Ph136 Solution 2

2.8 (a) $A^{\alpha\beta\gamma} g_{\beta\rho} S_{\gamma\lambda} g^{\rho\delta} g^{\lambda}{}_{\alpha} = A^{\alpha\beta\gamma} g_{\beta\rho} g^{\rho\delta} g^{\lambda}{}_{\alpha} S_{\gamma\lambda}$

$$= A^{\lambda\delta\gamma} S_{\gamma\lambda}$$

(b) $g_{\alpha\beta} g^{\alpha\beta} = (-1)^2 + 1 + 1 + 1 = 4$

(c) The expression $A_{\alpha}{}^{\beta\gamma} S_{\alpha\gamma}$ is a sum over γ since it's a repeating index which appears both upstairs and downstairs. In contrast, α only appears downstairs, so we cannot apply the Einstein summation convention. In this case, there is no consistent interpretation which obeys our index-manipulation laws in spacetime.

In the case of $A_{\alpha}{}^{\beta\gamma} S_{\beta} T_{\gamma} = R_{\alpha\beta\delta} S^{\beta}$ we have a free index on the left hand side (α), while we have two free indices on the right hand side (α and δ). This would imply that a rank 1 tensor is equal to a rank 2 tensor, something is obviously not possible.

2.11 (a) The case of a photon

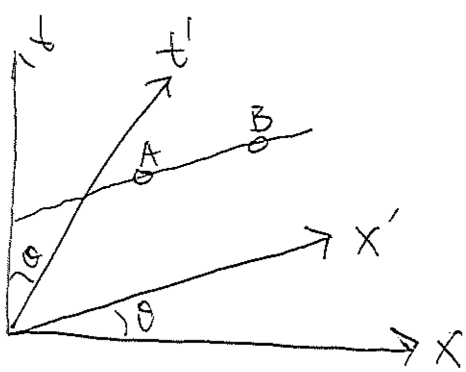
In frame F (in which the emitting atom appears to be moving with 3-velocity $\vec{v}$), $\vec{U} = (\gamma, \gamma\vec{v})$ with $\gamma = 1/\sqrt{1-v^2}$, and $\vec{p} = (E_F, E_F\vec{n})$. Then ~~using Eq. 1~~ we have the photon energy as measured by the emitting atom to be $E = -\vec{p}\cdot\vec{U} = E_F\gamma(1-\vec{v}\cdot\vec{n})$, i.e. $E_F/E = 1/[\gamma(1-\vec{v}\cdot\vec{n})]$.

(b) The case of a particle with finite rest mass m:

Now $\vec{U}$ is same as in the photon case, but $\vec{p} = (E_F, |p|\vec{n})$, where $|p| = \sqrt{E_F^2 - m^2}$. And we find $E = -\vec{p}\cdot\vec{U} = \gamma(E_F - \sqrt{E_F^2-m^2}\,\vec{v}\cdot\vec{n})$

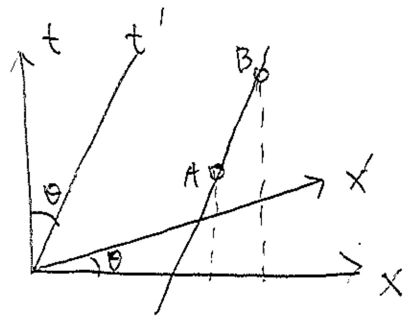
2.14 The spacetime diagrams are Fig. 1 through Fig. 6. In these figures, we use t', x' to denote $\bar{t}, \bar{x}$, and $\theta = \tan^{-1}\beta$

(a)



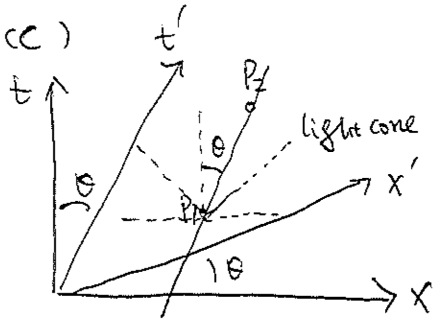
Events A and B are simultaneous in F. However because of the slope a $\bar{t}$=const line has in frame F, A will occur before B in frame $\bar{F}$ (A is the event that's "farther back".

(b)



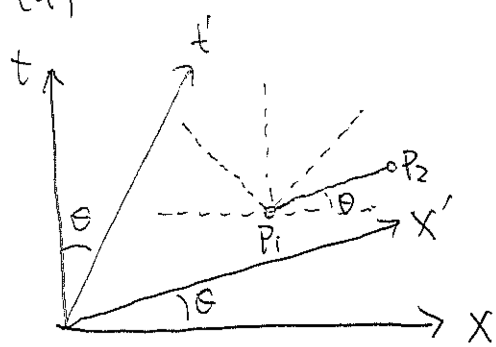
Events A and B occur at the same spatial location in $\bar{F}$ but not in F.

(c)



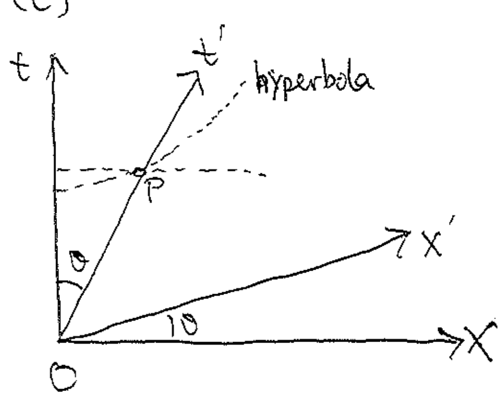
If P_1 and P_2 have a timelike separation, then P_2 lies inside the light cone and $\theta < 45°$. Hence in a boosted frame with $\beta = \tan\theta < 1$ the two events will occur at the same spatial location. In $\bar{F}$, $\sqrt{-\Delta s^2} = \Delta\tau = \Delta\bar{t}$

(d)



Analogously P_2 will lie outside of the light cone and hence the angle θ (between $\overrightarrow{P_1P_2}$ and the x-axis) is less than 45°. By boosting into $\bar{F}$ with $\tan\theta = \beta < 1$ we see that $\overrightarrow{P_1P_2}$ is parallel to the $\bar{x}$ axis, i.e in $\bar{F}$ these two events are simultaneous. And $\sqrt{\Delta s^2} = |\Delta\vec{x}|$.

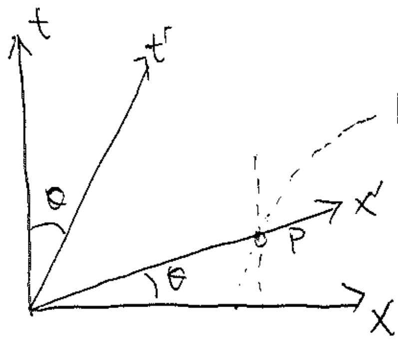
(e)



In the figure, the hyperbola is given by $t^2 - x^2 = \bar{t}^2$. Let's consider how much time will elapse as measured by observers in F and $\bar{F}$ between O and P. $\Delta\bar{t}^2 = \Delta t^2 - \Delta t^2 \tan^2\theta$
$= \Delta t^2(1-\beta^2)$

and thus $\Delta\bar{t} = \Delta t/\gamma$, time is slowed in a boosted frame.

(f)



In the figure, the hyperbola is given by $x^2 - t^2 = \bar{x}^2$. By analogous reasoning, $\Delta\bar{x}^2 = \Delta x^2 - \Delta x^2 \tan^2\theta$
$= \Delta x^2 (1-\beta^2)$.
thus $\Delta\bar{x} = \Delta x/\gamma$, i.e. objects in a boosted frame are contracted along the boost. Since there are no boosts along y and z, the length along those axe is unchanged.

2.26. (a) The stress-energy tensor should be a symmetric tensor made from $\vec{u}$, g, ρ and P, so it must be the form $T^{\alpha\beta} = A u^\alpha u^\beta + B g^{\alpha\beta}$.
In the local rest frame, $T^{jk} = P\delta^{jk}$ tells us $B = P$. $T^{00} = \rho$ tell us $A = \rho + B = \rho + P$; note $T^{0j} = 0$ is satisfied automatically. Thus we have derived the stress energy tensor.

(b) $$T = (\rho + P)\, \vec{u} \otimes \vec{u} + P g = P \mathcal{P} + \rho(\mathcal{P} - g)$$

where $\mathcal{P} = g + \vec{u} \otimes \vec{u}$, $\mathcal{P}(\_, \vec{A}) = \vec{A} + (\vec{A} \cdot \vec{u})\vec{u}$

$$T^{\alpha\beta}{}_{;\beta} = P_{;\beta}\mathcal{P}^{\alpha\beta} + P\mathcal{P}^{\alpha\beta}{}_{;\beta} + \rho_{;\beta} u^\alpha u^\beta + \rho P^{\alpha\beta}{}_{;\beta}$$

$$P^{\alpha\beta}{}_{;\beta} = u^\alpha{}_{;\beta} u^\beta + u^\alpha u^\beta{}_{;\beta} = a^\alpha + u^\alpha u^\beta{}_{;\beta}$$

$$\Rightarrow \nabla \cdot T = \mathcal{P}(\_, \nabla P) + \frac{d\rho}{d\tau}\vec{u} + (P + \rho)(\vec{a} + \vec{u}(\vec{\nabla} \cdot \vec{u}))$$

(b) Now we can write $\vec{u}\cdot(\nabla\cdot T)=0$ as

$$\underset{\substack{\parallel \\ 0}}{P(\vec{u},\nabla p)} - \frac{d\rho}{d\tau} - (P+\rho)(\nabla\cdot\vec{u}) = 0 \quad , \quad \vec{\nabla}\cdot\vec{u} = \frac{1}{V}\frac{dV}{d\tau}$$

$$\Rightarrow -\frac{d\rho}{d\tau} - \frac{P+\rho}{V}\frac{dV}{d\tau} = 0 \quad \Rightarrow \frac{d(\rho V)}{d\tau} = -P\frac{dV}{d\tau}$$

or $dE = d(\rho V) = -PdV$. This is ~~the~~ 1st law of thermodynamics for constant entropy.

(c) In the rest frame of the fluid, momentum conservation is given by $T^{i\nu}{}_{,\nu} = T^{i0}{}_{,0} + T^{ij}{}_{,j} = 0$, since T^{i0} and T^{ij} are the momentum density and momentum density flux, respectively. These are the spatial components of $\vec{\nabla}\cdot T = 0$. We know that in the rest frame of the fluid, the projection tensor $P = g + \vec{u}\otimes\vec{u}$ has the form $P_{\mu\nu} = \eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu$. Acting on a vector in the fluid rest frame, the projection operator yields $P_{\mu\nu}A^\nu = (0, A^i)$. Acting on tensor $P_{\mu\sigma}T^{\sigma\mu}{}_{,\nu} = (0, T^{i\nu}{}_{,\nu})$. Therefore, in the rest frame of the fluid, the law of momentum conservation is given by

$$P_{\sigma\alpha}T^{\alpha\beta}{}_{,\beta} = 0 \quad , \text{ or } \quad P(-, \vec{\nabla}\cdot T) = 0$$

(c) In part b, we found

$$\vec{\nabla}\cdot T = \mathcal{P}(\_,\vec{\nabla} p) + \frac{d\rho}{d\tau}\vec{u} + (p+\rho)(\vec{a} + \vec{u}(\vec{\nabla}\cdot\vec{u}))$$

and $$\vec{u}\cdot(\vec{\nabla}\cdot T) = -\frac{d\rho}{d\tau} - (p+\rho)(\vec{\nabla}\cdot\vec{u})$$

Therefore, the law of momentum conservation is

$$\mathcal{P}(\_,\vec{\nabla}\cdot T) = \vec{\nabla}\cdot T + \vec{u}[\vec{u}\cdot(\vec{\nabla}\cdot T)] = \mathcal{P}(\_,\vec{\nabla} p) + (p+\rho)\vec{a}$$
$$= 0$$

i.e. $(p+\rho)\vec{a} = -\mathcal{P}(\_,\vec{\nabla} p)$

This is the analogy of "$\vec{F} = m\vec{a}$" for a perfect fluid, where the inertial mass is $p+\rho$. The left side is the inertial mass of the fluid times the four acceleration, while the right hand side is "the "four force", which is the negative gradient of the projection of the pressure into fluid's 3-space.

Ph 136: Solution for Chapter 2

3.6 Observations of Cosmic Microwave Radiation from a Moving Earth [by Alexander Putilin]

(a)

$$
\begin{aligned}
&I_\nu = \frac{h^4\nu^3}{c^2}\mathcal{N} \\
&\mathcal{N} = \frac{g_s}{h^3}\eta = \frac{2}{h^3}\eta \ (for\ photons) \\
&\Longrightarrow I_\nu = \frac{2h\nu^3}{c^2}\eta = \frac{(2h/c^2)\nu^3}{e^{h\nu/kT_0}-1} \\
&in\ its\ mean\ rest\ frame.
\end{aligned}
$$

let $x = h\nu/kT_0$,

$$I_\nu = \frac{2(kT_0)^3}{h^2c^2}\frac{x^3}{e^x-1} = (3.0\times 10^{-15}\frac{erg}{cm^2})\frac{x^3}{e^x-1}$$

from Fig. 1, we see the intensity peak is at $x_m = 2.82$, which corresponds to $\nu_m = 1.6\times 10^{11}s^{-1}, \lambda_m = 0.19cm$.

(b) From chapter 1, we already know that the photon's energy as measured in the mean rest frame is $h\nu = -\vec{p}\cdot\vec{u}_0$, then (2.28) follows immediately.

(c) Let $\mathbf{n}$ be the direction at which the receiver points, and $\mathbf{v}$ be the earth's velocity w.r.t. to the microwave background, then in the earth's frame, $\vec{u}_0 = (1/\sqrt{1-\mathbf{v}^2}, -\mathbf{v}/\sqrt{1-\mathbf{v}^2})$, $\vec{p} = (h\nu, -h\nu\mathbf{n})$. Plugging the above expressions into (2.28), we find(let θ be the angle between $\mathbf{v}$ and $\mathbf{n}$)

$$
\begin{aligned}
I_\nu = \frac{2h\nu^3}{c^2}\eta = \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT}-1} \\
with\ T = T_0\left(\frac{\sqrt{1-v^2}}{1-vcos\theta}\right)
\end{aligned}
$$

For small v, we can keep only terms linear in v and find $T \approx T_0(1+vcos\theta)$ which exhibits a dipolar anisotropy. And the maximal relative variation $\Delta T/T \approx (T(\theta=0) - T(\theta=\pi))/T_0 = 2v/c = 4\times 10^{-3}$.

3.3 Regimes of Particulate and Wave-like Behavior [by Jeff Atwell]

(a) The equations in the text can be used to relate the occupation number

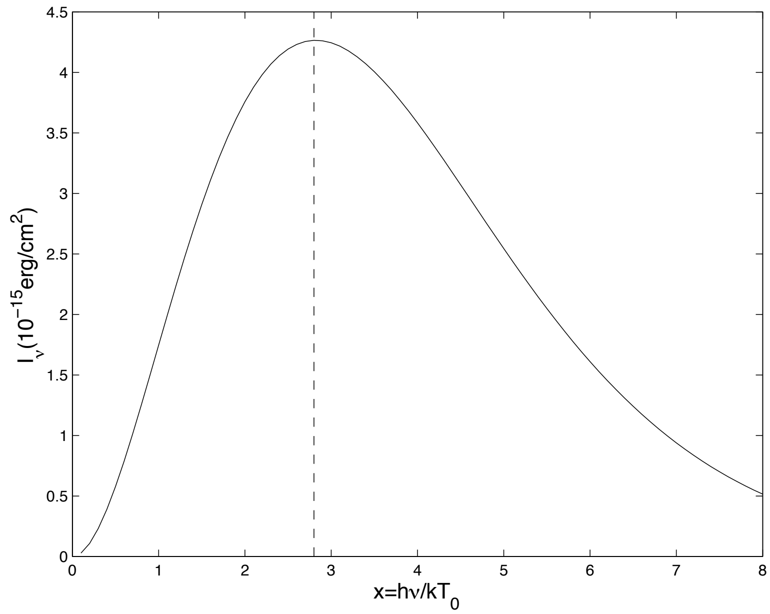


Figure 1: Ex. 2.6

to the specific intensity

$$\eta = \frac{h^3}{2}\mathcal{N} = \frac{c^2}{2h\nu^3} I_\nu.$$

We also have the equation

$$I_\nu = \frac{dE}{dAdtd\nu d\Omega}.$$

We are told that when the radiation reaches earth it has $dE/dA \sim 10^{-6}$ ergs/cm^2, and $dt = 1$ sec. We should take $d\Omega = (1000 \text{ km})^2/(10^{10} \text{ light years})^2$, and might take $d\nu \sim \nu$ (and one finds ν from $\mathcal{E} \sim 100keV$). With this I find $\eta \sim 2000$. This is not as small as what was expected, and implies that the photons behave more like waves in this case. The occupation number will stay the same as the photons propagate because the energy flux dE/dA dies out as $1/r^2$, and the solid angle subtended by the source dies out as $d\Omega \propto 1/r^2$, so $dE/dAd\Omega$ is independent of radius, as are all other quantities that enter into I_ν and then into η.

(b) We are given the total dE in terms of the mass of the sun. An observer at distance r from the source sees an energy flux $dE/dA = dE/(4\pi r^2)$, and sees this coming from a solid angle $d\Omega \sim \lambda^2/r^2$. Therefore, $dE/dAd\Omega \sim dE/(4\pi\lambda^2)$. Here $\lambda = c/\nu$, and $\nu \sim 1000$ Hz. We are also given $d\nu \sim 1500$ Hz, and $dt \sim 10^{-2}$ sec. Also, $g_s = 2$ for gravitons. With this I calculate $\eta = 10^{72}$, and so the gravitons will behave like a classical gravitational wave.

3.11 Specific Heat for Phonons in an Isotropic Solid [by Jeff Atwell]

(a) As with blackbody radiation, each mode in a solid is a harmonic oscillator with some well-defined frequency ν. Like all harmonic oscillators, this mode must have uniformly spaced energy levels, with an energy spacing

$$\epsilon = h\nu$$

When the mode is excited into its N'th energy level, we can regard it as having N quanta, i.e. N phonons; i.e., N will be its occupation number. Since the mode, like any harmonic oscillator, can have any occupation number $0 \leq N < \infty$, it must obey Bose-Einstein statistics rather than Fermi-Dirac statistics, and its quanta — its phonons — must be bosons.

(b) Since phonons are bosons, and $\mu_R = 0$, and each phonon has an energy $\mathcal{E} = h\nu$, the distribution functions will be the same as for blackbody radiation (equation (2.21)): the mean number of quanta N in a mode with frequency ν will be

$$\eta = \frac{1}{e^{h\nu/kT} - 1},$$

and the number density of phonons in phase space will be

$$\mathcal{N} = \frac{g_s}{h^3} \frac{1}{e^{h\nu/kT} - 1}.$$

In this expression we must think of ν as the frequency of a phonon whose energy is $\mathcal{E} = h\nu$ and whose momentum is $p = \mathcal{E}/c_s = h\nu/c_s$.

(c) To calculate the total energy in one type of sound wave (longitudinal or transverse), we integrate $\mathcal{N}\mathcal{E}$ over phase space, i.e. we multiply by the volume V of our solid and we integrate over momentum space using spherical coordinates so $d\mathcal{V}_p = 4\pi p^2 dp$:

$$\mathcal{E}_{Total} = \int \mathcal{N}\mathcal{E} V d\mathcal{V}_p = \int_0^\infty \left(\frac{g_s}{h^3} \frac{1}{e^{c_s p/kT} - 1} \right) (c_s p) 4\pi p^2 dp\ .$$

Change to the dimensionless variable $x = c_s p/kT$, to find

$$\mathcal{E}_{Total} = g_s \frac{4\pi k^4}{h^3 c_s^3} V T^4 \int_0^\infty \frac{x^3}{e^x - 1} dx.$$

The integral can be evaluated in terms of the Bernoulli number [Eq. (2.48c) of the text, plus Table 2.1]. The final result is

$$\mathcal{E}_{Total} = g_s \frac{4\pi^5 k^4}{15 h^3 c_s^3} V T^4,$$

which is what we wanted to show.

(d) To get the heat capacity, differentiate $\mathcal{E}_{Total}$ with respect to T:

$$C_V = 4a_s T^3 V.$$

(e) The phonon frequency and wavelength are related by $\nu = c_s/\lambda$, so the thermal distribution function is given by

$$\eta = \frac{1}{e^{hc_s/\lambda kT} - 1} = \frac{1}{e^{\lambda_T/\lambda} - 1}.$$

From this we can see that when $\lambda \ll \lambda_T$, $\eta \ll 1$; and for $\lambda \sim \lambda_T$, $\eta \sim 1$; and for $\lambda \gg \lambda_T$, $\eta \gg 1$.

The atomic spacing a_o puts a lower limit on the wavelengths of the phonons, $\lambda_{\rm min} = 2a_o$, corresponding to an upper limit on their energies:

$$\mathcal{E}_{max} = h\nu_{\rm max} = hc_s/\lambda_{\rm min} = \frac{hc_s}{2a_o}\ .$$

This may be safely ignored at low temperature. But when

$$kT \sim \mathcal{E}_{\rm max},$$

the computation will fail. $T_D \sim \mathcal{E}_{\rm max}/k$ is the Debye temperature. Once we hit the Debye temperature, the total number of modes saturates to some number $N_{\rm modes}$, as adding higher frequency (lower λ) modes would force us to exceed the fundamental lattice spacing.

When $kT > \mathcal{E}_{\rm max}$, every mode should hold an energy $\sim kT$, and so then $\mathcal{E}_{Total} \sim g_s N_{\rm modes} kT$, where $N_{\rm modes}$ is the total number of modes of vibration of the solid. This is the "equipartition theorem" in thermal physics. Then $C_V = d\mathcal{E}_{\rm total}/dT = g_s N_{\rm modes} k$ will be independent of temperature.

3.9 Equation of State for Electron-Degenerate Hydrogen [by Alexander Putilin]

Mean occupation number of electron gas:

$$\eta = \frac{1}{e^{\frac{\tilde{E}-\tilde{\mu}_e}{kT}} + 1},\ \ \tilde{E}^2 = p^2 + m_e^2$$

Gas is degenerate if $\tilde{\mu}_e - m_e >> kT$. In this limit $\eta(\tilde{E})$ looks like Fig. 2

The width of the "transition" region where $\eta(\tilde{E})$ goes from 0 to 1 is $\sim kT$, so in the limit $\tilde{\mu}_e - m_e >> kT$ we can approximate $\eta(\tilde{E})$ by step function: $\eta(\tilde{E}) = 1$ for $\tilde{E} < \tilde{\mu}_e$; $\eta(\tilde{E}) = 0$ for $\tilde{E} > \tilde{\mu}_e$.

The number density n is given by

$$\begin{aligned} n &= \int_0^\infty 4\pi\mathcal{N}p^2 dp = 4\pi\frac{2}{h^3}\int_0^\infty \eta p^2 dp \\ &= \frac{8\pi}{h^3}\int_0^{p_F} p^2 dp = \frac{8\pi}{3h^3}p_F^3 \end{aligned}$$

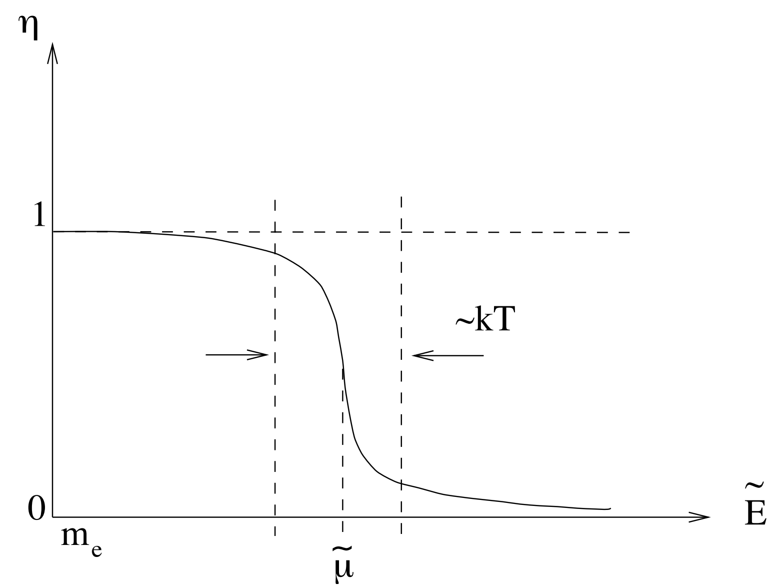


Figure 2: Ex. 2.9

where $p_F = \sqrt{\tilde{\mu}_e^2 - m_e^2}$.
The energy density $\rho = \rho_p + \rho_e$. Protons are nonrelativistic so $\rho_p = m_p n = 8\pi m_p p_F^3/3h^3$. while

$$\begin{aligned}
\rho_e &= 4\pi \int_0^\infty \mathcal{N}\tilde{E}p^2 dp = \frac{8\pi}{h^3}\int_0^{p_F} \sqrt{p^2 + m_e^2}\, p^2 dp \\
\rho_e &\approx m_e n,\ if\ p_F << m_e \text{ (non-relativistic case);} \\
\rho_e &\approx \frac{2\pi}{h^3} p_F^4,\ if\ p_F >> m_e \text{ (ultra-relativistic case)}
\end{aligned}$$

In both cases $\rho_e << \rho_p$, provided that $p_F << m_p$, i.e. protons remain non-relativistic. Thus

$$\rho \approx \rho_p = \frac{8\pi m_p}{3h^3} p_F^3 = \frac{8\pi m_p}{3(h/m_e)^3} x^3, \quad x = \frac{p_F}{m_e}$$

Now turn to pressure. Electron's pressure

$$\begin{aligned}
P_e &= \frac{4\pi}{3}\int_0^\infty \mathcal{N}\tilde{E}^{-1} p^4 dp = \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4 dp}{\sqrt{p^2 + m_e^2}} \\
&= \frac{8\pi m_e^4}{3h^3}\int_0^x \frac{z^4 dz}{\sqrt{z^2+1}}\ (let\ z = \frac{p}{m_e}) \\
&= \frac{\pi m_e^4}{h^3}\psi(x),\ where\ \psi(x) = \frac{8}{3}\int_0^x \frac{z^4 dz}{\sqrt{1+z^2}}
\end{aligned}$$

Using Mathematica we find

$$\psi(x) = sinh^{-1}x - x\left(1 - \frac{2x^2}{3}\right)\sqrt{1+x^2}$$

$$for\ x << 1,\ \psi(x) \approx \frac{8}{15}x^5;\ for\ x >> 1,\ \psi(x) \approx \frac{2}{3}x^4$$

Proton pressure $P_p = nkT << P_e$ in both cases. Thus

$$P \approx P_e = \frac{\pi m_e^4}{h^3}\psi(x)$$