Ph 136: Solution 5 for Chapter 5 and 6

5.5 Electron-Positron Equilibrium at "Low" Temperatures [by Xinkai Wu], [modified by Kip Thorne and Dan Grin]

(a) The reaction equation $e^- + p \longrightarrow e^- + p + e^- + e^+$ gives $\tilde{\mu}_{e^-} + \tilde{\mu}_p = 2\tilde{\mu}_{e^-} + \tilde{\mu}_p + \tilde{\mu}_{e^+}$, which implies $\tilde{\mu}_{e^-} + \tilde{\mu}_{e^+} = 0$, i.e.

$$\boxed{\tilde{\mu}_{e^-} = -\tilde{\mu}_{e^+}.}$$

(b) In what follows we shall use $\tilde{\mu}_-$ to denote $\tilde{\mu}_{e^-}$, and $\tilde{\mu}_+$ to denote $\tilde{\mu}_{e^+}$. The distribution function(density in phase space) for positrons and electrons (both having $g_s = 2$) is given by

$$\mathcal{N}_\pm = \frac{2}{h^3}\eta_\pm = \frac{2}{h^3}\frac{1}{e^{(\tilde{E}-\tilde{\mu}_\pm)/kT}+1} \tag{1}$$

$$\tilde{E} \approx mc^2 + \mathbf{p}^2/2m \tag{2}$$

$$\boxed{\Rightarrow \mathcal{N}_\pm = \frac{2}{h^3}\frac{1}{e^{(\frac{\mathbf{p}^2}{2m}+mc^2-\tilde{\mu}_\pm)/kT}+1}}$$

The density in coordinate space $n_\pm$ is given by $\int d^3\mathbf{p}\mathcal{N}_\pm$. We know that $n_- > n_+$ because while positrons and electrons are created in pairs we also have ionization electrons from hydrogen atoms. Thus we must have $\tilde{\mu}_- > \tilde{\mu}_+$. This inequality combined with $\tilde{\mu}_- + \tilde{\mu}_+ = 0$ gives

$$\boxed{\tilde{\mu}_- > 0, \ \tilde{\mu}_+ < 0.}$$

(c) In the dilute-gas regime, $\eta_\pm \approx exp\left[-(\frac{\mathbf{p}^2}{2m}+mc^2-\tilde{\mu}_\pm)/kT\right]$. It's trivial to perform the momentum-space integral and find

$$n_\pm = \frac{2}{h^3}(2\pi mkT)^{3/2}exp\left(\frac{\tilde{\mu}_\pm - mc^2}{kT}\right) \tag{3}$$

$\eta \ll 1$ means $(\tilde{\mu}_\pm - mc^2) \ll -kT$ (see Chapter 2), i.e.

$$n_\pm \ll \frac{2}{h^3}(2\pi mkT)^{3/2} = 4.8 \times 10^{27}\left(\frac{T}{10^8 K}\right)^{3/2} \text{cm}^{-3} \tag{4}$$

The hydrogen mass density is $\rho \approx n_- m_p$ (because $m_e \ll m_p$ and also electron-positron pair generation is negligible, thus by charge neutrality $n_- = n_p$) Thus the dilute-gas region is given by

$$\boxed{\rho \ll 8 \times 10^3 \left(\frac{T}{10^8 \text{K}}\right)^{3/2} \text{g} \cdot \text{cm}^{-3}.}$$

(d) Define $x \equiv e^{\tilde{\mu}_-/kT}$, then $e^{\tilde{\mu}_+/kT} = 1/x$. And using the expression for $n_\pm$ we found in part (c), we get

$$n = n_- - n_+ = \frac{2}{h^3}(2\pi mkT)^{3/2}e^{-mc^2/kT}(x - 1/x) \tag{5}$$

$$\text{define } y \equiv \frac{1}{4}n\lambda^3 e^{mc^2/kT}, \quad \text{where } \lambda \equiv \frac{h}{\sqrt{2\pi mkT}} \tag{6}$$

$$\Rightarrow 2y = (x - 1/x) \tag{7}$$

$$\Rightarrow x = y + (1 + y^2)^{1/2} \tag{8}$$

$$\tag{9}$$

$$\boxed{\text{Thus, } \frac{n_+}{n} = \frac{1/x}{x - 1/x} = \frac{1}{2y\left[y + (1+y^2)^{1/2}\right]}}$$

(e) [by Kip Thorne] The amount of phase space available to the ionization electrons, per ionization electron, is of order $V_{phase} \sim (2\pi mkT)^{3/2}\frac{m_p}{\rho}$. If one pair forms per ionization electron (our criterion for significant pair formation), then each positron will also have available to it the phase volume V_{phase}. When this V_{phase} is very large (i.e. when ρ is very small, at fixed $T \sim 10^8$K), then the positron is almost completely unaware of the presence of the electrons, so the probability of its meeting an electron and annihilating is very small. This means that, although pair production occurs only very rarely (because of the very few number of photons with sufficient energy to produce a pair), once it has occured, on average the positron lives for a very long time. It is this balancing of the long life, due to the huge phase space per particle, against the rarity of pair production, that enables significant pairs to form at T as low as $\sim 10^8$K.

5.4 Latent Heat and the Clausius-Clapeyron Equation [by Xinkai Wu]

(a) For fixed temperature and pressure, the change in the Gibbs potential is

$$dG = \tilde{\mu}_a dN_a + \tilde{\mu}_b dN_b = (\tilde{\mu}_a - \tilde{\mu}_b)dN_a \tag{10}$$

where we've used the fact that $dN_b = -dN_a$.

By the minimum principle for the Gibbs potential, N_a will decrease when $\tilde{\mu}_a > \tilde{\mu}_b$, and increase when $\tilde{\mu}_a < \tilde{\mu}_b$. As a consequence, if phases a and b are in equilibrium with each other, one must have $\tilde{\mu}_a = \tilde{\mu}_b$ (subtracting $m_{H_2O}c^2$ from both sides one finds $\mu_a = \mu_b$.) The Gibbs potential G is a function of (T, P, N), so the chemical potential $\mu = \left(\frac{\partial G}{\partial N}\right)_{T,P}$ is also a function of (T, P, N). However, μ is an intensive quantity and consequently can't depend on the extensive quantity N, namely μ is a unique function of T and P. The above reasoning is valid for any phase a. In a two-phase region, we must have $\mu_a(T, P) = \mu_b(T, P)$, which is one equation for two variables T and P, giving a line in the $T - P$

plane. In a three-phase region, $\mu_a(T, P) = \mu_b(T, P)$, *and* $\mu_b(T, P) = \mu_c(T, P)$, which are two equations for the two variables T and P, giving a point in the $T - P$ plane.

(b) The melting curve $P = P(T)$ is determined by $\mu_{\text{ice}}(T, P) = \mu_{\text{water}}(T, P)$. Then taking the temperature derivative of the above identity along the melting curve gives

$$\left(\frac{\partial \mu_{\text{ice}}}{\partial T}\right)_P + \left(\frac{\partial \mu_{\text{ice}}}{\partial P}\right)_T \left(\frac{dP}{dT}\right)_{\text{melt}} = \left(\frac{\partial \mu_{\text{water}}}{\partial T}\right)_P + \left(\frac{\partial \mu_{\text{water}}}{\partial P}\right)_T \left(\frac{dP}{dT}\right)_{\text{melt}} \quad (11)$$

$$\Rightarrow \left(\frac{dP}{dT}\right)_{\text{melt}} = \left[\left(\frac{\partial \mu_{\text{water}}}{\partial T}\right)_P - \left(\frac{\partial \mu_{\text{ice}}}{\partial T}\right)_P\right] \Big/ \left[\left(\frac{\partial \mu_{\text{ice}}}{\partial P}\right)_T - \left(\frac{\partial \mu_{\text{water}}}{\partial P}\right)_T\right] \quad (12)$$

Using the Maxwell relations that one reads off of $dG = -SdT + VdP + \mu dN$, we find

$$\left(\frac{\partial \mu_I}{\partial T}\right)_P = -\left(\frac{\partial S_I}{\partial N_I}\right)_{T,P} = -m_{\text{H}_2\text{O}} s_I \quad (13)$$

$$\left(\frac{\partial \mu_I}{\partial P}\right)_T = \left(\frac{\partial V_I}{\partial N_I}\right)_{T,P} = m_{\text{H}_2\text{O}}/\rho_I \quad (14)$$

where I denotes either phase (ice or water), s_I, ρ_I are the entropy per unit mass and density for phase I, respectively. Thus

$$\left(\frac{dP}{dT}\right)_{\text{melt}} = \frac{-m_{\text{H}_2\text{O}}(s_{\text{water}} - s_{\text{ice}})}{\frac{m_{\text{H}_2\text{O}}}{\rho_{\text{ice}}} - \frac{m_{\text{H}_2\text{O}}}{\rho_{\text{water}}}} \quad (15)$$

$$= (s_{\text{water}} - s_{\text{ice}}) \frac{\rho_{\text{ice}}\rho_{\text{water}}}{\rho_{\text{ice}} - \rho_{\text{water}}} \quad (16)$$

$$\boxed{\Rightarrow \left(\frac{dP}{dT}\right)_{\text{melt}} = \frac{\Delta q_{\text{melt}}}{T}\left(\frac{\rho_{\text{ice}}\rho_{\text{water}}}{\rho_{\text{ice}} - \rho_{\text{water}}}\right)}$$

(c) In equilibrium we still have $\mu_{\text{water}}(T, P_{\text{water}}) = \mu_{\text{vapor}}(T, P_{\text{vapor}})$. However, now P_{water} is no longer equal to P_{vapor}. Instead, $P_{\text{water}} = P_{\text{total}} = P_{\text{vapor}} + P_{\text{gas}}$. So we have $\mu_{\text{water}}(T, P_{\text{total}}) = \mu_{\text{vapor}}(T, P_{\text{vapor}})$. Fixing T and differentiating the above identity w.r.t. P_{total}, we have

$$\left(\frac{\partial \mu_{\text{water}}}{\partial P_{\text{total}}}\right)_T = \left(\frac{\partial \mu_{\text{vapor}}}{\partial P_{\text{vapor}}}\right)_T \left(\frac{dP_{\text{vapor}}}{dP_{\text{total}}}\right)_T \quad (17)$$

$$\Rightarrow \left(\frac{dP_{\text{vapor}}}{dP_{\text{total}}}\right)_T = \left(\frac{\partial \mu_{\text{water}}}{\partial P_{\text{total}}}\right)_T \Big/ \left(\frac{\partial \mu_{\text{vapor}}}{\partial P_{\text{vapor}}}\right)_T \quad (18)$$

$$= \frac{m_{\text{H}_2\text{O}}}{\rho_{\text{water}}} \Big/ \frac{m_{\text{H}_2\text{O}}}{\rho_{\text{vapor}}} \quad (19)$$

$$\boxed{\Rightarrow \frac{dP_{\text{vapor}}}{dP_{\text{total}}} = \rho_{\text{vapor}}/\rho_{\text{water}}.}$$

5.11 . Fluctuation of Temperature and Volume in a Ideal Gas

(a) For Gibbs ensemble, $\rho = e^{(G-\varepsilon+PV)/k_BT}$

So the probability density $\dfrac{dP}{dVdT} = Const \; e^{(G(\bar{T},\bar{V},N)-G(T,V,N))/k_BT}$

$$= Const \; e^{-G/k_BT}$$

Taylor expand the expression to 2nd order and note that 1st order terms vanish due to equilibrium conditon

$$\Rightarrow G(T,V,N) = G(\bar{T},\bar{V},N) + \frac{1}{2}\frac{\partial^2 G}{\partial T^2}(T-\bar{T})^2 + \frac{1}{2}\frac{\partial^2 G}{\partial V^2}(V-\bar{V})^2 + \frac{\partial^2 G}{\partial T\partial V}(V-\bar{V})(T-\bar{T})$$

(b) $dG = \left(\dfrac{\partial G}{\partial T}\right)dT + \left(\dfrac{\partial G}{\partial V}\right)dV + \left(\dfrac{\partial G}{\partial N}\right)dN$

$$= -SdT + PdV + HdN$$

$dE = \left(\dfrac{\partial E}{\partial S}\right)dS + \left(\dfrac{\partial E}{\partial V}\right)dV + \left(\dfrac{\partial E}{\partial N}\right)dN$

$$= TdS - PdV + HdN$$

$\Rightarrow \quad \dfrac{\partial^2 G}{\partial T^2} = \dfrac{\partial S}{\partial T} = \dfrac{C_v}{T_b} \quad \left(\dfrac{\partial^2 G}{\partial V^2}\right) = \left(\dfrac{\partial P}{\partial V}\right) = \dfrac{1}{K}$

$\dfrac{\partial^2 G}{\partial T\partial V} = 0$

5.11 (c) It's physically reasonable because they are

Independent degrees of freedom

(d) Obviously $C_V \propto N$ as $V \propto N$ and $E \propto N$

Also $\quad k \propto N$

$\Rightarrow \quad \sigma_V \propto \sqrt{N} \quad$ and $\quad \sigma_T \propto \dfrac{1}{\sqrt{N}}$

$\Rightarrow \quad \dfrac{\sigma_V}{V} \propto \dfrac{1}{\sqrt{N}} \quad$ and $\quad \dfrac{\sigma_T}{T} \propto \dfrac{1}{\sqrt{N}}$

It's reasonable because for larger system the classical limit is better, hence the relative deviation from equilibrium is smaller.