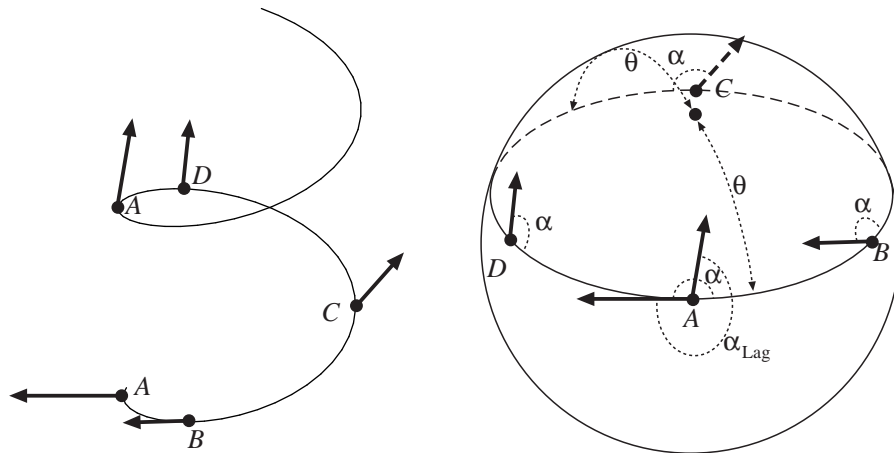
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[12]Berry (1990).

Fig. 7.15: (a) The ray along the optic axis of a helical loop of optical fiber, and the polarization vector $\hat{\mathbf{f}}$ that is transported along the ray by the geometric-optics transport law $d\hat{\mathbf{f}}/ds = -\hat{\mathbf{k}}(\hat{\mathbf{f}} \cdot d\hat{\mathbf{k}}/ds)$. The ray's propagation direction $\hat{\mathbf{k}}$ makes an angle $\theta = 73^{\circ}$ to the vertical direction. (b) The trajectory of $\hat{\mathbf{k}}$ on the unit sphere (a circle with polar angle $\theta = 73^{\circ}$), and the polarization vector $\hat{\mathbf{f}}$ that is parallel transported along that trajectory. The polarization vectors in drawing (a) are deduced from the parallel transport law of drawing (b). The lag angle $\alpha_{\mathrm{lag}} = 2\pi(1 - \cos\theta) = 1.42\pi$ is equal to the solid angle contained inside the trajectory of $\hat{\mathbf{k}}$ (the $\theta = 73^{\circ}$ circle).

If the fiber and ray are straight, then the propagation law (7.83) keeps $\hat{\mathbf{f}}$ constant. If the fiber and ray are circular, then the propagation law (7.83) causes $\hat{\mathbf{f}}$ to rotate in such a way as to always point along the generator of a cone as shown in Fig. 7.14 (a). This polarization behavior, and that for any other ray shape, can be deduced with the aid of a unit sphere on which we plot the ray direction $\hat{\mathbf{k}}$ [Fig. 7.14 (b)]. For example, the ray directions at ray locations C and H [drawing (a)] are as shown in drawing (b). Notice, that the trajectory of $\hat{\mathbf{k}}$ around the unit sphere is a great circle. This is because the ray in physical space is a closed circle. If, instead, the fiber and ray were bent into a helix (Fig. 7.15), then the trajectory on the unit sphere would be a smaller circle.

On the unit sphere we also plot the polarization vector $\hat{\mathbf{f}}$ — one vector at each point corresponding to a ray direction. Because $\hat{\mathbf{f}} \cdot \hat{\mathbf{k}} = 0$, the polarization vectors are always tangent to the unit sphere. Notice that each $\hat{\mathbf{f}}$ on the unit sphere is identical in length and direction to the corresponding one in the physical space of drawing (a).

For the circular, closed ray of Fig. 7.14 (a), the parallel transport law keeps constant the angle α between $\hat{\mathbf{f}}$ and the trajectory of $\hat{\mathbf{f}}$ [drawing (b)]. Translated back to drawing (a), this constancy of α implies that the polarization vector points always along the generators of the cone whose opening angle is $\pi/2 - \alpha$, as shown.

For the helical ray of Fig. 7.15 (a), the propagation direction $\hat{\mathbf{k}}$ rotates, always maintaining the same angle θ to the vertical direction, and correspondingly its trajectory on the unit sphere of Fig. 7.15 (b) is a circle of constant polar angle θ. In this case (as one can see, e.g., with the aid of a large globe of the Earth and a pencil that one transports around a circle

of latitude $90^\circ - \theta$), the parallel transport law dictates that the angle α between $\hat{\mathbf{f}}$ and the circle *not* remain constant, but instead rotate at the rate

$$d\alpha/d\phi = \cos\theta . \tag{7.84}$$

Here ϕ is the angle (longitude on the globe) around the circle. (This is the same propagation law as for the direction of swing of a Foucault Pendulum as the earth turns, and for the same reason: the gyroscopic action of the Foucault Pendulum is described by parallel transport of its plane along the earth's spherical surface.)

In the case $\theta \simeq 0$ (a nearly straight ray), the transport equation (7.84) predicts $d\alpha/d\phi = 1$: although $\hat{\mathbf{f}}$ remains constant, the trajectory of $\hat{\mathbf{k}}$ turns rapidly around a tiny circle about the pole of the unit sphere, so α changes rapidly—by a total amount $\Delta\alpha = 2\pi$ after one trip around the pole. For an arbitrary helical pitch angle θ, the propagation equation (7.84) predicts that during one round trip α will change by an amount $2\pi \cos\theta$ that lags behind its change for a tiny circle (nearly straight ray) by the lag angle $\alpha_{\text{Lag}} = 2\pi(1 - \cos\theta)$, which is also the solid angle $\Delta\Omega$ enclosed by the path of $\hat{\mathbf{k}}$ on the unit sphere:

$$\alpha_{\text{Lag}} = \Delta\Omega . \tag{7.85}$$

For the circular ray of Fig. 7.14, the enclosed solid angle is $\Delta\Omega = 2\pi$ steradians, so the lag angle is 2π radians, which means that $\hat{\mathbf{f}}$ returns to its original value after one trip around the optical fiber, in accord with the drawings in the figure.

By itself, the relationship $\alpha_{\text{Lag}} = \Delta\Omega$ is merely a cute phenomenon. However, it turns out to be just one example of a very general property of both classical and quantum mechanical systems when they are forced to make slow *adiabatic* changes described by circuits in the space of parameters that characterize them. In the more general case one focuses on a phase lag, rather than a direction-angle lag. We can easily translate our example into such a phase lag:

The apparent rotation of $\hat{\mathbf{f}}$ by the lag angle $\alpha_{\text{Lag}} = \Delta\Omega$ can be regarded as an advance of the phase of one circularly polarized component of the wave by $\Delta\Omega$ and a phase retardation of the other circular polarization by the same amount. This implies that the phase of a circularly polarized wave will change, after one circuit around the fiber's helix, by an amount equal to the usual phase advance $\Delta\varphi = \int \mathbf{k} \cdot d\mathbf{x}$ (where $d\mathbf{x}$ is displacement along the fiber) *plus an extra* **geometric** *phase change* $\pm\Delta\Omega$, where the sign is given by the sense of circular polarization. This type of geometric phase change is found quite generally when classical vector or tensor waves propagate through backgrounds that change slowly, either temporally or spatially; and the phases of the wave functions of quantum mechanical particles with spin behave similarly.

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EXERCISES

Exercise 7.14 *Derivation: Parallel-Transport*

Use the parallel-transport law 7.83 to derive the relation 7.84.

Exercise 7.15 *Example: Martian Rover*

A Martian Rover is equipped with a single gyroscope that is free to pivot about the direction perpendicular to the plane containing its wheels. In order to climb a steep hill on Mars without straining its motor, it must circle the summit in a decreasing spiral trajectory. Explain why there will be an error in its measurement of North after it has reached the summit. Could it be programmed to navigate correctly? Will a stochastic error build up as it traverses a rocky terrain?

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Bibliographic Note

Most texts on optics are traditional in outlook, which is entirely appropriate for a subject with an extensive history that underlies its contemporary technological applications. The most comprehensive of these traditional texts with on extensive discussion of geometric optics is Born & Wolf (1970). A standard text written from a more modern perspective is Hecht (1987). A clear, brief contemporary monograph is Welford (1990). An excellent summary of catastrophe optics, covering both the geometric limit and diffractive effects is Berry (1982).

Bibliography

Born, M. & Wolf, E. 1970 *Principles of Optics* Oxford: Oxford University Press

Berry, M. V. 1990 Physics Today 43:12 34

Berry, M. V. & Upstill, C. 1980 Progress in Optics 18

Blandford, R. D. & Narayan, R. 1992 *Cosmological Applications of Gravitational Lenses* Ann. Rev. Astr. Astrophys. 30 311

Goldstein, H. 1980 *Classical Mechanics* New York: Addison Wesley

Guenther, R. 1990 *Modern Optics* New York: Wiley

Hecht, E. 1987 *Optics* New York: Wiley

Klein, M. V. & Furtak, T. E. 1986 *Optics* New York: Wiley

Schneider, P., Ehlers, J. and Falco, E.E. 1999 *Gravitational Lenses* Berlin: Springer-Verlag

Welford, W. T. 1990 *Optics* Oxford: Oxford Science Publications

Box 7.3
Important Concepts in Chapter 6

- General Concepts

    - Dispersion relation – Sec. 7.2.1, Ex. 7.2, Eq. (7.37)
    - Phase velocity and group velocity – Eqs. (7.2)
    - Wave Packet, its motion and spreading, and its internal waves – Sec. 7.2.2
    - Quanta associated with geometric-optics waves – Secs. 7.2.2 and 7.3.2

- General formulation of geometric optics: Sec. 7.3.3 and the following:

    - Eikonal (geometric optics) approximation – beginning of Sec. 7.3
    - Bookkeeping parameter for eikonal approximation – Box 7.2
    - Hamilton's equations for rays – Eqs. (7.25)
    - Connection to quantum theory – Sec. 7.3.2
    - Connection to Hamilton-Jacobi theory – Ex. 7.9, Eq. (7.26a)
    - Propagation law for amplitude (conservation of quanta) – Eqs. (7.33)
    - Fermat's principle, Eq. (7.45)
    - Breakdown of geometric optics – Sec. 7.3.5

- EM waves in dielectric medium, sound waves in fluid or isotropic solid, and gravitational waves in a weak, Newtonian gravitational field

    - Lagrangian, wave equation, energy density and energy flux – Ex. 7.4, Eqs. (7.17)–(7.19)
    - Dispersion relation $\Omega = C(\mathbf{x}, t)k$ – Eq. (7.23)
    - Ray equation in second–order form – Eq. (7.49)
    - Fermat's principle for rays – Eqs. (7.47),
    - Snell's law for rays in a stratified medium – Eq. (7.50)
    - Conservation of phase along a ray – Eq. (7.28)
    - Propagation law for amplitude – Eq. (7.24)

        * Geometric phase – Sec. 7.6.2

    - Parallel propagation law for polarization vector – Eq. (7.83)
    - Paraxial optics – Sec. 7.4

        * Matrix formalism for rays – Secs. 7.4, 7.4.1
        * Application to charged particles in storage ring – Sec. 7.4.2

    - Multiple images, crossing of rays, coalescence of images and caustics – Sec. 7.5.1
    - Magnification at a caustic – Eq. (7.80)