Solution 13

Stokes Flow:

(a) $\nabla P = \rho \vec{g} + \eta \nabla^2 \vec{v}$

$$\nabla \times \vec{\omega} = -\nabla^2 \vec{v} + \nabla(\nabla \cdot \vec{\omega}) = \left(\frac{\rho \vec{g} - \nabla P}{\eta}\right) + \nabla(\nabla \cdot \vec{\omega})$$

$\Rightarrow \nabla \times \nabla \times \nabla \times V = 0$, $\nabla \cdot \vec{v} = 0$ because It's incompressible

(b) This is proven in Solution 12

(c) The differential equation we have is

$$r^2 \frac{d^4 u}{dr^4} + 4r \frac{d^3 u}{dr^3} - 4 \frac{d^2 u}{dr^2} = 0$$

If $u \propto r^n \Rightarrow n = 3, 1, -2, 0$

So $u = A r^3 + B r + \frac{C}{r^2} + D$

(d) $\underset{\sim}{v}(r \to \infty) = V \hat{z}$, $\underset{\sim}{v}(r = a) = 0$

$$\Rightarrow \begin{cases} u(a) = 0 \\ u'(a) = 0 \end{cases} , \quad A = 0 , \quad B = V\sqrt{\pi/3}$$

$$\Rightarrow u(r) = \frac{a^3 V}{2r^2}\sqrt{\frac{\pi}{3}} + V\sqrt{\frac{\pi}{3}}\, r - \frac{3a}{2}\sqrt{\frac{\pi}{3}}\, V$$

Solution 13

13.19

$$\begin{cases} T^{0\mu} = (\rho+P)u^0u^\mu + Pg^{0\mu} \\ T^{0\mu}{}_{;\mu} = \gamma\rho_{;\mu}u^\mu + \gamma P_{;\mu}u^\mu + \gamma(\rho+P)u^\mu{}_{;\mu} \end{cases}$$

$$\Rightarrow \quad u^i(\rho+P)_{;j} = -(\rho+P)u^i{}_{;j}$$

Also $(\rho_0 u^\mu)_{;\mu} = u^\mu \rho_{0;\mu} + \rho_0 u^\mu{}_{;\mu} = 0$

$$\Rightarrow \quad u^j(\rho_0)_{;j} = -\rho_0 u^j{}_{;j}$$

$$\Rightarrow \frac{dB}{dt} = u_j \frac{\partial B}{\partial x^j} = u^j \gamma \left\{ \frac{1}{\rho_0}(P_{;j} + \rho_{;j}) - \frac{1}{\rho_0^2}(\rho+P)\rho_{0;j} \right\}$$

$$= 0$$

So B is a concerved quantity

14.4

$$\rho \frac{dv}{dt} = -\nabla P - \rho g + \eta \nabla^2 v$$

Here we can ignore the gravitational term

and the viscous term

Solution 13

14.4 $\quad \Delta P \sim \Delta(\frac{1}{2}\rho v^2) = \rho v \Delta v$

$$\text{So } F_L = L\int \Delta P \, dx = L\rho v \int \Delta v \, dx$$

$$= L\rho v \int \vec{v} \cdot d\underset{\sim}{x} = L\rho v \Gamma$$

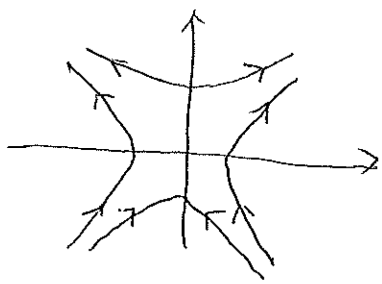
Since the momentum of the air is going downwards, it must be a lifting force due to momentum conservation.

14.1

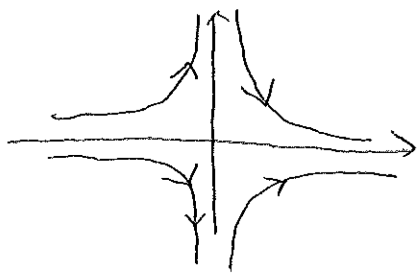
(a)

$\nabla \cdot V = 2y$

$\nabla \times V = 0$

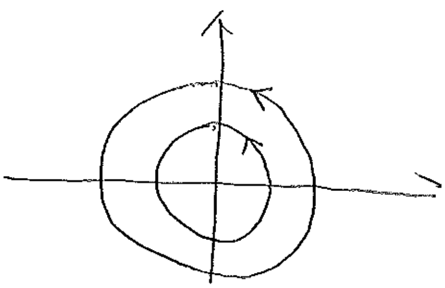


(b) $\nabla \cdot V = 0$, $\nabla \times V = -2(x+y)\hat{z}$



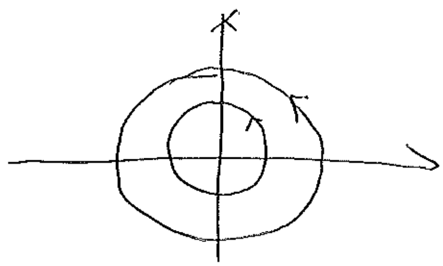
(c) $\nabla \cdot V = 0$

$\nabla \times V = 2\hat{z}$



(d) $\nabla \cdot V = 0$

$\nabla \times V = 0$



Solution for Chapter 14

(compiled by Nate Bode, solutions by credited authors)
February 12, 2009

A

13.10 Winds and ocean currents in the north Atlantic [by unknown]

(a) The balance of the friction of the wind with the ocean and the Coriolis force builds an Ekman layer in the Sargasso Sea area. See BT-15.4.4 for the discussion of the surface currents' direction.

The layer thickness is $\delta_E = \sqrt{\frac{\nu}{\Omega}}$.

$$\Omega = \Omega_e \sin(latitute) = \frac{2\pi}{24 \times 3600s} \times 0.5 \simeq 3.6 \times 10^{-5} s^{-1} \ . \tag{1}$$

The friction of the wind with the water is achieved by turbulence, taking $\nu \simeq 0.01 \sim 0.1$ as the tubulent viscosity(see BT-13.5.3), $\delta_E = \sqrt{\frac{\nu}{\Omega}} \simeq 16$ m $\sim$ 50 m.

The Encyclopedia Britannica states that the trade winds have an average speed of 5 km/s. Because these winds are very consistent the surface should come in to equilibrium with the winds and should be traveling at roughly the same speed. We saw in the chapter that the transverse velocity is also of the same order as the parallel velocity, so we expect the fluid to be moving towards the gyre at roughly the same speed. Equating the initial kinetic energy of a fluid element and the potential energy gives a crude limit as to the height of the gyre:

$$h \sim \frac{1}{2}\frac{{v_{\text{transverse}}}^2}{g} \sim \frac{1}{2}\frac{25 \text{ m}^2/\text{s}^2}{9.8 \text{ m/s}^2} \approx 1.28 \text{ m} \tag{2}$$

(b) When the water moves towards the center of the Sargasso sea,it piles up and builds the gyre with an outward horizontal pressure gradient. This gradient gives rise to the deep ocean geostrophic flow.

$$2\mathbf{\Omega} \times \mathbf{v} = -\frac{\nabla P'}{\rho}, \qquad P = \rho \mathbf{g} h(x,y), \tag{3}$$

where $h(x,y)$ is the height of ocean surface above an equipotential of

$$\Phi + \frac{1}{2}(\mathbf{\Omega^2} \times \mathbf{r})^\mathbf{2} \ . \tag{4}$$

The y-component in Eq.(3) is

$$2\mathbf{\Omega} v_y = \frac{1}{\rho}\frac{dP'}{dx} = \mathbf{g}\frac{dh(x,y)}{dx}, \qquad \Rightarrow v_y \simeq \frac{\mathbf{g}}{2\Omega}\frac{\Delta h}{\Delta x} \tag{5}$$

For Gulf Stream, $\Delta h \sim 1.5$ m (the height of the gyre), $\Delta x \sim 150$ km (the distance over which the sargasso Sea rises), then $v_y \simeq 1.4$ m/sec $\simeq 5$ km/hr is the speed of the deep ocean current.

(c) Without continents we would see the current travels parallel to lines of latitude. With the continents as the barriers, There are circular flows in every ocean basin. They flow clockwise in the North Atlantic and North Pacific and flow counterclockwise in the southern hemisphere.

B

15.10 Teacup [by Guodong Wang '03, modified by Geoffrey Lovelace]

(a) In the water's rotating reference frame, $\mathbf{v} = \mathbf{0}$, according to the Navier-Stokes equation in the rotating frame (eq. BT-13.53),

$$\nabla P' = 0 \Rightarrow P' = P + \rho[\mathbf{g}z - \frac{1}{2}\Omega^2\varpi^2] = constant \tag{6}$$

Taking $z = 0, \varpi = 0$ as the center of the water's top surface, the constant at (0,0) becomes the air pressure above the tea cup, namely P_0. So,

$$P(\varpi, z) = P_0 - \rho\mathbf{g}z + \frac{1}{2}\rho\Omega^2\varpi^2, \tag{7}$$

where the z-axis' direction is upward.

At the top surface of the water, $z(\varpi)$, the pressure $P(\varpi, z(\varpi))$ is P_0. Therefore the surface of the water is parabolic,

$$z(\varpi) = \frac{\Omega^2}{2\mathbf{g}}\varpi^2. \tag{8}$$

(b) The thickness of the Ekman layer at the bottom of the cup is

$$\delta_E = \sqrt{\frac{\nu}{\Omega}} \sim \sqrt{\frac{10^{-6}\ \mathrm{m^2/s}}{1/\mathrm{s}}} \sim 1\mathrm{mm}. \tag{9}$$

Because the bulk flow is geostrophic, the fluid velocity in the Ekman layer can be described as an Ekman spiral. To obtain an analytic expression for the spiral, begin by using the axial symmetry of the problem: examine the velocity profile only for some choice of ϕ (I am using (ϖ, ϕ, z cylindrical coordinates). Specifically, if $\mathbf{w} \equiv \mathbf{v} - \mathbf{V}$, where $\mathbf{v}$ is the velocity of the fluid at some point in the boundary layer, and $\mathbf{V}$ is the velocity of the fluid in the bulk, then, if $w = w_x + iw_y$, where w_x and w_y are the Cartesian components of $\mathbf{w}$, then the complex number w satisfied by Eq. (13.64) (see the text for derivation and details)

$$\frac{d^2w}{dz^2} = \left(\frac{1+i}{\delta_e}\right)^2 w. \tag{10}$$

As a second order ODE, this equation needs two boundary conditions, which for our case, are as follows: i) the fluid in the bulk is at rest in the rotating frame (implying $V = 0$ and $w = v$) so, $w(z/\delta_e \to \infty) \to 0$; and ii) There is also a no-slip boundary condition between the fluid and the bottom of the cup, so, in the rotating frame, $w(z = 0) = i \cos\phi - \sin\phi$. The solution satisfying these boundary conditions is (with velocity measured in units of $\Omega\varpi$)

$$w = v = v_x + iv_y \tag{11}$$

$$q \equiv \frac{z}{\delta_e} \tag{12}$$

$$v_x = e^{-q} \sin(\phi - q) \tag{13}$$

$$v_y = -e^{-q} \cos(\phi - q). \tag{14}$$

When $\phi = 0$, then $v_x = v_\varpi$ and $v_y = v_\phi$. Thus

$$v_\varpi = -e^{-q} \sin q \tag{15}$$

$$v_\phi = -e^{-q} \cos q. \tag{16}$$

This solution, an Ekman spiral, is drawn in in Fig. 1. Clearly, there is an inward radial component of the velocity in the boundary layer.

We can estimate the radial velocity of fluid in the Ekman layer on dimensional grounds. Deep in the boundary layer, the Ekman spiral diagram implies that the tangential and radial velocities are of the same order of magnitude. Therefore,

$$v_\varpi \simeq v_\theta = \Omega\varpi \sim \Omega L \sim 1 \text{ s}^{-1} \times 0.1 \text{ m} \sim 0.1 \text{ m/s} \tag{17}$$

The mass flux that is carried by $v_\varpi \sim \Omega L$ is

$$\rho\delta_E 2\pi L v_\varpi \sim 60 \text{ g/s} \tag{18}$$

(c) The pressure gradient is found by differentiating Eq. (4):

$$dP/d\varpi = \rho\Omega^2\varpi \tag{19}$$

This gradient is independent of z, so this pressure gradient is independent of height going into the boundary layer. If $dP/d\varpi$ did vary significantly with height once you reach the boundary layer, then $(d/dz)(dP/d\varpi)$ would be significant. Changing the order of the derivatives would then imply that $(d/d\varpi)(dP/dz)$

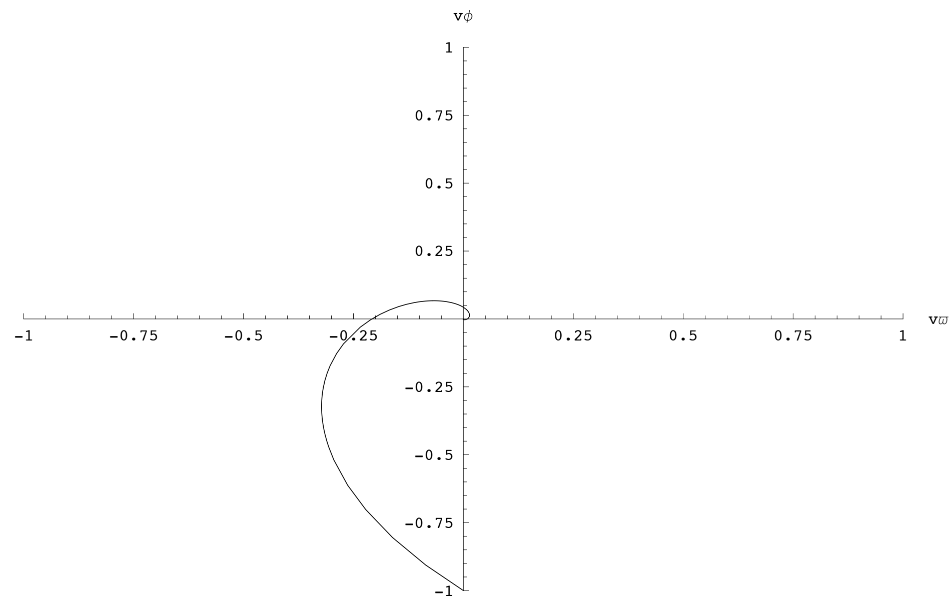


Figure 1: The Ekman spiral for fluid moving in the teacup. The radial and tangential velocities, v_ϖ and v_ϕ, respectively, are plotted in units of $\Omega\varpi$. When $q = z/\delta_e = 0$, $v_\varpi = 0$ and $v_\phi = -1$. As $q \to \infty$, the velocity vanishes.

is significant, which would imply that there would be a non-negligible, vertical force on the boundary layer that depended on ϖ. But there should be no normal force on the boundary layer, because such a force would deform the layer, and since the boundary layer is very thin, any variation in its shape is vanishingly small. It follows that we can take the radial pressure gradient to be independent of z, even into the boundary layer.

Therefore, the bulk radial pressure gradient should be the same as the pressure gradient in the boundary layer. In an inertial frame, force balance (i.e., the Navier-Stokes equation) is

$$\frac{d\mathbf{v}}{dt} \propto -\nabla P \propto -\mathbf{e}_{\varpi}. \tag{20}$$

Therefore, there is inward radial flow.

(d) After the boundary-layer water reaches the center of the cup, it moves upward and joins the bulk water. (The only place for the (incompressible) fluid to go when it reaches the center is up into the bulk.) Assuming a geostrophic flow (which we shall check), the flow in the bulk should move outward toward the side wall in such a way that the velocity gradients have no z components (in the bulk, the Rossby and Ekman numbers are small, so the Taylor-Proudman theorem holds). In other words, the fluid travels radially outward in Taylor columns.

The water is incompressible, so water must leave the center of the bulk at the same rate that fluid upwells into the center of the bulk., its outward speed must be smaller than the inward speed of the bottom boundary layer by the ratio of their thickness: $v_{\varpi}^{\text{bulk}} = -v_{\varpi}^{\text{boundary layer}}(L/\delta_E) \sim (10 \text{ cm/s}) \times 0.01 = 1$ mm/s. The Rossby number for this bulk motion is $\text{Ro} = \frac{V}{L\Omega} \sim 0.01$, so the geostrophic approximation is reasonable.

The fluid must return to the boundary layer somehow. Our treatment of the bulk and of the boundary layer neglected the vertical velocities. But we can qualitatively get at the answer as follows:

As fluid in the boundary layer is swept toward the center, the fluid just above (which is at the bottom of a Taylor column) falls down into the boundary layer. Therefore, the entire column gradually sinks as it travels outward. I say "gradually" because the velocity of the Taylor column is far less than the velocity of the overall circulation (of order $L\Omega$). An observer in an inertial frame would see tea leaves in the boundary layer upwell quickly in the center and then gradually spiral back down to the boundary layer.

Thus, a fiducial fluid element has the following circulation pattern: i) flow radially to the center of the boundary layer; ii) upwell in the center of the fluid; iii) flow in a Taylor column with an outward velocity; eventually, the element stops flowing upward and begins to flow back down, until iv) it re-enters the boundary layer. Note that the flow lines should have no sharp corners, since the forces on the fluid (and thus the fluid's acceleration) should be continuous.

I sketch the flow lines in Fig. 2.

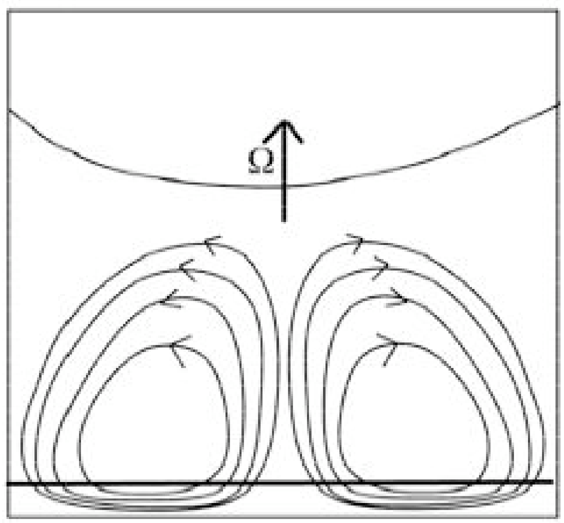


Figure 2: A sketch of the circulation patterns in the teacup as seen in a rotating reference frame. The bulk fluid (outside the boundary layer) is rotating nearly rigidly, but the fluid gradually descends back to the boundary layer. Note that the characteristic speed of this circulation pattern is much lower than $\Omega\varpi$, the velocity of the rotating fluid.

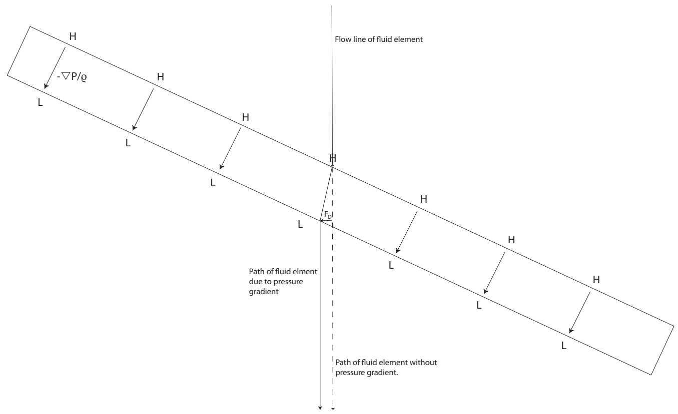


Figure 3: Normally a fluid particle would travel in a straight line as noted by the initial flow line and the dashed line. However, in the presence of an askew column a high pressure zone forms upstream of the column and a low pressure zone forms downstream so that a pressure gradient forms which is orthogonal to the column's axis of symmetry. This pressure force will move the fluid element off of its intial trajectory and give it a slight horizontal component. This horizontal component will carry with it a drag force which will be proportional in magnitude to the displacement of the fluid element off of its initial path. Obviously when the flow is perpendicular to the column this displacement is 0 as it is when the flow is parallel to the column. Therefore there is an intermediate angle when the displacement is maximized which is likely around $\pi/4$.

(e) The total mass of the water in the cup is $M \sim \pi L^2 \cdot L$, so it can mix much of the water in a time scale of

$$t_E \sim \frac{M}{\rho \delta_E 2\pi L v_\varpi} \sim \frac{L\delta_E}{\nu} \sim \frac{L}{\sqrt{\nu\Omega}}. \tag{21}$$

Let $L = 0.1$ m, $\Omega = 1$ s^{-1}, $\nu = 10^{-6}$ m^2/s, $t_E \sim 100$ s. The angular momentum of the water is conserved in the bulk. Only in the bounary layer at the bottom, the angular momentum loses by the friction. So the time scale of the mass mixing is also the time scale for the bulk flow to slow down.

C

High and low Reynolds number self-propelling devices [by Nate Bode]

(a) Let us first consider low Reynolds number flows. In this case the flow is reversible, so any reversed motion will leave you right where you began as in the case of the flapping paddle. In the case of high Reynolds number flows the viscosity is not important and the only relevant terms are those of the Euler equations, the pressure and inertial terms. This indicates that motion will be dominated by inertial interactions and the motion forwards will be essentially due to the momentum flux behind the paddle. If the paddle is of length L, width W and oscillates such that $\theta(t) = \theta_{\max} \sin \omega t$ then the displaced volume of material is approximately $WL^2\theta_{\max}$, and the velocity of the fluid is approximately $L\omega$. Therefore the force that the paddle produces is of order

$$F = \frac{\Delta P}{\Delta t} = \text{Mass} \times \text{Velocity} \times \omega = \rho W L^3 \theta_{\max} \omega^2 \tag{22}$$

Of course, this would only be true for small $\theta_{\max}$. One might think that given a motor and a boat, all one would have to do is increase the length of the paddle to speed up the boat, but obviously the motor would need to have a similar increase in power to be able to accommodate such a change in the length of the paddle.

(b) Intuitively it is obvious that in a high Reynolds number flow the corkscrew is not going to go anywhere. In this case the thin wire of the corkscrew can not change the momentum of the water by any considerable amount and therefore can not receive any propulsion. However, when the Reynolds number is low the inertial term is irrelevant and the viscous drag forces become the dominant factor. To consider how it moves let us view a small segment of column moving through an angled flow as in Fig. **??**. See the figure for a discussion of the forces on the corkscrew.As done in the text we expect the horizontal drag force per unit length to be $F_{Dz} = C_D \frac{1}{2} \rho V^2 d \sin \alpha$ where d is the diameter of the column, C_D is a drag constant of order unity, and α is the angle of displacement. Then we would expect the total force on the corkscrew with an optimal choice of angle to be

$$F = F_{Dz} L \sim \rho V^2 d L \tag{23}$$

D

14.1 Spreading of a Laminar Wake Behind a Sphere [by Alexander Putilin '00]

(b) Now with the cylinder replaced by the sphere, the cross section perpendicular to the flow is two dimensional and momentum conservation then implies:

$$\Delta v \cdot w^2 = \text{const} \tag{24}$$

namely $\Delta v \propto w^{-2}$.

The x component of the Navier-Stokes equation gives the same relation in the sphere case as in the cylinder case, $w \propto x^{1/2}$.

Combining these we get $\Delta v \propto x^{-1}$.

14.4 Turbulent wake behind a sphere [by Alexei Dvoretskii '99]

(b) The turbulent wake works in much the same way as its laminar counterpart, except that we should replace the intrinsic molecular viscosity ν with the kinematic turbulent viscosity $\nu_t \sim \Delta\bar{v}w$. The x component of the Navier-Stokes equation then gives the familiar relation

$$w \sim \left(\frac{\nu_t x}{V}\right)^{1/2} \sim \left(\frac{\Delta\bar{v}wx}{V}\right)^{1/2} \tag{25}$$

regardless of whether it's a cylinder or a sphere.

Now for the sphere, conservation of momentum implies that $\Delta\bar{v} \sim w^{-2}$.

Combining these we find $w \sim \text{const} \cdot x^{1/3}$. Using the fact that when $x \sim d$, $w \sim d$, we determine that const $\sim d^{2/3}$ and thus $w \sim d^{2/3}x^{1/3}$. Also we get $\Delta\bar{v} \sim \text{const} \cdot x^{-2/3}$. Using the fact that when $x \sim d$, $\Delta\bar{v} \sim V$, we can determine the const and find $\Delta\bar{v} \sim V\,(d/x)^{2/3}$.

14.2 Spreading of a 2-dimensional laminar jet [by H.W. Lee and Kip Thorne]

This problem is pretty much parallel to the analysis in Section 13.4, except that the width w of the jet and its speed v_x now scale w.r.t. x differently from those in Section 13.4. (because now the ambient fluid is at rest and we have a nozzle ejecting fluid out.)

(a) This argument goes the same as that on Page 20 of Section 13.4. The Navier-Stokes equation reads

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \frac{-\boldsymbol{\nabla}P}{\rho} + \nu\boldsymbol{\nabla}^2\mathbf{v} \tag{26}$$

The y component of the N-S equation shows that the pressure difference $\Delta P \sim \rho v_x^2 w^2/x^2$. Recall that we'll use the x component of the N-S equation to find the velocity profile. So we plug the above expression for ΔP into the x component of the N-S equation and find there the ratio between the $\frac{\boldsymbol{\nabla}P}{\rho}$ term and the $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ term (which is of the same order as the $\nu\nabla^2\mathbf{v}$ term) is $\sim \frac{w^2}{x^2} \ll 1$. Thus the pressure gradient term is indeed negligible for our purpose.

(b) The balance between the x components of the $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ term and the $\nu\nabla^2\mathbf{v}$ term gives the familiar result

$$w \sim \left(\frac{\nu x}{v_x}\right)^{1/2} \tag{27}$$

Now the conservation of momentum along the x direction requires

$$v_x^2 w = \text{const}, \text{ i.e. } \; v_x \sim w^{-1/2} \tag{28}$$

Combining these we find

$$w \sim x^{2/3}, \quad v_x \sim x^{-1/3} \tag{29}$$

(c) Give the stream function the following trial form

$$\zeta = a x^p f(\xi) \tag{30}$$

where the normalization a and the index p are to be determined, and $f(\xi)$ is a function of the dimensionless number $\xi \equiv b x^{-2/3} y$ with $b \equiv \left(\frac{\mathcal{F}}{48\rho\nu^2}\right)^{1/3}$.

Then we find

$$v_x = \frac{\partial \zeta}{\partial y} = a x^p f' \frac{\xi}{y} \tag{31}$$

$$v_y = -\frac{\partial \zeta}{\partial x} = a x^{p-1} \left(\frac{2}{3}\xi f' - p f\right) \tag{32}$$

Plugging these expressions into the x component of the N-S equation and throwing away terms subleading in the small parameter $\xi^2/(b^2 x^{2/3})$ (i.e. y^2/x^2, recalling that the jet is assumed to be very "thin"), we get

$$\frac{1}{3} a^2 b^2 x^{-\frac{7}{3}+2p} \left[(-2+3p) f'^2 - 3p f f''\right] = a b^3 \nu x^{-2+p} f''' \tag{33}$$

thus to have a self-similar solution we must satisfy

$$-\frac{7}{3} + 2p = -2 + p, \quad \text{i.e. } \; p = \frac{1}{3} \tag{34}$$

and the N-S equation becomes

$$f''' + \frac{a}{3b\nu}(f'^2 + f f'') = 0 \tag{35}$$

which can rewritten as

$$f''' + \frac{a}{3b\nu}(f f')' = 0 \tag{36}$$

Integrating once we get

$$f'' + \frac{a}{3b\nu} f f' + C_1 = 0 \tag{37}$$

We have the boundary conditions $v_y(y=0) = 0 \Rightarrow f(0) = 0$ and $\frac{\partial v_x}{\partial y}(y=0) = 0 \Rightarrow f''(0) = 0$, using which tells us $C_1 = 0$.

Integrating again gives

$$f' + \frac{a}{6b\nu} f^2 - C_2 = 0 \tag{38}$$

solving which gives

$$f = \sqrt{\frac{6b\nu}{a} C_2} \, \tanh \left[\sqrt{\frac{C_2 a}{6b\nu}} (\xi + C_3) \right] \tag{39}$$

The boundary condition $f(0) = 0$ gives $C_3 = 0$. Using this result we find

$$v_x = (C_2 a) b x^{-1/3} \text{sech}^2 \left(\sqrt{\frac{C_2 a}{6b\nu}} \xi \right) \tag{40}$$

Now using the normalization condition

$$\mathcal{F} = \int_{-\infty}^{+\infty} \rho v_x^2 dy \tag{41}$$

we find

$$C_2 a = \left(\frac{3\mathcal{F}}{4\rho\sqrt{6\nu}} \right)^{2/3} \frac{1}{b} \tag{42}$$

which when plugged into the expression for v_x gives the final answer

$$v_x = \left(\frac{3\mathcal{F}^2}{32\rho^2 \nu x} \right)^{1/3} \text{sech}^2(\xi) = \left(\frac{3\mathcal{F}^2}{32\rho^2 \nu x} \right)^{1/3} \text{sech}^2 \left(\left[\frac{\mathcal{F}}{48\rho\nu^2 x^2} \right]^{1/3} y \right) \tag{43}$$

14.5 Spreading of a 2-dimensional turbulent jet [by H.W. Lee and Kip Thorne]

(a) By now the following analysis should be very familiar to us:

x component of the N-S equation gives $w \sim \left(\frac{\nu_t x}{v_x} \right)^{1/2}$; Conservation of momentum gives $v_x \sim w^{-1/2}$; and we take $\nu_t \sim v_x w$. Combining these three facts we easily get

$$w \sim x, \quad v_x \sim x^{-1/2} \tag{44}$$

E

14.3 Reynolds Stress and Weak Turbulence Theory [by Alexei Dvoretskii '99]

(a) Let's write the Navier-Stokes equation

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \nu\rho\nabla^2\mathbf{v}$$

Decompose the velocity into a steady and a small fluctuating part

$$\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$$

And insert into the Navier-Stokes equation.

$$\rho\frac{\partial}{\partial t}\delta\mathbf{v}+\rho(\bar{\mathbf{v}}\cdot\nabla)\bar{\mathbf{v}}+\rho(\bar{\mathbf{v}}\cdot\nabla)\delta\mathbf{v}+\rho(\delta\mathbf{v}\cdot\nabla)\bar{\mathbf{v}}+\rho(\delta\mathbf{v}\cdot\nabla)\delta\mathbf{v} = -\nabla P+\nu\rho\nabla^2\bar{\mathbf{v}}+\nu\rho\nabla^2\delta\mathbf{v} \tag{45}$$

Taking the time average and using $\overline{\delta\mathbf{v}} = 0$ get

$$\rho(\bar{\mathbf{v}}\cdot\nabla)\bar{\mathbf{v}} = -\rho\overline{(\delta\mathbf{v}\cdot\nabla)\delta\mathbf{v}} - \nabla\bar{P} + \nu\rho\nabla^2\bar{\mathbf{v}}$$

The first term on the right-hand side can be rewritten as $-\nabla\cdot\mathbf{T}_R$ where $\mathbf{T}_R = \rho\overline{\delta\mathbf{v}\otimes\delta\mathbf{v}}$.

(b) To find the evolution of this tensor we take its time derivative

$$\frac{\partial \mathbf{T}_R}{\partial t} = \rho\overline{\frac{\partial\delta\mathbf{v}}{\partial t}\otimes\delta\mathbf{v}} + \rho\overline{\delta\mathbf{v}\otimes\frac{\partial\delta\mathbf{v}}{\partial t}}$$

Since $\frac{\partial\delta\mathbf{v}}{\partial t}$ involves averages of double products of velocity fluctuations, the time derivative of the velocity tensor will contain tensors that are time averages of triple products of velocity fluctuations. If we were to consider the time evolution of those tensors, because of the non-linearity of the equations, we'd have to consider such tensors of higher and higher rank. To close the sequence it would be necessary to truncate it by specifying a priori the tensors of some rank.

(c) We can rewrite the time-averaged Navier-Stokes equation as

$$-\nabla\bar{P} = \rho\overline{(\delta\mathbf{v}\cdot\nabla)\delta\mathbf{v}} - \nu\rho\nabla^2\bar{\mathbf{v}} + \rho(\bar{\mathbf{v}}\cdot\nabla)\bar{\mathbf{v}} \tag{46}$$

and plug it back in into the full Navier-Stokes, note that $P = \bar{P} + \delta P$, equation (14.22a) then follows immediately.

(d) Multiplying by $\delta\mathbf{v}$ and taking the time average we get

$$\bar{\mathbf{v}}\cdot\nabla(\frac{1}{2}\rho\overline{\delta v^2}) + \mathbf{T}_R^{ij}\bar{v}_{i,j} + \nabla\cdot\overline{\left(\frac{1}{2}\rho\delta v^2\delta\mathbf{v} + \delta P\delta\mathbf{v}\right)} = \nu\rho\overline{\delta\mathbf{v}\cdot(\nabla^2\delta\mathbf{v})}$$

Regroup terms

$$\bar{\mathbf{v}} \cdot \nabla(\frac{1}{2}\rho\overline{\delta v^2}) + \nabla \cdot \overline{\left(\frac{1}{2}\rho\delta v^2\delta\mathbf{v} + \delta P\delta\mathbf{v}\right)} = \nu\rho\overline{\delta\mathbf{v} \cdot (\nabla^2\delta\mathbf{v})} - \mathbf{T}_R^{ij}\bar{v}_{i,j}$$

The terms on the left hand side are the convective time derivative and the divergence of the flow of turbulent energy density typical of conservation laws. On the right hand side are possible sources of energy or its dissipation. In this case the first term is energy dissipation due to molecular viscosity and the second term is due to energy exchange between the ordered and turbulent motion.

(e) This can be seen if we take the Navier-Stokes equation and perform a similar transformation to get the law of ordered motion energy conservation

$$\nabla \cdot ((\frac{1}{2}\rho\bar{v}^2)\bar{\mathbf{v}}) + \nabla \cdot (P\bar{\mathbf{v}}) = \nu\rho\bar{\mathbf{v}} \cdot \nabla^2\bar{\mathbf{v}} - \bar{\mathbf{v}} \cdot \nabla\mathbf{T}_R$$

For incompressible fluid the full divergence $\nabla \cdot (\bar{\mathbf{v}}\mathbf{T}_R) = 0$ and so we can rewrite

$$\nabla \cdot ((\frac{1}{2}\rho\bar{v}^2)\bar{\mathbf{v}}) + \nabla \cdot (P\bar{\mathbf{v}}) = \nu\rho\bar{\mathbf{v}} \cdot \nabla^2\bar{\mathbf{v}} + \mathbf{T}_R^{ij}\bar{v}_{i,j}$$

We see that, indeed, the last term describes the exchange of energy between ordered and turbulent motion.