Solution for Problem Set 19 (Ch 18)

(compiled by Nate Bode)
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A

18.1 Boundary of Degeneracy [by Alexander Putilin '00]

We'll ignore factors of order unity in what follows.

(a) $l = n_e^{-1/3} \gg \lambda_{dB} = \hbar/(momentum) \simeq \hbar/\sqrt{m_e kT}$, which immediately gives $n_e \ll (m_e kT)^{3/2}/h^3$.

(b) Using the uncertainty principle, $\Delta x \simeq h/\Delta p \simeq h/\sqrt{m_e kT} \ll n_e^{-1/3}$, which again reduces to $n_e \ll (m_e kT)^{3/2}/h^3$.

(c) The quantum mechanical zero-point energy is given by $(\Delta p)^2/m_e \simeq h^2/(l^2 m_e) \simeq h^2/(n_e^{-2/3} m_e) \ll kT$, which reduces to $n_e \ll (m_e kT)^{3/2}/h^3$.

18.6 Parameters for various plasmas [by Henry Huang '98]

Text eq. (18.10) $\lambda_D = \left(\frac{\epsilon_0 kT}{ne^2}\right)^{1/2} = 69 \left(\frac{T/1K}{n/1\text{m}^{-3}}\right)^{1/2}$ m.

Text eq. (18.11) $N_D = n\frac{4\pi}{3}\lambda_D^3 = 1.4 \times 10^6 \frac{(T/1\text{K})^{3/2}}{(n/1\text{m}^{-3})^{1/2}}$.

Text eq. (18.13) $f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\left(\frac{ne^2}{\epsilon_0 m_e}\right)^{1/2} = \frac{56.4}{2\pi}\left(n/1\text{m}^{-3}\right)^{1/2}$ Hz.

Text eq. (18.21) $t_D^{ee} = \frac{1}{\nu_D^{ee}} = \frac{1}{2.5\times 10^{-5}}\left(n/1\text{m}^{-3}\right)^{-1}(T/1\text{K})^{3/2}(\ln\Lambda/10)^{-1}$ s.
Using the fact that $\Lambda = \frac{9}{2}N_D$ (see Exercise 18.2 part (a)), we get $t_D^{ee} = 4 \times 10^4 \left(n/1\text{m}^{-3}\right)^{-1}(T/1\text{K})^{3/2}\left(\ln(\frac{9N_D}{2})/10\right)^{-1} s$.

And we only need to know T and n to get numerical values of these.

(a) Atomic bomb.

Text eq. (16.57) gives $T \sim 4\times 10^4\,(t/1\text{ms})^{-1.2}$ K; at $t = 1$ms, $T \sim 4\times 10^4$K. The discussion above eq. (16.57) gives $\rho \sim 5\text{kg/m}^3$, which means $n \sim \frac{\rho}{\mu m_p} \sim \frac{5\text{kg/m}^3}{29\times 1.66\times 10^{-27}\text{kg}} \sim 10^{26}\text{m}^{-3}$.

(b) Space shuttle

Box 16.2 gives $T \sim 9000\text{K}$. And since shuttle moves at $\sim 7000\text{m/s} \gg$ sound speed 280m/s (all given in Box 16.2), we can use eq. (16.45a) which says $\frac{\rho_1}{\rho_2} \simeq \frac{\gamma-1}{\gamma+1}$ and gives $\rho_2 \sim 5\rho_1$ taking $\gamma \sim 1.5$. The density at the altitude of 70km is $\rho_1 \sim \rho_{\text{ground}} \exp(-70\text{km}/8\text{km}) \sim 10^{-4}\text{kg/m}^3$, which gives $n \sim \frac{\rho_2}{\mu m_p} \sim \frac{5\rho_1}{\mu m_p} \sim 10^{22}\text{m}^{-3}$.

(c) Expanding universe

Text Fig. 16.1 gives: at recombination threshold, $\log T \sim 3.5 \Rightarrow T \sim 10^{3.5}\text{K} \sim 3 \times 10^3\text{K}$. Also from chapter 26, $\rho \propto T^3 \Rightarrow \rho_{\text{then}} = \rho_{\text{now}} \left(\frac{T_{\text{then}}}{T_{\text{now}}}\right)^3 \sim 10^{-29}\text{g/cm}^3 \left(\frac{3\times 10^3\text{K}}{3\text{K}}\right)^3 \sim 10^{-20}\text{g/cm}^3$. So we get $n \sim \frac{\rho_{\text{then}}}{m_p} \sim 10^{10}\text{m}^{-3}$.

Plugging the above values of T and n into the equations on the top of this page, we get

	λ_D(m)	N_D	f_p(Hz)	t_D(s)
A-bomb	1×10^{-9}	1	9×10^{13}	2×10^{-14}
Shuttle	7×10^{-8}	10	9×10^{11}	9×10^{-12}
Universe	4×10^{-2}	2×10^{6}	9×10^{5}	4×10^{-1}

B

18.5 Stopping of alpha particles [by Alexander Putilin '00]

First calculate the energy loss of an α-particle in a Coulomb collision with an electron (with impact parameter b). Consider the collision in the electron's rest frame. We can approximate the trajectory of the α-particle by a straight line (take this to be the x-axis). Then the momentum change of the electron is given by an integral of force over time:

$$F_y = F \sin\theta = \frac{2e^2}{4\pi\epsilon_0(b^2+x^2)} \frac{x}{\sqrt{b^2+x^2}} \tag{1}$$

$$\Delta p_e = \int F_y dt = \int_{-\infty}^{+\infty} \frac{dx}{v} \frac{e^2 x}{2\pi\epsilon_0(b^2+x^2)^{3/2}} = \frac{e^2}{\pi\epsilon_0 v b}. \tag{2}$$

Then the energy loss is

$$\Delta E = -\frac{\Delta p_e^2}{2m_e} = -\left(\frac{e^2}{\pi\epsilon_0}\right)^2 \frac{1}{2m_e v^2 b^2} = -\left(\frac{e^2}{\pi\epsilon_0}\right)^2 \frac{m_\alpha}{4m_e E b^2}, \tag{3}$$

where $m_\alpha = 4m_p$ is the mass of an α-particle and $E = \frac{1}{2}m_\alpha v^2$ is the energy of an α-particle. When an α-particle travels a distance $d\ell$, it loses energy:

$$dE = \Delta E \cdot (\text{number of collisions}) = \int_{b_{\min}}^{b_{\max}} \Delta E \cdot n_e \cdot 2\pi b \cdot db \cdot d\ell. \tag{4}$$

$$\frac{dE}{d\ell} = -\int_{b_{\min}}^{b_{\max}} \left(\frac{e^2}{\pi\epsilon_0}\right)^2 \frac{m_\alpha}{4m_e E b^2} \cdot n_e \cdot 2\pi b \cdot db = -\frac{\pi n_e m_\alpha}{2m_e E}\left(\frac{e^2}{\pi\epsilon_0}\right)^2 \ln\Lambda, \tag{5}$$

where $\Lambda = b_{\max}/b_{\min}$. To estimate $b_{\max}$, notice that electrons in plastic are not free but rather are bounded in atoms. It means that there is no Debye shielding and so a reasonable estimate for $b_{\max}$ is the atomic spacing: $b_{\max} \sim n_e^{-1/3} \sim 2\cdot 10^{-10}$ m. For $b_{\min}$, use the usual formula

$$b_{\min} = \text{Max}\left[b_o = \frac{2(2e^2)}{4\pi\epsilon_0 m_\alpha v^2}, \frac{\hbar}{m_\alpha v}\right]. \tag{6}$$

We see that $\ln\Lambda$ depends on the energy E, but since $\Lambda \gg 1$ and $\ln\Lambda$ varies slowly for large Λ, we can assume $\ln\Lambda$ to be constant equal to its initial value at $E = E_0 = 100\,\text{MeV}$. So, $b_{\min} \approx \frac{\hbar}{m_\alpha\sqrt{2E_0/m_\alpha}} \approx 2.5\cdot 10^{-16}\,m$, and so finally $\ln\Lambda \approx 13$. Integrating equation (1), we get

$$\frac{1}{2}\left(E_0^2 - E(\ell)^2\right) = \frac{\pi n_e m_\alpha}{2m_e}\left(\frac{e^2}{\pi\epsilon_0}\right)^2 \ln\Lambda\cdot\ell. \tag{7}$$

The range ℓ is defined by $E(\ell) = 0$, so

$$\ell = \left(\frac{\pi\epsilon_0}{e^2}\right)^2 \frac{m_e E_0^2}{\pi m_\alpha n_e \ln\Lambda}. \tag{8}$$

Plugging in the numbers, we find $\ell \approx 0.5\,\text{cm}$.

18.7 Equilibration Time for a Globular Cluster [by unknown author]

(a) For single deflections when $b \le b_0$, $\sigma = \pi b_0^2$. While for cumulative deflections, in which each deflection has $b \gg b_0$, then $\Delta E = -(b_0/b)^2 E$ for each deflection. Since we are interested in the case where the test star has high kinetic energy compared to the field stars, then we add up ΔE linearly.

$$\frac{\Delta E}{E} = -\int_{b_{\min}}^{b_{\max}} \left(\frac{b_0}{b}\right)^2 nvt2\pi b db = -2\pi vtnb_0^2 \ln\left(\frac{b_{\max}}{b_{\min}}\right) \tag{9}$$

where $b_{\min}$ is b_0 and $b_{\max}$ is $R =$ the radius of the star cluster. So the energy change timescale is dominated by cumulative deflections:

$$t_E = \frac{1}{2\pi b_0^2 nv\ln\Lambda} \tag{10}$$

In the gravitational case, $b_0 = 2Gm/v^2$ and we get

$$t_E = \frac{v^3}{8nG^2m^2 \ln \Lambda} \tag{11}$$

To estimate this, use the Virial theorem(e.g. Goldstein) which says that the cluster's kinetic energy is half the potential energy, so $\frac{1}{2}Nmv^2 \sim \frac{1}{2}\frac{G(Nm)^2}{R}$, $\Rightarrow v \sim \left(\frac{GNm}{R}\right)^{1/2}$.

$$\ln \Lambda = \ln \frac{R}{b_0} = \ln \frac{R}{2Gm/v^2} = \ln N = \ln 10^6 = 14 \tag{12}$$

and

$$t_E = \frac{N^{3/2}}{14n(8GmR^3)^{1/2}} \tag{13}$$

Also we can put in $n = N/(\frac{4\pi}{3}R^3)$ to get

$$t_E = \frac{\frac{4\pi}{3}N^{1/2}R^{3/2}}{14(8Gm)^{1/2}} = 4 \times 10^{17}\text{s} = 1.3 \times 10^{10}\text{yr} \tag{14}$$

which is about the age of the universe.

(b) The cluster will try to develop a distribution function that is a function of the constant of motion (so it satisfies the collisionless Boltzmann equation, i.e. Liouville's theorem of chapter 2). The velocity distribution will try to become isotropic, so

$$\mathcal{N} = \frac{dN}{d^3xd^3p} = f(E) = f(m\Phi + \frac{1}{2}mv^2) \tag{15}$$

where Φ is the gravitational potential which is less than zero. In true equilibrium, this $f(E)$ should become an exponential, so $\mathcal{N} = C\exp(-E/kT)$. However, only stars with $E < 0$ are gravitationally bound in the cluster; those with $E > 0$ escape and fly away. This means that $\mathcal{N} = 0$ for $E > 0$ and $\mathcal{N} \simeq C\exp(-E/kT)$ for $E < 0$.

Stellar encounters then keep kicking stars, occasionally, to energies $E > 0$, and those stars evaporate from the cluster. Since the evaporated stars have larger energy than average, the rest of the cluster keeps shrinking and becoming more and more tightly bound.

C

18.4 Dependence on thermal equilibration on charge and mass [by Alexander Putilin '00]

The ion equilibration rate for a pure He^3 plasma is derived by the same method as proton-proton equilibration rate. We start with electron-electron equilibration rate (B.T. eq. (18.27))

$$\nu_{ee} = \frac{n_e \sigma_T c \ln\Lambda}{2\sqrt{\pi}} \left(\frac{kT}{m_e c^2}\right)^{-3/2} \tag{16}$$

$$= \frac{n_e c \ln\Lambda}{2\sqrt{\pi}} \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 \left(\frac{kT}{m_e c^2}\right)^{-3/2} \tag{17}$$

Replace electron charge, density, and mass with corresponding values for He^3 ions: $e \to 2e$, $n_e \to n_{\rm He} = \frac{1}{2} n_e$, $m_e \to m_{\rm He} = 3m_p$. We get

$$\nu_{\rm He\ He} = \frac{n_{\rm He} c \ln\Lambda}{2\sqrt{\pi}} \frac{8\pi}{3} \left(\frac{4e^2}{4\pi\epsilon_0 m_{\rm He} c^2}\right)^2 \left(\frac{kT}{m_{\rm He} c^2}\right)^{-3/2} \tag{18}$$

$$= \frac{16}{\sqrt{3}} \sqrt{\frac{m_e}{m_p}} \frac{n_{\rm He} \sigma_T c \ln\Lambda}{2\sqrt{\pi}} \left(\frac{kT}{m_e c^2}\right)^{-3/2} \tag{19}$$

$$= \frac{16}{\sqrt{3}} \cdot 5.8 \times 10^{-7} {\rm s}^{-1} \left(\frac{n_{\rm He}}{1{\rm m}^{-3}}\right) \left(\frac{T}{1{\rm K}}\right)^{-3/2} \left(\frac{\ln\Lambda}{10}\right) \tag{20}$$

and we have $n_{\rm He} = \frac{1}{2} n_e = 0.5 \times 10^{20} {\rm m}^{-3}$, $T = 10^8$K. Now estimate $\ln\Lambda$. $\Lambda = \frac{\lambda_D}{b_{\rm min}}$, with $\lambda_D = \left(\frac{\epsilon_0 kT}{n_{\rm He}(2e)^2}\right)^{1/2} = 4.9 \times 10^{-5}$m, and $b_{\rm min} = {\rm Max}[b_0 = \frac{2(2e)^2}{m_{\rm He} v^2}, \frac{\hbar}{m_{\rm He} v}]$, where $v \simeq \sqrt{\frac{3kT}{m_{\rm He}}}$. We find $b_0 = 4.4 \times 10^{-13}$m and $\frac{\hbar}{m_{\rm He} v} = 2\times 10^{-14}$m. Thus we take $b_{\rm min} = 4.4\times 10^{-13}$m, which gives $\ln\Lambda = \ln \frac{4.9\times 10^{-5}}{4.4\times 10^{-13}} \simeq 18$.

Plugging it into the formula for $\nu_{\rm He\ He}$ we get

$$\nu_{\rm He\ He} \simeq 500 \ {\rm s}^{-1} \tag{21}$$

18.11 Adiabatic indices for rapid compression of a magnetized plasma [by unknown author]

(a) The amount of momentum that passes through a surface ΔA normal to the z direction per time Δt is $m_e v_z$ for each electron, and only those electrons (with velocity v_z) which are in the region of volume $\Delta A v_z \Delta t$ pass through, so the total amount is $n_e m_e \langle v_z^2 \rangle \Delta A \Delta t$. Since T_{zz} is this number divided by $\Delta A \Delta t$, then

$$P_{e\|} = T_{zz} = n_e m_e \langle v_z^2 \rangle.$$

Similarly,

$$P_{e\perp} = n_e m_e \langle v_x^2 \rangle = n_e m_e \langle v_y^2 \rangle,$$

and since $\langle v_x^2 \rangle + \langle v_y^2 \rangle = \langle |v_\perp|^2 \rangle$, then

$$P_{e\perp} = \frac{1}{2} n_e m_e \langle |v_\perp|^2 \rangle.$$

(b) From Box 10.1, we see that

$$\Theta = S_{xx} + S_{yy} + S_{zz}$$

and

$$\Sigma_{zz} = \frac{2}{3} S_{zz} - \frac{1}{3} \left(S_{xx} + S_{yy} \right).$$

Invert to get that $S_{zz} = \frac{1}{3}\Theta + \Sigma_{zz}$ and $S_{xx} + S_{yy} = \frac{2}{3}\Theta - \Sigma_{zz}$, so that one sees that

$$\frac{d\ell/dt}{\ell} = \frac{dS_{zz}}{dt} = \frac{1}{3}\frac{d\Theta}{dt} + \frac{d\Sigma_{zz}}{dt} = \frac{1}{3}\theta + \sigma^{jk} b_j b_k$$

and

$$\frac{dA/dt}{A} = \frac{d(S_{xx} + S_{yy})}{dt} = \frac{2}{3}\frac{d\Theta}{dt} - \frac{d\Sigma_{zz}}{dt} = \frac{2}{3}\theta - \sigma^{jk} b_j b_k.$$

(c) The amount of kinetic energy corresponding to motion in the z direction in the fluid element is $n_e A \ell \frac{1}{2} m_e \langle v_\|^2 \rangle$. Due to energy conservation, if the element expands, doing work at rate $P\, d(volume)/dt = n_e m_e \langle v_\|^2 \rangle A (d\ell/dt)$, then the energy must drop accordingly, so

$$n_e A \ell \frac{1}{2} m_e \frac{d\langle v_\|^2 \rangle}{dt} = -n_e m_e \langle v_\|^2 \rangle A \frac{d\ell}{dt},$$

so

$$\frac{1}{\langle v_\|^2 \rangle} \frac{d\langle v_\|^2 \rangle}{dt} = -\frac{2}{\ell}\frac{d\ell}{dt}.$$

Following the same argument for the perpendicular contribution to the energy gives

$$n_e A \ell \frac{1}{2} m_e \frac{d\langle v_\perp^2 \rangle}{dt} = -\frac{1}{2} n_e m_e \langle v_\perp^2 \rangle \ell \frac{dA}{dt},$$

so

$$\frac{1}{\langle v_\perp^2 \rangle} \frac{d\langle v_\perp^2 \rangle}{dt} = -\frac{1}{A}\frac{dA}{dt}.$$

Due to particle number conservation, $n_e A\ell$ is a constant (equals the number of particles in the fluid element). So setting $d(n_e A\ell)/dt$ to zero, and dividing both sides by $n_e A\ell$, yields:

$$\frac{1}{n_e}\frac{dn_e}{dt} = -\frac{1}{\ell}\frac{d\ell}{dt} - \frac{1}{A}\frac{dA}{dt}.$$

(d) Using the above results,

$$\frac{1}{P_{e\parallel}}\frac{dP_{e\parallel}}{dt} = \frac{1}{n_e}\frac{dn_e}{dt} + \frac{1}{\langle v_\parallel^2\rangle}\frac{d\langle v_\parallel^2\rangle}{dt} = -3\frac{d\ell/dt}{\ell} - \frac{dA/dt}{A} = -\frac{5}{3}\theta - 2\sigma^{jk}b_j b_k,$$

$$\frac{1}{P_{e\perp}}\frac{dP_{e\perp}}{dt} = \frac{1}{n_e}\frac{dn_e}{dt} + \frac{1}{\langle v_\perp^2\rangle}\frac{d\langle v_\perp^2\rangle}{dt} = -\frac{d\ell/dt}{\ell} - 2\frac{dA/dt}{A} = -\frac{5}{3}\theta + \sigma^{jk}b_j b_k.$$

(e) When there is no expansion along B, we can set the $d\ell = 0$. Using the results from part (d) and mass conservation (which says that $d\ln A = -d\ln\rho$) respectively,

$$d(\ln P_{e\perp}) = -2d(\ln A) = 2d(\ln\rho)$$

or

$$\frac{\partial(\ln P_{e\perp})}{\partial(\ln\rho)} = 2.$$

And similarly,

$$d(\ln P_{e\parallel}) = -d(\ln A) = d(\ln\rho)$$

or

$$\frac{\partial(\ln P_{e\parallel})}{\partial(\ln\rho)} = 1.$$

When there is no expansion perpendicular to B, we can set the $dA = 0$. Using the results from part (d) and mass conservation respectively,

$$d(\ln P_{e\perp}) = -d(\ln\ell) = d(\ln\rho)$$

or

$$\frac{\partial(\ln P_{e\perp})}{\partial(\ln\rho)} = 1.$$

And similarly,

$$d(\ln P_{e\parallel}) = -3d(\ln\ell) = 3d(\ln\rho)$$

or

$$\frac{\partial(\ln P_{e\parallel})}{\partial(\ln\rho)} = 3.$$

(f) Finally, using the results of part (d),

$$d(\ln(P_\perp^2 P_\parallel)) = 2d(\ln P_\perp) + d(\ln P_\parallel) = -5d(\ln A) - 5d(\ln\ell) = 5d(\ln n_e),$$

so, integrating gives

$$P_\perp^2 P_\parallel \propto n_e^5.$$

Also,

$$d(\ln P_\perp) = -d(\ln \ell) - 2d(\ln A) = d(\ln n_e) - d(\ln A) = d(\ln n_e) + d(\ln B),$$

because the flux AB through a circle is constant. So, integrating gives

$$P_\perp \propto n_e B.$$

D

18.8 Thermoelectric transport coefficients [by Jeff Atwell]

(a) Basically a temperature gradient creates an electric current and an electric field causes heat flow because the carriers of both currents (electric and heat) are electrons, which always carry both energy and charge.

Suppose that initially the electrons on the left side of a room are hotter than those on the right side of the room. This means that initially the electrons on the left side are moving faster, on average. If we then let the room equilibrate, the electrons initially on the left side will penetrate faster to the right side, on average, then the right side electrons penetrate to the left side. Then we would say that both heat and charge have flowed from left to right (because electrons carry both energy and charge). Now, after the room has equilibrated, let's suppose that we turn on an electric field which causes the electrons to accelerate to the right. This clearly will cause heat to flow to the right.

(b) The distribution function $f(\mathbf{x}, \mathbf{v})$ is defined by the relation

$$f(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} = \text{Number of particles in } d\mathbf{x} d\mathbf{v}.$$

Recall from Chapter 2 the Boltzmann transport equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f + \frac{d\mathbf{v}}{dt} \cdot \boldsymbol{\nabla}_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}.$$

Now recall from exercise 2.13 that it is often valid to use the "collision-time approximation":

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{f - f_0}{t_D},$$

where f_0 is the distribution function in thermal equilibrium. We are interested in the steady state, so that $\partial f/\partial t = 0$, and so now the Boltzmann transport equation reads

$$\mathbf{v} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f + \frac{d\mathbf{v}}{dt} \cdot \boldsymbol{\nabla}_{\mathbf{v}} f = -\frac{f - f_0}{t_D}.$$

For simplicity, suppose there is an electric field E in the x direction and a temperature gradient dT/dx. Then the transport equation becomes

$$\frac{eE}{m}\frac{\partial f}{\partial v_x} + v_x \frac{\partial f}{\partial x} = -\frac{f - f_0}{t_D}.$$

Rewriting this, we have

$$f = f_0 - t_D \left(\frac{eE}{m}\frac{\partial f}{\partial v_x} + v_x \frac{\partial f}{\partial x} \right).$$

We now assume weak fields and small temperature gradients, that is, we assume $(f - f_0)/f_0 \ll 1$. To this approximation

$$f = f_0 - t_D \left(\frac{eE}{m}\frac{\partial f_0}{\partial v_x} + v_x \frac{\partial f_0}{\partial x} \right). \tag{22}$$

For the Maxwell-Boltzmann distribution, f_0 is a function of the energy $\mathcal{E}$ and the temperature T, so

$$\frac{\partial f_0}{\partial x} = \frac{\partial f_0}{\partial T}\frac{dT}{dx},$$

and

$$\frac{\partial f_0}{\partial v_x} = \frac{\partial f_0}{\partial \mathcal{E}}\frac{d\mathcal{E}}{dv_x} = m v_x \frac{\partial f_0}{\partial \mathcal{E}}.$$

If we suppose that $dT/dx = 0$, then equation 22 reduces to

$$f = f_0 - e t_D E v_x \frac{\partial f_0}{\partial \mathcal{E}}.$$

The electric current density is given by

$$J_x = \int e v_x f d^3\mathbf{v} = -t_D e^2 E \int v_x^2 \frac{\partial f_0}{\partial \mathcal{E}} d^3\mathbf{v},$$

as $\int v_x f_0 d^3\mathbf{v} = 0$ because f_0 is an even function of the velocity component v_x. Similarly, the heat flux is given by

$$q_x = \int \mathcal{E} v_x f d^3\mathbf{v} = -t_D e E \int \mathcal{E} v_x^2 \frac{\partial f_0}{\partial \mathcal{E}} d^3\mathbf{v}.$$

We work with the Maxwell-Boltzmann distribution

$$f_0 = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-mv^2/2kT}.$$

Notice that

$$\frac{\partial f_0}{\partial \mathcal{E}} = -\frac{1}{kT} f_0,$$

so

$$J_x = \frac{t_D e^2 E}{kT} \int v_x^2 f_0 d^3\mathbf{v},$$

and

$$q_x = \frac{t_D e E}{kT} \int \mathcal{E} v_x^2 f_0 d^3\mathbf{v}.$$

Doing the integrals on mathematica using this f_0, and dropping coefficients of order unity, I find:

$$J_x \sim \frac{ne^2 t_D}{m} E,$$

and

$$q_x \sim \frac{ne t_D kT}{m} E,$$

which agrees with equations (18.33). Now, from (18.31):

$$\kappa_e \sim \frac{e^2}{\sigma_T cm \ln\Lambda} \left(\frac{kT}{mc^2}\right)^{3/2} \sim \left(\frac{mc^2}{e^2}\right)^2 \frac{e^2}{cm\ln\Lambda} \left(\frac{kT}{mc^2}\right)^{3/2} \sim \frac{(kT)^{3/2}}{e^2\sqrt{m}\ln\Lambda},$$

where I have used (18.28) for the Thomson cross section. Then from (18.33):

$$\kappa_e \sim \frac{n t_D e^2}{m} \sim \frac{ne^2}{m} \frac{m^2 (kT/m)^{3/2}}{ne^4 \ln\Lambda} \sim \frac{(kT)^{3/2}}{e^2\sqrt{m}\ln\Lambda},$$

where I have used (18.21) for t_D. Notice that these two κ_e expressions agree.

(c) When $E = 0$, equation 22 reads

$$f = f_0 - t_D v_x \frac{dT}{dx}\frac{\partial f_0}{\partial T}.$$

Then we have

$$J_x = \int e v_x f d^3\mathbf{v} = -t_D e \frac{dT}{dx} \int v_x^2 \frac{\partial f_0}{\partial T} d^3\mathbf{v},$$

and

$$q_x = \int \mathcal{E} v_x f d^3\mathbf{v} = -t_D \frac{dT}{dx} \int \mathcal{E} v_x^2 \frac{\partial f_0}{\partial T} d^3\mathbf{v}.$$

Again, doing the integrals on mathematica, I find agreement with equations (18.34). Now, from (18.31):

$$\kappa \sim \frac{kc}{\sigma_T \ln\Lambda} \left(\frac{kT}{mc^2}\right)^{5/2} \sim \left(\frac{mc^2}{e^2}\right)^2 \frac{kc}{\ln\Lambda} \left(\frac{kT}{mc^2}\right)^{5/2} \sim \frac{k(kT)^{5/2}}{e^4\sqrt{m}\ln\Lambda},$$

where I have again used (18.28) for the Thomson cross section. Then from (18.34):

$$\kappa \sim \frac{n t_D k^2 T}{m} \sim \frac{nk^2 T}{m} \frac{m^2 (kT/m)^{3/2}}{ne^4 \ln\Lambda} \sim \frac{k(kT)^{5/2}}{e^4\sqrt{m}\ln\Lambda},$$

where I have again used (18.21) for t_D. Notice that these two κ expressions agree.

(d)

$$\frac{\alpha\beta}{\kappa_e\kappa} \sim \frac{\left(\frac{e}{kT}\kappa\right)\left(\frac{kT}{e}\kappa_e\right)}{\kappa_e\kappa} \sim 1 \approx 0.581,$$

and

$$\alpha T + \frac{kT}{e}\kappa_e \sim \frac{e}{kT}\kappa T + \frac{kT}{e}\frac{e}{kT}\beta \sim \frac{e}{k}\kappa + \beta \sim \frac{enkTt_D}{m} + \beta \sim \beta + \beta \approx \beta.$$

(e) Look at the heat flow when $J_x = \kappa_e E + \alpha\frac{dT}{dx} = 0$, or when $E = -\frac{\alpha}{\kappa_e}\frac{dT}{dx}$. It follows that

$$q_x = -\kappa\frac{dT}{dx} - \beta E = -\left(1 - \frac{\alpha\beta}{\kappa_e\kappa}\right)\kappa\frac{dT}{dx}.$$

18.12 Mirror machine [by Alexander Putilin '00]

(a) Since $\mu = mv_\perp^2/(2B)$ is conserved, then since B increases by a factor of 10 from center to end, then $v_\perp^2$ must increase by a factor of 10 as a particle goes from center to edge. So

$$v^2_{\perp,\text{final}} = 10v^2_{\perp,\text{initial}},$$

and by conservation of energy

$$v^2_{\|,\text{final}} = v^2_{\|,\text{initial}} - 9v^2_{\perp,\text{initial}}.$$

So, they escape when $v^2_{\|,\text{final}} > 0$, so only those particles released that have, initially, $v_\|^2 > 9v_\perp^2$ escape. That is, only a fraction given by

$$\int_{\tan^{-1}3}^{\pi/2} \cos\alpha d\alpha = 1 - \frac{3}{\sqrt{10}} = 0.0513$$

escape.

(b) The distribution function $dN/d|x|$ as a function of the pitch angle α, averaging over particles at all locations throughout the bottle, will be zero beyond $\tan^{-1}3$ (because these particles would have escaped). For small values of α, $dN/d|x|$ will be pretty flat, which corresponds to the fact that at $\alpha = 0$ (the mirror points), α changes linearly with time for a particle. The distribution function will drop off rapidly as the value $|\alpha| = \tan^{-1}3$ is approached.

If we just look at the middle of the bottle, then assuming the particles are released continuously in time (over the time period of many cycles for the particles to bounce from one mirror point to another), then for all times later, the distribution will remain isotropic there, except for the removal of all particles with $\sin\alpha > 0.95 = \sin\tan^{-1}3$.

(c) Since the diffusion time is independent of α, and the hole is so large in α space (from $\tan^{-1} 3$ to $\pi/2$, about 0.32 radians, which is $(0.32)/(\pi/2) = 1/5$ of α space), then one out of five collisions scatter a particle out of the bottle, so it leaks out (e-folds) in approximately time $5t_D$.

(d) Wouldn't it seem more reasonable that particles shouldn't diffuse with diffusion time independent of α but rather $\sin\alpha$ to account for solid angle properly? In such case, then in part (c), all occurrences of α should be replaced by $\sin\alpha$, and since $\sin\alpha$ goes from 0 to 1 particles that scatter into $\sin\alpha > 0.95$ get lost. In which case, 1/20 of all particles scatter out of the bottle in time t_D (as opposed to 1/5 as computed in part (c)), so the e-folding time is $20t_D$.