Solution 16

16.13 For Quadrupolar radiation

$$P \sim \frac{\overline{\dot{q}^2}}{4\pi\rho c}\left(\frac{b}{\lambda}\right)^4$$, Now we also know

$$b \sim \ell,\ q \sim \rho\ell^2 v,\ \ell/\lambda \sim \frac{v}{c},\ \omega \sim \frac{v}{\ell},\ \ell \sim \frac{1}{k}$$

As implied by the Kolmogorov Spectrum

$$v_k \sim k^{-1/3},\quad \Rightarrow P \sim v^8 \ell^2 \sim k^{-14/3} \sim \omega^{-7/2}$$

The ratio between radiated energy and dissipated energy:

$$\frac{P_{rad}}{P_{dis}} \sim \frac{1}{\omega^{7/2}\int dk\, u_k} \sim \frac{1}{\omega^{7/2}\int dk\, k^{-5/3}} \sim v^5 \sim M^5$$

16.15 Center the coordinates on the ball, $r=0$ is at the ball's Center, the surface is displaced according to $r_{ball} = a + \xi(t)$

Force balance equation: $m\ddot{\xi}(t) + m\omega_0^2\xi(t) = -4\pi a^2 \delta P(a)$

Now $\delta P(r) = -\rho\frac{\delta\psi}{\partial t}$ and $\psi = \frac{f(t-\epsilon r/c)}{r}$ ($\varepsilon = 1$ for outgoing wave, $= -1$ for ingoing wave)

$$\Rightarrow \delta P(a) = -\rho\left.\frac{\partial\psi}{\partial t}\right|_{r=a} = -\rho\frac{d f(t-\varepsilon a/c)}{dt} = -a\rho\dot{\Phi}(t)$$

$$\Rightarrow \quad \ddot{\xi}(t) + \omega_0^2 \xi(t) = \frac{4\pi\rho a^3}{m}\dot{\phi}(t) = k\dot{\phi}(t) \qquad ①$$

velocity continuity at the surface

$$\Rightarrow \quad \dot{\xi}(t) = \left.\frac{\partial\psi}{\partial r}\right|_{r\to a} = -\phi(t) - \frac{\varepsilon a}{c}\dot{\phi}(t) \qquad ②$$

$$\left.\begin{matrix} ① \\ ② \end{matrix}\right\} \Rightarrow \quad a\varepsilon\dddot{\xi} + c(1+k)\ddot{\xi} + a\varepsilon\omega_0^2\dot{\xi} + c\omega_0^2\xi = 0$$

If $\xi = e^{\Lambda t}$ $\Rightarrow$ $a\varepsilon\Lambda^3 + c(1+k)\Lambda^2 + a\varepsilon\omega_0^2\Lambda + c\omega_0^2 = 0$

In the slow motion limit, the solutions are

$$\begin{cases} \omega_\pm = \mp i\omega - \frac{1}{2}\varepsilon\tau\omega^2 \\ \omega_1 = -\frac{k}{\varepsilon\tau} + \varepsilon\tau\omega^2 = -\frac{\varepsilon k}{\tau} + \varepsilon\tau\omega^2 \approx -\frac{\varepsilon k}{\tau} \end{cases}$$

Where $\omega = \omega_0/\sqrt{1+k}$, $\tau = \frac{k}{1+k}\frac{a}{c}$

$$\text{So} \quad \begin{cases} \xi_\pm = \exp(\pm i\omega t)\exp(-\varepsilon\tau\omega^2 t/2) \\ \xi_1 \propto \exp(-k\varepsilon t/\tau) \end{cases}$$

3. (16.22) Using the force balance equation in the bubble's rest frame

$$F_A \sim \rho g V, \quad F_v \sim \rho v^2 S, \quad V \propto r^3, \quad S \propto r^2$$

$$\Rightarrow \quad v \sim \sqrt{g r}$$

4. $$c^{2/(\gamma-1)} v = (\gamma K)^{1/(\gamma-1)} \dot{m}/A, \quad \frac{c^2}{\gamma-1} + \frac{v^2}{2} = \frac{c_1^2}{\gamma-1}$$

For $\gamma = 3$

$$c v = \sqrt{3K}\, \frac{\dot{m}}{A}, \quad c^2 + v^2 = c_1^2$$

$$\Rightarrow \quad \frac{3K\dot{m}^2}{A^2} \frac{1}{v^2} + v^2 = c_1^2$$

$$\Rightarrow \quad v^2 = \frac{c_1^2}{2}\left(1 \pm \sqrt{1 - \frac{12K\dot{m}^2}{A^2 c_1^2}}\right)$$

It's easy to verify the phase diagram

Solution 16

6. (a) $\frac{\partial h}{\partial t} + \frac{\partial (hv)}{\partial x} = 0$, $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g\frac{\partial h}{\partial x} = 0$

$$\Rightarrow \frac{\partial (v \pm 2\sqrt{gh})}{\partial t} + (v \pm \sqrt{gh})\frac{\partial (v \pm 2\sqrt{gh})}{\partial x} = 0$$

The Riemann invariants : $J_\pm \equiv v \pm 2\sqrt{gh}$

with charateristic speed : $V_\pm \equiv v \pm \sqrt{gh}$

$$\left(\frac{\partial}{\partial t} + V_\pm \frac{\partial}{\partial x}\right) J_\pm = 0$$

(b) Here $v = \sqrt{2gh}$ - Constant, So the water at the peak of the wave moves faster than the water in the bottom. This causes the leading edge of the water to steepen.

(c) $J_+ = v + 2\sqrt{gh} = 2\sqrt{gh_0}$

For the leftward charateristic C_-, we obtain

$$x = (v - \sqrt{gh})t = (2\sqrt{gh_0} - 3\sqrt{gh})t$$

So $$h(x,t) = \frac{h_0}{9}\left(2 - \frac{x/t}{\sqrt{gh_0}}\right)^2$$

and $$v(x,t) = \frac{2}{3}\left(x/t + \sqrt{gh_0}\right)$$

at $-\sqrt{gh_0}t < x < 2\sqrt{gh_0}\,t$