Solution 12

13.5 (a) $\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho$

$$\Rightarrow \phi(r) = \frac{2\pi G\rho r^2}{3} + \text{Const}$$

(b) A general solution of Laplace's equation is

$$\sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta) \quad , \mu = \sin(\text{latitude}) = \cos\theta$$

for $r \to 0$, it should not blow up, so $B_n = 0$. $P_1(\cos\theta)$ is not symmetric about $\theta = \pi/2$, so

$$\phi_{in} = \frac{2\pi G\rho r^2}{3} + A r^2 P_2(\mu)$$

(c) outside: $\phi_{out} = B_0 r^{-1} + B_2 r^{-3} P_2(\cos\theta)$, $B_0 = -GM$

(d) $\phi_{cent} = -\frac{1}{2}(\Omega \times r)^2$

(e) $\phi_{\theta=0} = \phi_{\theta=\pi/2}$

$$-\frac{GM}{R_e} - \frac{3}{4}\Omega^2 R_e^2 - \frac{1}{2}\Omega^2 R_e^2 = -\frac{GM}{R_p}$$

$$\Rightarrow R_e - R_p \simeq \frac{5}{4}\frac{\Omega^2 R^2}{g}$$

Solution 12

13.9 (a) $\nabla B = \nabla(\frac{1}{2}v^2 + h + \phi)$

$$[\nabla(\tfrac{1}{2}v^2)]_{,i} = v_j \frac{\partial v_j}{\partial x^i}$$

$$dh = Tds + \frac{dP}{\rho} \Rightarrow \nabla h = T\nabla s + \frac{\nabla P}{\rho}$$

$$(\nabla B)_i = v_j \frac{\partial v_j}{\partial x^i} + \left(T\nabla s + \frac{\nabla P}{\rho} + \nabla\phi\right)_i$$

Euler equation: $(\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla P}{\rho} - \nabla\phi$

$$\Rightarrow \left(\frac{\nabla P}{\rho} + \nabla\phi\right)_i = -v_j \frac{\partial v_i}{\partial x_j}$$

$$\Rightarrow (\nabla B)_i = (T\nabla s)_i + v_j \frac{\partial v_j}{\partial x^i} - v_j \frac{\partial v_i}{\partial x^j}$$

$$\Rightarrow \nabla B = T\nabla s + v \times \omega$$

(b) s is almost const in a tornado $\nabla B = v \times \omega \sim v_{max}^2/R$

$\Delta B \sim v_{max}^2$, $B = \frac{1}{2}v^2 + h$, $dh = Tds + \frac{1}{\rho}dP$

$\Delta B = 1/\rho\, \Delta P \sim v_{max}^2 \Rightarrow \Delta P \sim \rho v_{max}^2$

$v_{max} \sim 200$ miles/hr, $\rho \sim 1.2\ kg/m^3 \Rightarrow \Delta P \sim 10^4\ Pa$

Big enough to explode a house!

Solution 12

13.13
$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla \cdot F = 0 \\ V \cdot \left(\dfrac{\partial(\rho V)}{\partial t} + \nabla \cdot T \right) = 0 \end{cases}$$

If we expand out the equations, we can get

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho\phi)}{\partial t} + V\cdot\nabla(\rho h) + \rho h \Theta + (V\cdot\nabla)(\rho\phi) + \rho\phi\Theta - \rho\frac{\partial\phi}{\partial t}$$
$$- (V\cdot\nabla)P + \rho\vec{V}\cdot\vec{g} = 0$$

Or $$\frac{\partial(\rho v)}{\partial t} + (V\cdot\nabla)(\rho h) + \rho h\Theta - (V\cdot\nabla)P = 0$$

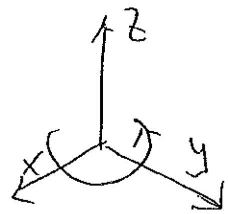
$$\begin{cases} du = Tds - Pd\left(\frac{1}{\rho}\right) \\ dh = Tds + \dfrac{dP}{\rho} \end{cases} \qquad \text{So } u = h - P/\rho$$

$$\Rightarrow \frac{\partial\rho}{\partial t}u + \rho T\frac{ds}{dt} + \frac{P}{\rho}\frac{\partial\rho}{\partial t} + h((V\cdot\nabla)\rho) + \rho h\Theta = 0$$

or $$\frac{\partial\rho}{\partial t}h + \rho T\frac{ds}{dt} + h((\vec{V}\cdot\vec{\nabla})\rho) + h(\rho\Theta) = 0$$

Contiuality equation $\rightarrow$ $\rho T\frac{ds}{dt} = 0$

13.15 $V = \vec{\Omega} \times \vec{X}$



$$\Rightarrow \begin{cases} V_x = \Omega y \\ V_y = \Omega x \end{cases}$$

$$\omega = \nabla \times \vec{V}$$

$$\int_S \omega \, d^2A = \int_{\partial S} \vec{V} \cdot dx$$

$$\Rightarrow \quad \omega = \frac{2\pi r \Omega r}{\pi r^2} = 2\Omega$$

So the vorticity is twice of the average angular velocity