Ph136a Solution 1.

1. Tensor Basis

$$T(\vec{A},\vec{B},\vec{C}) = A_i B_j C_k T(e_i, e_j, e_k) = T_{ijk}\, e_i(\vec{A})\, e_j(\vec{B})\, e_k(\vec{C})$$

$$\Rightarrow T_{ijk} = T(e_i, e_j, e_k)$$

2. The Slot Naming Index Notation

(a) $A_i B_{jk} \longrightarrow A(\_) \otimes B(\_,\_)$

$A_i B_{ji} \longrightarrow B(\_, A)$

$S_{ijk} = S_{kji} \longrightarrow S(A,B,C) = S(C,B,A)$

$A_i B_i = A_i B_j g_{ij} \longrightarrow A(B) = g(A,B)$

(b) $T(\_,\_,A) \longrightarrow T_{ijk} A_k$

$T(\_, S(B,\_),\_) \longrightarrow T_{ijk} S_{\ell j} B_\ell$

3. Rotation in the x-y plane

(a)
$$\begin{cases} e_{\bar{x}} = e_x \cos\phi + e_y \sin\phi \\ e_{\bar{y}} = -\sin\phi\, e_x + \cos\phi\, e_y \\ e_{\bar{z}} = e_z \end{cases} \Rightarrow R_{\bar{p}i} = \begin{pmatrix} \cos\phi & \sin\phi & \\ -\sin\phi & \cos\phi & \\ & & 1 \end{pmatrix}$$

(b) $\begin{cases} x_{\bar{1}} = \cos\phi\, x + \sin\phi\, y \\ x_{\bar{2}} = -\sin\phi\, x + \cos\phi\, y \\ x_{\bar{3}} = z \end{cases} \Rightarrow x_{\bar{p}} = R_{\bar{p}i}\, x_i$

(c) $A_{\bar{x}} + i A_{\bar{y}} = A_x(\cos\phi - i\sin\phi) + A_y(i\cos\phi + \sin\phi)$

$= e^{-i\phi}(A_x + i A_y)$

(d) Using $h_{\bar{\alpha}\bar{\beta}} = R_{\bar{\alpha}i} R_{\bar{\beta}j} h_{ij}$

$\Rightarrow \begin{cases} h_{\bar{x}\bar{x}} = (\cos^2\phi - \sin^2\phi) h_{xx} + 2\cos\phi\sin\phi\, h_{xy} \\ h_{\bar{x}\bar{y}} = \cos 2\phi\, h_{xy} - \sin 2\phi\, h_{xx} \\ h_{\bar{x}\bar{x}} + i h_{\bar{x}\bar{y}} = e^{-2i\phi}(h_{xx} + i h_{xy}) \end{cases}$

4. Cross Product and Curl

(a) $[\nabla\times(\nabla\times\vec{A})]_i = \varepsilon_{ijk}\nabla_j(\varepsilon_{klm}\nabla_l A_m)$

$= \varepsilon_{ijk}\varepsilon_{lmk}\nabla_j\nabla_l A_m = \nabla_i(\nabla_m A_m) - \nabla_l(\nabla_l A_i)$

$\Rightarrow \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

(b) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) \rightarrow$

$\varepsilon_{ijk}A_jB_k\varepsilon_{ilm}C_lD_m = A_jC_jB_kD_k - A_jD_jB_kC_k$

$\Rightarrow (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

(c) $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) \rightarrow (\varepsilon_{\ell jk}A_jB_k)\varepsilon_{i\ell m}(\varepsilon_{mnp}C_nD_p)$

$= \varepsilon_{\ell jk}A_jB_kC_iD_\ell - \varepsilon_{\ell jk}A_jB_kC_\ell D_i$

$\Rightarrow (\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = (\vec{D}\cdot(\vec{A}\times\vec{B}))\vec{C} - (\vec{C}\cdot(\vec{A}\times\vec{B}))\vec{D}$

5. Integral over a sphere

(a) It's straight forward to see the two legs are $a\,d\theta\,\hat{e}_\theta$ and $a\sin\theta\,d\phi\,\hat{e}_\phi$

$\Rightarrow d\vec{\Sigma} = a\,d\theta\,\hat{e}_\theta \times a\sin\theta\,d\phi\,\hat{e}_\phi$

$= \epsilon(-,\hat{e}_\theta,\hat{e}_\phi)a^2\sin\theta\,d\theta\,d\phi$

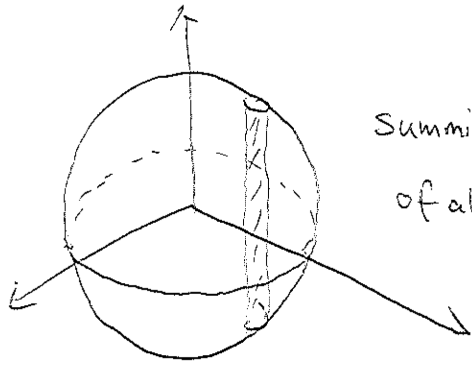
(b) $e_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\phi$

$\vec{A}\cdot d\vec{\Sigma} = a^3\cos^2\theta\sin\theta\,d\theta\,d\phi\,\hat{e}_r\cdot(\hat{e}_\theta\times\hat{e}_\phi)$

$= a\cos^2\theta\,\epsilon(\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi)\,a^2\sin\theta\,d\theta\,d\phi$

(c) $\epsilon(\hat{e_r},\hat{e}_\theta,\hat{e}_\phi)$ because $\{\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi\}$ ~~are~~ is a right-handed orthonormal basis

(d) $\int \vec{A}\cdot d\Sigma = \frac{4\pi a^3}{3} =$



Summing over the volumn of all the tubes

6. Equation of Motion for a perfect fluid

(a). $T_{jk} = \begin{pmatrix} j\text{- component of force per unit area} \\ \text{across a surface perpendicular to } \vec{e}_k \end{pmatrix}$

$$= P\delta_{jk} + \rho v_j v_k$$

$$\Rightarrow \overleftrightarrow{T} = Pg + \rho \vec{v} \otimes \vec{v}$$

(b) $\frac{d}{dt}$ (mass in V) $= -\oint_S$ mass flow $\cdot d\Sigma$

$$\Rightarrow \frac{d}{dt}\int_V \rho \, dv = -\oint \rho \vec{v} \cdot d\vec{\Sigma} = -\int_V \nabla \cdot (\rho \vec{v}) \, dv$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

(c) $$f(\vec{x}, t) = f'(\vec{x} + \vec{v}t, t)$$

$$\Rightarrow \frac{df}{dt} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right) \vec{f}'$$

(d) $$\frac{\partial \rho}{\partial t} + \vec{v} \cdot (\nabla \rho) + \rho(\vec{\nabla} \cdot \vec{v}) = \frac{d\rho}{dt} + (\vec{\nabla} \cdot \vec{v})\rho$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dt} = -\vec{\nabla} \cdot \vec{v}$$

(e) $$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot(\overleftrightarrow{T}) = 0$$

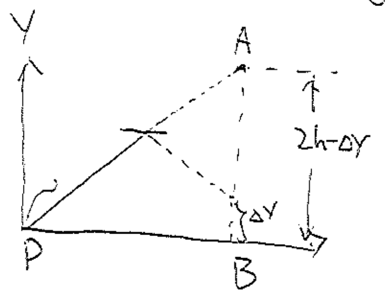
Here $\overleftrightarrow{T} = Pg + \rho\vec{v}\otimes\vec{v}$

$$\Rightarrow \quad \rho\frac{\partial\vec{v}}{\partial t} + \frac{\partial\rho}{\partial t}\vec{v} = -\nabla P - [(\vec{v}\cdot\nabla)\rho]\vec{v} - \rho(\nabla\cdot\vec{v})\vec{v} - \rho(\vec{v}\cdot\nabla)\vec{v}$$

$$\Rightarrow \quad \rho\frac{d\vec{v}}{dt} = -\nabla P$$

7. Derivation of invariance interval Δs^2

(a). The (angle of reflection) = (angle of incidence) should be true regardless of the mirror's motion in x direction.



$PA = c\Delta t$, $PB = \Delta x$; $c=1$

$$\Rightarrow -(\Delta t)^2 + \Delta x^2 = BA^2 = -(2h-\Delta y)^2$$

$$\Rightarrow (\Delta s)^2 = -(2h-\Delta y)^2 + \Delta y^2$$

Similarly we have $(\Delta s')^2 = -(2h'-\Delta y')^2 + \Delta y'^2$

(b) $\Delta y' = \Delta y$, $h = h'$ because of the symmetry between prime and unprimed frame $\rightarrow$ $\Delta s^2 = \Delta s'^2$

8. Another derivation of the invariance interval Δs^2

(a) For a linear invertible transformation

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} , \quad (\Delta \bar{s})^2 = (\bar{t}\ \bar{x}\ \bar{y}\ \bar{z})\, M^T \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} M \begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

The matrix $M^T \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} M$ has the same signature as $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

(b) Constant speed of light

$\rightarrow$ Considering two points on the light trajectory

$$-t^2 + x^2 + y^2 + z^2 = 0 \Leftrightarrow -\bar{t}^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2$$

(c) Since $\Delta s^2 = A$ asymptotes $\Delta s^2 = 0$ for large distance away from the
(d) origin, $\Delta \bar{s}^2 = B$ should have the same behavior

$$\Rightarrow \quad \Delta \bar{s}^2 = -\bar{t}^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2$$

(e)(f) The light cone (or the asymptotic limit) are the same

$\Rightarrow \Delta \bar{s}^2 = B/A\, \Delta s^2$ also due to symmetry

$\Rightarrow B/A = 1$