Solution 15

16.1 Fluid motions in gravity waves [by Xinkai Wu 2002]
(a) The potential function for the velocity field is given by eq. (15.3)

$$\psi(x,z,t) = \psi_0 \cosh[k(z+h_0)] \exp[i(kx-\omega t)], \tag{1}$$

from which we find the velocity $\mathbf{v} = \nabla\psi$ (upon taking the real part):

$$v_x = -k\psi_0 \cosh[k(z+h_0)] \sin(kx-\omega t);\ v_y = 0;\ v_z = k\psi_0 \sinh[k(z+h_0)] \cos(kx-\omega t). \tag{2}$$

Thus we see that at a given depth z, the fluid element undergoes elliptical motion.
(b) From the velocity field found in the previous part, we see that the longitudinal and vertical diameters are given by

$$D_l = 2k\psi_0 \cosh[k(z+h_0)];\quad D_v = 2k\psi_0 \sinh[k(z+h_0)], \tag{3}$$

with their ratio being

$$\frac{D_v}{D_l} = \tanh[k(z+h_0)]. \tag{4}$$

(c) For a deep-water wave, $kh_0 >> 1$, and we see that

$$D_l \approx k\psi_0 \exp[k(z+h_0)];\quad D_v \approx k\psi_0 \exp[k(z+h_0)], \tag{5}$$

namely, the ellipses are all circles with diameters dying out exponentially with depth.
(d) Although individual fluid elements move in circles as time passes, the velocity field at any fixed moment of time is not circular; rather, it has a shape that is more nearly like (roughly like) a set of hyperbolae, which is vorticity free. More specifically, in the deep water case at $t = 0$, $v_x = -\alpha e^{kz} \sin kx$, $v_z = \alpha e^{kz} \cos kx$ for some constant α. The lines in x,z space to which this velocity field is tangent look like UUUUUU , not like OOOO. If they looked like OOOO the field would contain vorticity; since it looks like UUUU they do not contain vorticity.
(e) For a shallow-water, $kh_0 << 1$, we see that

$$D_l \approx 2k\psi_0 \left[1 + \frac{1}{2}k^2(z+h_0)^2\right] \approx 2k\psi_0 \tag{6}$$

$$D_v \approx 2k\psi_0 k(z+h_0) \approx 0, \tag{7}$$

namely the motion is (nearly) horizontal and independent of height z.
(f) The N-S equation in our case is

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\nabla P}{\rho} + \mathbf{g}_e. \tag{8}$$

Writing the pressure as the sum of an unperturbed part P_0 and a perturbation δP

$$P(x,z,t) = P_0(z) + \delta P(x,z,t) \tag{9}$$

and plug it into the z component of the N-S equation, we get

$$\begin{aligned} \frac{\partial P_0}{\partial z} &= -\rho g_e & (10) \\ \frac{\partial \delta P}{\partial z} &= -\rho \frac{\partial v_z}{\partial t}. & (11) \end{aligned}$$

Integrating the second equation from $z = 0$ to z gives

$$\delta P(z) = \delta P(z=0) - \int_0^z \rho \frac{\partial v_z}{\partial t}, \tag{12}$$

where the first term corresponds to the weight of the overlying fluid, and the second term corresponds to the contribution from the vertical acceleration of the fluid.

Using the expression for v_z worked out in part (a) as well as the boundary condition

$$\delta P(z=0) = \rho g_e \xi = -\rho \frac{\partial \psi}{\partial t}|_{z=0} = -\rho \omega \psi_0 \sin(kx - \omega t) \cosh[kh_0], \tag{13}$$

where we have used eq. (15.5) for ξ, the surface's vertical displacement from equilibrium, we find that

$$\begin{aligned} \delta P(x,z,t) &= -\rho\omega\psi_0 sin(kx - \omega t)\cosh[kh_0] & (14) \\ & \quad -\rho\omega\psi_0 \sin(kx - \omega t)\left\{\cosh[k(z+h_0)] - \cosh[kh_0]\right\} & (15) \\ &= -\rho\omega\psi_0 \sin(kx - \omega t)\cosh[k(z+h_0)]. & (16) \end{aligned}$$

For shallow water, the vertical fluid acceleration term (the second term in the first line of the above expression) is $\propto kh_0$ and can be ignored, and the pressure is basically determined by the first term, i.e. the weight of the overlying fluid. For general depth the second term is no longer negligible.

16.2 Shallow-water waves with variable depth; tsunamis. [by Xinkai Wu 2000]

We can treat this as a 2-dimensional problem, i.e., only consider the horizontal components of the velocity, which are almost independent of z. In what follows, ∇ is the 2-dimensional derivative operator.

(a) The mass per unit area is $\rho(h_0 + \xi)$, and the mass flux per unit length is $\rho(h_0 + \xi)\mathbf{v} \approx \rho h_0 \mathbf{v}$, to the first order in perturbation. Then by mass conservation,

$$\begin{aligned} & \partial[\rho(h_0 + \xi)]/\partial t + \nabla \cdot (\rho h_0 \mathbf{v}) = 0 \\ & \Rightarrow \partial \xi/\partial t + \nabla \cdot (h_0 \mathbf{v}) = 0, & (17) \end{aligned}$$

assuming constant ρ.

The Navier-Stokes equation in this case is:

$$\partial \mathbf{v}/\partial t = -\nabla P/\rho + \mathbf{g}_e,$$

whose vertical component tells us $P = \rho g_e(\xi - z)$, and whose horizontal components then tell us

$$\partial \mathbf{v}/\partial t = -g_e \nabla \xi. \tag{18}$$

Applying ∂_t to both sides of eqn. (21) and then plugging in eqn (22), we get

$$\partial^2 \xi/\partial^2 t - g_e \nabla \cdot (h_0 \nabla \xi) = 0. \tag{19}$$

(b) Plugging a plane wave solution to the wave equation, we get the dispersion relation

$$\omega = k\sqrt{g_e h_0}\sqrt{1 - i\frac{\nabla h_0}{h_0} \cdot \frac{\mathbf{k}}{\mathbf{k}^2}}, \tag{20}$$

the imaginary part in the square root is of order $\lambda/L << 1$, with λ being the wave length and L being the scale over which h_0 varies, and we've used the fact the h_0 is varying very slowly. Thus the dispersion relation is

$$\omega = \Omega(\mathbf{k}, \mathbf{x}) \approx k\sqrt{g_e h_0(\mathbf{x})}. \tag{21}$$

Our intuition for geometric optics thus tells us that water waves propagate in the ocean with a "refractive index" $n \propto h_0^{-1/2}$. Hence as waves approach a beach, their propagation directions get closer and closer to the normal direction, by Snell's law, since smaller h_0 corresponds to larger n.

(c) What you have to do is create some kind of mountain on the ocean floor (which gives smaller h_0 and thus larger n) with the shape of a lens, with Japan and LA at the object point and image point, respectively.

(d) The energy flux must be independent of location. The energy flux in this case is $F = -gh_o\dot{\xi}\nabla\xi$ (Eq. 15.94 in the notes). In magnitude this is $(1/2)gh_o\omega k\xi_o^2$, where ξ_o is the wave amplitude. The wave frequency ω will be conserved but the wave number k will not be conserved as the waves come in toward shore, so we need to look at how k changes as well as how h_o changes. Now $\omega/k = \sqrt{gh_o}$ so $k \propto 1/\sqrt{h_o}$, which means $F \propto \sqrt{h_o}\xi_o^2$. Since F is conserved, as h_o decreases, $\xi_o \propto 1/h_o^{1/4}$ increases.

For a tsunami the wavelength is roughly 100 km out at sea where the depth is 4km. Near shore at depth of, say, 10 meters, the amplitude has grown from about 1 meter to about $1\,\text{meter} \times (400)^{1/4} \sim 5\,\text{meters}$ and the wavelength has decreased to $100\,\text{km}/\sqrt{400} \sim 5\,\text{km}$ which is so large that geometric optics is now failing (the change of depth near shore is generally on a lengthscale short compared to 5 km). To propagate on inward we must abandon geometric optics, but the amplitude will continue to grow and the wavelength will continue to decrease.

16.7 Boat waves [by Xinkai Wu 2000]

(a) Consider a plane wave with frequency ω_0 and wave vector $\mathbf{k_0}$ as measured in the water's frame, and ω and $\mathbf{k}$ as measured in the boat's frame. For an observer with position $\mathbf{x_0}$ in the water's frame and $\mathbf{x}$ in the boat's frame, the phase he measures is $\mathbf{k_0} \cdot \mathbf{x_0} - \omega_0 t$ in terms of water's frame variables and $\mathbf{k} \cdot \mathbf{x} - \omega t$ in terms of boat's frame variables.

Since the phase is invariant under the change of reference frame, we can equate the above two expressions and then differentiate both sides with respect to t, noting that for an observer moving together with the boat, $d\mathbf{x_0}/dt = \mathbf{u}, d\mathbf{x}/dt = 0$, we get:

$$\omega = \omega_0 - \mathbf{k_0} \cdot \mathbf{u}. \tag{22}$$

By looking at Fig 15.3 and use $\mathbf{k}$ to denote the wave vector as measured in the water's frame(as the text does), we get

$$\omega = \omega_0 + uk\cos\phi. \tag{23}$$

(b) θ is the angle between $\mathbf{V}_{g0} - \mathbf{u}$ and $\mathbf{u}$, elementary trigonometry then gives $\tan\theta = V_{g0}\sin\phi/(u + V_{g0}\cos\phi)$. For a stationary wave pattern $\omega = 0$, so using the $\omega(k)$ we got in part (a), we see that

$$\omega_0(k) = -uk\cos\phi. \tag{24}$$

(c) For capillary waves,

$$\omega_0 \approx \sqrt{(\gamma/\rho)k^3},$$
$$V_{g0} = \partial\omega_0/\partial k = (3/2)\sqrt{(\gamma/\rho)k}. \tag{25}$$

Plugging these and $u = -\omega_0/(k\cos\phi)$ into the expression for $\tan\theta$, we get $\tan\theta = (3\tan\phi)/(1 - 2\tan^2\phi)$. A capillary wave pattern for a given θ exists only when we can find some $\phi \in (\pi/2, \pi)$ (i.e. only forward waves can contribute to the pattern) satisfying the above equation. And it's easy to show that indeed for any θ we can find such a ϕ given by:

$$\boxed{\tan\phi = (-3 - \sqrt{9 + 8\tan^2\theta})/(4\tan\theta) \text{ when } \theta < \pi/2.}$$

and

$$\boxed{\tan\phi = (-3 + \sqrt{9 + 8\tan^2\theta})/(4\tan\theta) \text{ when } \theta > \pi/2.}$$

(d) For gravity waves, $\omega_0 \approx \sqrt{g_e k}$, and $V_{g0} = (1/2)\sqrt{g_e/k}$, and we get $\tan\theta = (-\tan\phi)/(1 + 2\tan^2\phi)$. Only when $\theta < \arcsin(1/3)$ can we find some $\phi \in (\pi/2, \pi)$ satisfying this equation, which is:

$$\boxed{\tan\phi = (-1 \pm \sqrt{1 - 8\tan^2\theta})/(4\tan\theta) \text{ (both solutions are valid).}}$$

This means that the gravity-wave pattern is

$$\boxed{\text{confined to a trailing wedge with an opening angle } \theta_{gw} = 2\arcsin(1/3).}$$

16.6 For capillary waves the dispersion relation is

$$\omega^2 = gk + \gamma k^3/\rho$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{1}{2\omega}\left(g + 3\gamma k^2/\rho\right) = \frac{g + \frac{3k^2\gamma}{\rho}}{\sqrt{gk + \gamma k^3/\rho}} = V_g$$

$$\frac{dV_g}{dk} = 0 \Rightarrow k^4 + \frac{2g\rho k^2}{\gamma} - \frac{g^2\rho^2}{3\gamma^2} = 0$$

$$\Rightarrow k = \sqrt{\frac{g\rho}{\gamma}} \cdot \sqrt{-1 + \frac{2}{\sqrt{3}}}$$

For water, $\lambda \sim 4\text{cm}$ and $V_g|_{min} \sim 0.18\,\text{m/s}$