Physics 136b, Week 3: Elastodynamics

(Dated: due January 25, 2012)

The maximum number of points you can get for this assignment is **30**, although you could choose to do problems that worth more than 30 points.

This week we discussed elastodynamics, covering 12.4–12.5 of Blandford and Thorne (BT). [See version version 1112.1.K of Chapter 12.]

1. **Reflection and Transmission of Normal, Longitudinal Waves at a Boundary** [15 Points]

   Exercise 12.10 of BT.

2. **Vector Spherical Harmonics** [15 Points]

   Let us define the following, vector spherical harmonics

$$\vec{Y}^{lm} = \vec{e}_r Y^{lm}, \quad \vec{\Psi}^{lm} = r\vec{\nabla} Y^{lm}, \quad \vec{\Phi}^{lm} = \vec{r} \times \vec{\nabla} Y^{lm} \tag{1}$$

   where Y^{lm} is the scalar spherical harmonics.

   (a) Sketch the flow described by these vector spherical harmonics.

   (b) Prove the following identities:

$$\vec{\nabla} \cdot \left(f\vec{Y}^{lm}\right) = \left(f' + 2f/r\right) Y^{lm} \tag{2}$$

$$\vec{\nabla} \cdot \left(f\vec{\Psi}^{lm}\right) = -\frac{l(l+1)}{r} f Y^{lm} \tag{3}$$

$$\vec{\nabla} \cdot \left(f\vec{\Phi}^{lm}\right) = 0 \tag{4}$$

   Here f is $f(r)$, a function of radius alone, same below. Show that

$$w(t,r)\vec{\Phi}^{lm}, \quad l(l+1)\frac{u(t,r)}{r}\vec{Y}^{lm} + \left[\partial_r u(t,r) + \frac{u(t,r)}{r}\right]\vec{\Psi}^{lm} \tag{5}$$

   are good for representing transverse modes of an elastic sphere.

   (c) Prove that

$$\vec{\nabla} \times \left(f\vec{Y}^{lm}\right) = -f/r\vec{\Phi}^{lm} \tag{6}$$

$$\vec{\nabla} \times \left(f\vec{\Psi}^{lm}\right) = (f' + f/r)\vec{\Phi}^{lm} \tag{7}$$

$$\vec{\nabla} \times \left(f\vec{\Phi}^{lm}\right) = -\frac{l(l+1)}{r} f\vec{Y}^{lm} - (f' + f/r)\vec{\Psi}_{lm} \tag{8}$$

   Show that

$$\partial_r z(t,r)\vec{Y}^{lm} + \frac{z}{r}\vec{\Psi}^{lm} \tag{9}$$

   is good for representing the longitudinal deformation of an elastic sphere.

   (d) Prove the following identity:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\xi}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\xi}) - \nabla^2 \vec{\xi} \tag{10}$$

   [Hint: Here I found it easier to first write this using the index notation, then use the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]

3. **Elastic waves propagating inside an elastic sphere.** [10 Points]

   (a) Show that in the frequency domain, w satisfies

$$\frac{\omega^2}{c_T^2} w + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial w}{\partial r}\right) - \frac{l(l+1)}{r^2} w = 0 \tag{11}$$

   Show that $w = j_l(k_T r)$ (spherical Bessel function) and $n_l(k_T r)$ (spherical Neumann function) are both solutions to this differential equation, and show that we should pick the Bessel function, because it makes the motion regular at the origin. What is the starting l of this series of solutions?

   (b) Show that u satisfies the same differential equation as w, and that z satisfies

$$\frac{\omega^2}{c_L^2} z + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) - \frac{l(l+1)}{r^2} z = 0 \tag{12}$$

4. **Boundary conditions and modes of an elastic sphere** [15 Points]

   So far we had been able to get by without using tensors in spherical polar coordinate system. However, in order to get the boundary condition and get the spectrum of the sphere, we need some tensor analysis.

   (a) Write down $\vec{\xi} = w(r)\vec{\Phi}^{lm}$ in the orthonormal spherical polar basis. Compute

$$T_{r\theta} = \frac{1}{2}\left(\nabla_r \xi_\theta + \nabla_\theta \xi_r\right) \tag{13}$$

   Show that $T_{rr} = 0$ automatically, and that both $T_{r\theta} = 0$ and $T_{r\phi} = 0$ are given by the same equation for w. Calculate the first few such eigenmodes of an elastic sphere, write this in terms of the ratio between the radius of the sphere and the wavelength of the transverse wave. Also write down the asymptotic mode frequencies.

   (b) Write down T_{rr}, $T_{r\theta}$ and $T_{r\phi}$ for each of u and z. Show that in general, u and z must both be present in order for the boundary condition $T_{rj} = 0$ be satisfied. Write down the condition that governs the eigenfrequencies of these u-z modes.