Ph 136: Solution 3 for Chapter 3

3.14 Solar Heating of the Earth: The Greenhouse Effect [by Alexander Putilin]

(a) The energy per unit time per unit frequency emitted by the surface element dA of the sun into the solid angle $d\Omega$ centered around unit vector $\hat{\mathbf{n}}$ is (see Fig. 1) $d\tilde{E}/dt = I_\nu dA cos\theta d\Omega d\nu$. And the total energy flux is thus

$$F = \int I_\nu cos\theta d\Omega d\nu = \int_0^{\pi/2} 2\pi sin\theta d\theta \int_0^{+\infty} d\nu\ cos\theta \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT_\odot} - 1}$$
$$= \frac{2\pi k^4 T_\odot^4}{c^2 h^3} \int_0^{+\infty} \frac{x^3 dx}{e^x - 1} \quad (let\ x = h\nu/kT_\odot)$$

The value of the above integral is $\pi^4/15$, thus we find

$$F = \sigma T_\odot^4, \quad where\ \sigma = \frac{2\pi^5}{15} \frac{k^4}{h^3 c^2}$$

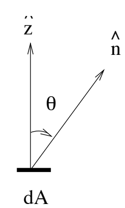

Figure 1: Ex. 2.14a

(b) (See Fig. 2). Similarly, the flux arriving at the earth is given by

$$F_e = \int I_\nu cos\theta d\Omega d\nu = \int_0^{\theta_0} 2\pi sin\theta cos\theta d\theta \int I_\nu d\nu = sin^2\theta_0 \sigma T_\odot^4$$

From Fig.4, we see $sin\theta_0 = R_\odot/r$. Thus

$$F_e = \sigma T_\odot^4 \left(\frac{R_\odot}{r}\right)^2$$

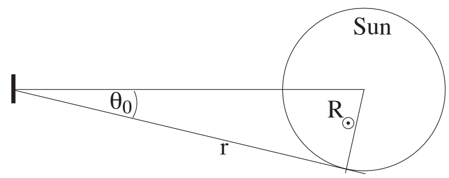

Figure 2: Ex. 2.14b

(c) (See Fig. 3) Radiated power $\left(d\tilde{E}/dt\right)_{radiated} = \sigma T_{\oplus}^4 4\pi R_{\oplus}^2$, while absorbed power $\left(d\tilde{E}/dt\right)_{absorbed} = F_e \int_{\theta=0}^{\theta=\pi/2} R_{\oplus}^2 cos\theta d\Omega = \pi F_e R_{\oplus}^2$. Then in thermal equilibrium, $\left(d\tilde{E}/dt\right)_{radiated} = \left(d\tilde{E}/dt\right)_{absorbed}$ immediately tells us

$$T_{\oplus} = T_{\odot}\left(\frac{R_{\odot}}{2r}\right)^{1/2} = 280K$$

Figure 3: Ex. 2.14c

(d) If we take albedo A into account, $\left(d\tilde{E}/dt\right)_{absorbed} = (1-A)\pi R_{\oplus}^2 F_e$, while $\left(d\tilde{E}/dt\right)_{radiated}$ remains the same. Thus we get

$$T_{\oplus} = T_{\odot}\left(\frac{\sqrt{1-A}\; R_{\odot}}{2r}\right)^{1/2} = 255K$$

(e) As given, the problem's numbers yield an insanely high temperature for the Earth's surface temperature. We will use a Greenhouse factor of $G = 0.58$. Due to Greenhouse Effect, $\left(d\tilde{E}/dt\right)_{radiated} = G \cdot 4\pi R_{\oplus}^2 \sigma T_{\oplus}^4$, and then $\left(d\tilde{E}/dt\right)_{radiated} = \left(d\tilde{E}/dt\right)_{absorbed}$ gives us $T_{\oplus} = 293K$.

3.15 Olber's Paradox and Solar Furnace [by Alexander Putilin]

Place an observer at some spot on the earth and choose some arbitrary direction $\hat{\mathbf{n}}$. (See Fig. 3)

Since the universe is assumed to be flat, it must be infinite in space and time so the observer will see some star in that direction.
Vlasov equation then gives $I_\nu(\hat{\mathbf{n}})/\nu^3 = I_\nu/\nu^3|_{at\ the\ star's\ surface}$. And since there's no gravitational and Doppler shifts in a flat stationary universe: $I_\nu(\hat{\mathbf{n}}) = I_\nu(at\ the\ star's\ surface)$.
The energy flux received by the observer is (see Ex. 2.11)

$$F = \int I_\nu cos\theta d\Omega d\nu = \int_0^{\pi/2} 2\pi sin\theta d\theta \int_0^{+\infty} d\nu\ cos\theta \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT_{star}} - 1}$$
$$= \sigma T^4,\ where\ T^4 =< T_{star}^4 >= \sum_{i=1}^{N} \frac{1}{N} T_i^4$$

the summation is over all the visible stars and T_i is the temperature of the i-th star.
The hotter stars will dominate in this sum, so that $T \approx T_{hotter\ stars} \approx 10^4 K$.
When the earth come into thermal equilibrium $F = \sigma T_\oplus^4$, so its surface temperature will be $T_\oplus = T \approx 10^4 K$.
We are protected from being from fried because the universe is not stationary but rather is expanding (and has finite lifetime). The stars first formed when the universe was about 2 billion years old (at a redshift ~ 5). When we look out beyond that point, we see no more stars or galaxies. This means that only a small fraction of our sky is actually covered by stellar surfaces.
Now let's talk about solar furnace(see Fig. 4). We can use a lens of large

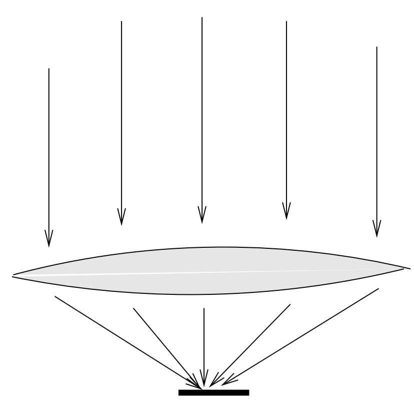

Figure 4: Ex. 2.15: Solar Furnace

diameter D and small focal length $f << D$ to focus the sun's rays. At the spot where the rays are focused, the specific intensity I_ν is the same as at the surface of the sun:

$$I_\nu = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT_\odot} - 1}$$

And the energy flux $F = \int I_\nu cos\theta d\Omega d\nu$, where the integration over the solid angle is almost over the whole upper hemisphere: $0 \leq \theta \leq \pi/2$. Thus we find $F = \sigma T_\odot^4$. So at equilibrium the temperature of the spot is $T = T_\odot$.
The effect of the lens is that it enlarges the image of the sun so that the image is spread over almost all the sky.

3.20 Neutron Diffusion in a Nuclear Reactor [by Jeff Atwell]

(a) Denote the distribution function by $n_E(E, t)$.
Use $n_s, n_a, \sigma_s, \sigma_a$ to denote the density of scattering centers (i.e. moderator atoms), absorbing centers (i.e. ^{238}U atoms), the scattering cross section, and the absorption cross section, respectively.
A neutron with speed $v = \sqrt{2E/m}$ has a probability of getting scattered per unit time $vn_s\sigma_s$, and a probability of getting absorbed per unit time $vn_a\sigma_a$.
Now we must use the energy decrement during each scattering. We are given

$$\xi = -d(\ln E) \approx 0.158.$$

This may be rewritten easily as

$$dE = -\xi E.$$

It follows that the rate of a neutron slowing down is

$$\frac{dE}{dt} = -vn_s\sigma_s\xi E.$$

Then the "flux" q will be this quantity multiplied by the distribution function:

$$q = \frac{dN}{d^3xdt} = \left(\frac{dN}{d^3xdE}\right)\left(\frac{-dE}{dt}\right) = -n_E\frac{dE}{dt} = vn_s\sigma_s\xi En_E.$$

In the steady state, when $n_E(E, t) = n_E(E)$, q will also be a function of the energy only.

(b) Consider a tiny interval of energy δE at a neutron energy E that might or might not be in the energy range where absorption occurs. The rate per unit volume at which neutrons enter this interval via scattering from higher energies is

$$\frac{dn_{\text{enter}}}{dt} = q(E)\frac{\delta E}{\xi E}$$

[because q is the rate per unit volume at which neutrons scatter downward in energy through E, on average each neutron loses ξE of energy in its scattering, so the fraction of them that wind up in the tiny interval δE is $\delta E/(\xi E)$.] Neutrons leave δE due to both scattering and absorption. The number of neutrons per unit volume in δE is $n_E\delta E$, and they move with speed v, encountering $n_a\sigma_a$

absorbers per unit length of their motion, and encountering $n_s\sigma_s$ scatterers. Therefore, the rate at which they leave δE is

$$\frac{dn_{\rm leave}}{dt} = n_E \delta E v(n_a\sigma_a + n_s\sigma_s) \ .$$

Because the neutron distribution is in a steady state, the rates of entering and leaving δE must be equal, which means

$$q = n_E v \xi E(\sigma_s n_s + \sigma_a n_a) \ . \tag{1}$$

Thus, the relation between q and n_E is different in the absorbing region than in the non-absorbing region: it depends on $\sigma_a n_a$.
Next we shall derive a differential equation for q. We shall do so using the law of conservation of particles in energy space. Just as in physical space the number density n and flux S_j of particles satisfies the conservation law $\partial n/\partial t + \partial S_j/\partial x_j = (dn/dt$ due to particle creation or destruction), so similarly in energy space

$$\frac{\partial n_E}{\partial t} + \frac{\partial(-q)}{\partial E} = \frac{dn_E}{dt}_{\rm absorption} = -n_E v\sigma_a n_a \ .$$

Here the negative sign on q is because q is the flux *downward* in energy space, but the conservation law requires the flux *upward*. The first term on the left side of this equation vanishes because the neutron distribution is in a steady state. Rewriting the partial derivative as an ordinary derivative (because q depends only on energy E), we obtain

$$dq/dE = -n_E v \sigma_a n_a \ .$$

Combining with Eq. (1) for q, we obtain the claimed result:

$$\frac{d\ln q}{d\ln E} = \frac{E}{q}\frac{dq}{dE} = \frac{\sigma_a n_a}{(\sigma_s n_s + \sigma_a n_a)\xi} \ .$$

Outside the absorption region σ_a vanishes and the neutron flux in energy space is conserved, $q =$ constant. In the absorption region q decreases with energy as described by this equation.

(c) Integrate the differential equation from 7 eV to 6 eV:

$$\ln\left(\frac{q(E=7\,eV)}{q(E=6\,eV)}\right) = \frac{\sigma_a n_a}{\xi\,(\sigma_s n_s + \sigma_a n_a)} \ln{(7/6)}.$$

In order to maintain the chain reaction, we need $q(E=6\,eV)/q(E=7\,eV) \approx 1/2$. Now plug this in and solve for n_s/n_a:

$$\frac{n_s}{n_a} \approx \frac{\sigma_a}{\sigma_s}\left(\frac{\ln{(7/6)}}{\xi \ln 2} - 1\right) \approx 140,$$

where I have used $\sigma_s = 4.8$ barns and $\sigma_a = 1600$ barns.