Physics 136a, Week 7: Geometric Optics and Diffraction

(Dated: November 17, 2011; due **Wednesday November 23, 2011**)

The maximum number of points you can get for this assignment is **50**, although you could choose to do problems that worth more than 50 points.

This week we discussed fluctuations and geometric optics. This corresponds to Sections 7.3.5 – 7.6 of Blandford and Thorne (BT), Version 1107.2, and Sections 8.1 – 8.3, Version 1108.1.

1. Exercise 7.8, **Self-Focusing Optical Fibers** [15 Points]
2. Exercise 7.11, **Rays bouncing between two mirrors** [15 Points]
3. Exercise 7.12, **Stellar gravitational lens** [15 Points]
4. **Number of Images from a Gravitational Lens** [15 Points]

   Let us try to prove the formula

   $$N_+ - N_- = 1 \tag{1}$$

   for gravitational lens, where N_- is the number of images that preserves the source's parity, while N_+ is the number of those that flip parity.

   (a) If the phase delay φ is a function of sky location, and if we "compactify" infinity into one point, then show that the above formula is equivalent to

   $$N_{\rm max} + N_{\rm min} - N_{\rm saddle} = 2 \tag{2}$$

   where $C_{\rm max,min,saddle}$ are the number of maxima, minima and saddle points (we shall refer to these as critical points), respectively. Note that *infinity* is a maximum.

   (b) Note that now φ is defined on a domain that has the topology of a 2-D sphere. Let us "triangulate" this 2-D sphere, basically dividing it into very small triangles, each containing at most one critical point. For each triangle, let us travel along its counterclockwise (viewed from "above"), and compare how the gradient vector $\nabla\varphi$ has rotated with respect to our path (counterclockwise) after we complete one loop, and obtain a "deficit angle". Let us sum these deficit angles up for all triangles, argue that we will get a total deficit angle of

   $$\alpha = -2\pi(F - N_{\rm saddle} - N_{\rm max} - N_{\rm min}) - 4\pi N_{\rm saddle} = 2\pi(N_{\rm max} - N_{\rm saddle} + N_{\rm min} - F) \tag{3}$$

   Here F is the number of triangles. *The reason we use F here will become rather clear later.*

   [Hint: for triangles containing a maxima or a minima, there's no total difference angle between $\nabla\varphi$ and our trajectory, for triangles containing nothing, we have -2π, for triangles containing a saddle point, this is -4π.]

   (c) Calculate α using another way. To start with, show that for each edge shared by two triangles, their sense of counterclockwise would require us travel opposite directions. Show that this means deficit angles we accumulate along the edges will eventually cancel. This also means α can only accumulate at vertices, where the edges meet and change directions abruptly.

   (d) For each vertex, where the edges meet, let us label them by $j = 1, \ldots, V$, where V is the number of vertices. Also denote the number of edges at the j vertex as E_j. Show that another formula for α would be

   $$\alpha = -\sum_{j=1}^{V}(E_j\pi - 2\pi) \tag{4}$$

   Show further that this simplifies into

   $$\alpha = 2\pi(V - E) \tag{5}$$

(e) Show that we have, at the end,

$$F - E + V = N_{\text{max}} - N_{\text{saddle}} + N_{\text{min}} \tag{6}$$

Here $F - E + V$ is the Euler Characteristic χ of the complicated polyhedron made up by our triangles. It is the same as χ of a sphere, which is 2.

5. Exercise 8.3, **Triangular Diffraction Grating** [15 Points]

6. Exercise 8.4, **Derivation: Airy Pattern** [15 Points]