Ph136 Solution 6

6.6 (a) $\xi(r) = \langle \delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle = \lim_{V\to\infty} \frac{1}{V}\int_V dx\, \delta x\, \delta(x+r)$

$$= \lim_{V\to\infty} \frac{1}{V}\frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\, \delta_V(\vec{k})\,\delta_V(-\vec{k})$$

$$= \int \frac{d\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}} P(\vec{k})$$

The universe is isotropic $\Rightarrow \delta(\vec{x}) = \delta(|\vec{x}|) \Rightarrow P(\vec{k}) = P(k)$

Using the fact $\int_0^{\pi} d\theta \sin\theta \exp(-ikr\cos\theta) = 2\,\mathrm{Sinc}(kr)$

$$\Rightarrow \xi(r) = \int_0^{\infty} \frac{dk}{(2\pi^2)} k^2\, \mathrm{Sinc}(kr)\, P(k)$$

(b) Trivial

(c) $\delta_R(\vec{x}) = \frac{3}{4\pi R^3}\int_{r<R} dr\, \delta(x+r) = \int_V dr\, \delta(x+r)\, k(r)$

where $k(r) = \begin{cases} 3/4\pi R^3 & \text{if } r<R \\ 0 & \text{elsewhere} \end{cases}$

So $\sigma^2 = \langle \delta_R^2(x)\rangle = \lim_{V\to\infty} \int \frac{1}{V}\frac{d\vec{k}}{(2\pi)^3} |\delta_R(\vec{k})|^2$

By convolution theorem, $|\delta_R(k)|^2 = |\delta_V(k)|^2 |k(k)|^2$

Also using spherical coordinates, $\tilde{K}(k) = W(kR)$, where $W(x) = \frac{3(\mathrm{Sinc}x - \cos x)}{x^2}$

$$\Rightarrow \sigma^2 = \int_0^{\infty} \frac{dk}{(2\pi)} k^2 P(k)\, W^2(kR)$$

6.10 (a) $N(t) = \int K(t'-t)Y(t')dt' = \int K(t'')y(t''+t)dt''$

$$= \int K(t'')Y(t'')dt''$$

where we made a change of variable $t'' = t' - t$ and noted that Y is a Stationary random process. Thus, the statistical property is independent of time. Also $\overline{N(t)} = \langle N(t) \rangle = \int K\,\bar{y}\,dt = 0$

$$\Rightarrow \overline{N^2(t)} = \sigma_N^2 = \int_0^\infty S_N(f)\,df = \int_0^\infty |\tilde{K}(f)|^2 S_Y(f)\,df$$

(b) Note $K(-f) = K^*(f)$, $S(-f) = S^*(f)$

$$S = \int_0^\infty K(f)\,s^*(f)\,df + c.c$$

$$\Rightarrow \frac{S}{\langle N^2 \rangle^{1/2}} = \frac{\int_0^\infty K(f)\,s^*(f)\,df + c.c}{|\int_0^\infty |K(f)|^2 S_Y(f)\,df|^{1/2}}$$; Vary $K(f) \to K(f) + \delta K(f)$

$$\Rightarrow \delta\left(\frac{S}{\langle N^2 \rangle^{1/2}}\right) = \frac{S}{\langle N^2 \rangle^{1/2}} \left[\int_0^\infty df\,\delta K \left(\frac{s(f)}{S} - \frac{K(f)S_Y(f)}{2\langle N^2 \rangle} \right) + c.c \right]$$

$$\Rightarrow K(f) = \text{Const} \times \frac{s(f)}{S_Y(f)}$$

$K(f)$ is maximum is obvious, which can be checked by doing 2nd derivative.

6.10 (c) $$\frac{S^2}{\langle N^2\rangle} = \frac{\left(\int K(f)S^*(f)df + c.c\right)^2 df}{\int |K(f)|^2 S_Y(f)\, df}$$

$$= \frac{\left(\int_0^\infty \left(\frac{|S(f)|^2}{S_Y(f)} + c.c\right)df\right)^2}{\int_0^\infty \frac{S^2(f)}{S_Y(f)} df}$$

$$= 4\int_0^\infty \frac{|S(f)|^2}{S_Y(f)} df$$

7.2 (a) Taylor expanding $\Omega(k)$ to 2nd order and noticing $V_g = \frac{d\Omega}{dk}\Big|_{k_0} \Rightarrow \Omega = \omega_0 + V_g k + (dV_g/dk)\, k^2/2$

(b) Trivial

(c) Note

$$\int_{-\infty}^{\infty} dk \exp(-(a^2k^2 + ibk + ick^2)) = \left(\frac{\pi}{a^2+ic}\right)^{1/2} \exp\left(-\frac{b^2(a^2-ic)}{4(a^2+c^2)}\right)$$

Now $a^2 = \frac{1}{2\Delta k^2}$, $b = -(x - x_0 - V_g t)$, $c = \frac{1}{2}\frac{dV_g}{dk} t$

$$\Rightarrow |\psi| \propto \exp\left(-\frac{b^2a^2}{4(a^4+c^2)}\right] = \exp\left[-\frac{(x-x_0-V_g t)}{2L^2}\right]$$

with $L = \frac{1}{\Delta k}\sqrt{1 + \left(\frac{dV_g}{dk} 2\Delta k^2 t\right)^2}$

7.2 (d) At $t=0$, $L = \frac{1}{\Delta k}$. Recalling that $L \sim \Delta x$

$\Rightarrow \Delta x \, \Delta k \sim 1$ (uncertainty principal).

(e) $L(t=0) = \frac{1}{\Delta k}$, Solving $L(t) = 2L(0)$

$\Rightarrow t = \frac{\sqrt{3}}{\frac{dv_g}{dk} (\Delta k)^2}$. Now let's consider a wave packet travelling from Hawaii to california. Spreading less than 2 $\Rightarrow \left(\frac{dv_g}{dk}\right) \frac{1}{\Delta x^2} t < \sqrt{3}$

Also for waves on the surface of deep water : $\omega = \sqrt{gk}$

And $t = \frac{D}{v_g}$ where D being the distance from Hawaii $\rightarrow$ California

$D \sim 3 \times 10^3$ km, $k_0 = \frac{2\pi}{\lambda_0}$ and $\lambda_0 \sim 100$m

$$\Rightarrow \quad \Delta x > \sqrt{\frac{D \lambda_0}{4\pi \sqrt{3}}} \sim 4\text{ km} = 40 \lambda_0$$