Solution 19

20.1 (a) $n_s m_s\left(\frac{\partial V_s}{\partial t} + (V_s\cdot\nabla)V_s\right) = -\nabla P_s + n_s q_s(\vec{E}+\vec{V_s}\times\vec{B})$ ①

Also $\vec{V_s} = \vec{V}_{s\parallel} + \vec{V}_{s\perp}$ with $\begin{cases}\vec{V}_{s\perp}\cdot\vec{B}=0\\ \vec{V}_{s\parallel}\times\vec{B}=0\end{cases}$

Take the cross product of Eq. ① with $\vec{B}$ and note that in equilibrium $\partial_t = 0$

$$\Rightarrow V_{s\perp} = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{\nabla P_s\times\vec{B}}{q_s n_s B^2} - \frac{m_s}{q_s B^2}\left[(\vec{V_s}\cdot\nabla)\vec{V_s}\right]_\perp\times\vec{B}$$

(b) $\vec{j} = \sum_s q_s n_s \vec{V_s}$, for the diamagnetic term

$$j_\perp = -\frac{q_s n_s}{n_s q_s}\frac{\nabla P_s\times B}{B^2} = -\frac{\nabla P\times\vec{B}}{B^2}$$

(c) Similarly, $j_\perp' = -\sum_s q_s n_s \frac{m_s}{q_s B^2}\left[(V_s\cdot\nabla)V_s\right]\times\vec{B}$

$$= -\sum_s \frac{\rho_s}{B^2}\left[(\vec{V_s}\cdot\nabla)\vec{V_s}\right]\times\vec{B}$$

Since $\rho \gtrsim \rho_p \gg \rho_e$

$$\Rightarrow j_\perp' \simeq -\frac{\rho}{B^2}\left[(\vec{V}\cdot\nabla)\vec{V}\right]\times\vec{B}$$

Solution for Problem Set 19-20

(compiled by Dan Grin and Nate Bode)
April 16, 2009

A

19.4 Ion Acoustic Waves [by Xinkai Wu 2002]
(a) The derivation of these equations is trivial, so we omit it here.
(b) Write the proton density as $n = n_0 + \delta n$ and substitute it into the equations of part (a), keeping only terms linear in δn, u, and Φ. We find

$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial u}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} = -\frac{e}{m_p}\frac{\partial \Phi}{\partial z} \tag{2}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = -\frac{e}{\epsilon_0}\left(\delta n - \frac{n_0 e}{k_B T_e}\Phi\right). \tag{3}$$

Plugging the plane-wave solution where δn, u, and Φ are all of the form $\propto exp[i(kz - \omega t)]$ into the above linearized equations, we find

$$-i\omega\delta n + n_0 iku = 0 \tag{4}$$

$$-i\omega u = -\frac{e}{m_p} ik\Phi \tag{5}$$

$$-k^2\Phi = -\frac{e}{\epsilon_0}\left(\delta n - \frac{n_0 e}{k_B T_e}\Phi\right). \tag{6}$$

For the above algebraic equation to possess a solution, the determinant of its coefficient matrix must vanish, which gives

$$\omega^2\left(-k^2 - \frac{n_0 e^2}{\epsilon_0 k_B T_e}\right) + \frac{n_0 e^2}{\epsilon_0 m_p}k^2 = 0. \tag{7}$$

Solving this yields the dispersion relation

$$\omega = \omega_{pp}(1 + 1/k^2\lambda_D^2)^{-1/2}, \tag{8}$$

with $\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_0 e^2}\right)^{1/2}$ and $\omega_{pp} = \left(\frac{n_0 e^2}{\epsilon_0 m_p}\right)^{1/2}$.

For long-wavelength, $k\lambda_D << 1$, one finds

$$\omega \approx \omega_{pp} k\lambda_D = k\left(\frac{k_B T_e}{m_p}\right)^{1/2}, \tag{9}$$

which agrees with eq. (20.36).

19.6 Dispersion and Faraday rotation of pulsar pulses [by Alexei Dvoretskii 2000]
(a) The pulses are limited in time, so they must be composed of a range of frequencies. Such a wave packet will propagate at the group velocity. For high frequencies,

$$v_g = \frac{d\omega}{dk} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}} \approx c\left(1-\frac{\omega_p^2}{2\omega^2}\right). \tag{10}$$

The difference in propagation time is given by

$$\triangle t = \int dx \left[\frac{1}{v_g(\omega_H)} - \frac{1}{v_g(\omega_L)}\right].$$

Simplifying and assuming $\omega_H \gg \omega_L \gg \omega_p$ and constant, we get

$$\triangle t = \frac{L}{c}\frac{\omega_p^2}{2\omega_L^2}.$$

Using $n = 3\times 10^4 \mathrm{m}^{-3}$ we get $\omega_p = 10^4 \mathrm{s}^{-1}$, so indeed $\omega_p \ll \omega_L \ll \omega_H$. The distance to the pulsar is then $L \approx 3\times 10^{17}$ m.

(b) Consider the Faraday rotation

$$\frac{d\chi}{dx} = \frac{\omega_p^2\omega_c}{2\omega^2 c}.$$

The overall rotation angle is then given by

$$\triangle\chi = \frac{1}{2\omega^2 c}\int \omega_p^2\omega_c dx$$

and

$$\frac{\triangle\chi}{\triangle t} = \frac{e}{m}\frac{\int \omega_p^2 B_\parallel}{\int \omega_p^2 dx} = \frac{e}{m}\frac{\int nB_\parallel}{\int n dx} = \frac{e}{m} < B_\parallel > . \tag{11}$$

From here the interstellar magnetic field can be estimated to be
$< B_\parallel > \approx 1.7\times 10^{-6}$ G.

B

19.5 Ion Acoustic Solitons [by Keith Matthews 2005]
(a) You'll note that the u coefficient $(k_BT_e/m_p)^{1/2}$ is the characteristic velocity of the system, so let's name it v_c. Also define $\alpha \equiv k_BT_e/e$. Then

$$\begin{aligned} u &= v_c(\epsilon u_1 + \epsilon^2 u_2 + ...) \\ \Phi &= \alpha(\epsilon\Phi_1 + \epsilon^2\Phi_2 + ...). \end{aligned} \tag{12}$$

For an arbitrary field ψ,

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= -\alpha\frac{\sqrt{2}}{\lambda_D}\epsilon^{1/2}\frac{\partial\psi}{\partial\eta} + \sqrt{2}\omega_{pp}\epsilon^{3/2}\frac{\partial\psi}{\partial\tau} \\ \frac{\partial\psi}{\partial z} &= \frac{2\omega_{pp}}{\lambda_D}\epsilon^{1/2}\frac{\partial\psi}{\partial\eta}. \end{aligned} \tag{13}$$

i) From the continuity eqn at leading order $\epsilon^{3/2}$ we find

$$-\frac{\partial n_1}{\partial\eta} + \frac{\partial u_1}{\partial\eta} = 0, \tag{14}$$

and at $\epsilon^{5/2}$ order we have

$$-\frac{\partial u_2}{\partial\eta} + \frac{\partial(n_1u_1)}{\partial\eta} = 0. \tag{15}$$

ii) To leading order $\epsilon^{3/2}$ the equation of motion produces

$$\frac{\partial u_1}{\partial\eta} - \frac{e\alpha}{m_p{v_c}^2}\frac{\partial\Phi_1}{\partial\eta} = 0.$$

However $\frac{e\alpha}{m_p{v_c}^2} = 1$ so we have

$$\frac{\partial u_1}{\partial\eta} - \frac{\partial\Phi_1}{\partial\eta} = 0. \tag{16}$$

We note that $\lambda_D\omega_pp = v_c$ so to $\epsilon^{5/2}$ we have

$$-\frac{\partial u_2}{\partial\eta} + \frac{\partial u_1}{\partial\tau} + \frac{1}{2}\frac{\partial(u_1)^2}{\partial\eta} = -\frac{\partial\Phi_2}{\partial.\eta} \tag{17}$$

iii) $\frac{en_0{\lambda_D}^2}{\alpha\epsilon_0 2} = \frac{1}{2}$ so at order ϵ and ϵ^2 Poisson's eqn gives

$$n_1 = \Phi_1 \tag{18}$$

$$\frac{\partial^2\Phi_1}{\partial\eta^2} = -\frac{1}{2}(n_2 - \Phi_2 - \frac{1}{2}{\Phi_1}^2). \tag{19}$$

iv) From Eqs. (14), (16) and (18) we find

$$n_1 = u_1 = \Phi_1. \tag{20}$$

Taking $\frac{\partial}{\partial \eta}$ of Eq. (19) and invoking Eq. (20) gives

$$\frac{\partial^3 n_1}{\partial \eta^3} = -\frac{1}{2}\left(\frac{\partial n_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \eta} - \frac{1}{2}\frac{\partial (n_1)^2}{\partial \eta}\right). \tag{21}$$

We find that we can take care of the $\frac{\partial n_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \eta}$ term by adding Eqs. (15) and (17) and again invoking Eq. (20).

$$\frac{\partial n_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \eta} = 2\frac{\partial n_1}{\partial \tau} + \frac{3}{2}\frac{\partial (n_1)^2}{\partial \eta}. \tag{22}$$

Substituting Eq. (22) into (21) gives us

$$\frac{\partial^3 n_1}{\partial \eta^3} = -\frac{1}{2}\left(2\frac{\partial n_1}{\partial \tau} + n_1\frac{\partial n_1}{\partial \eta}\right). \tag{23}$$

which is just what the doctor ordered. Since $n_1 = u_1 = \Phi_1$ any of the three can solve this KdV equation.

19.9 Exploration of Modes in CMA Diagram [by Kip Thorne 2005]
(a) EM waves in a cold unmagnetized plasma can only propagate when their frequency exceeds the plasma frequency ω_p, as can be seen from their dispersion relation

$$\omega = \sqrt{\omega_p^2 + c^2 k^2}.$$

Thus $\omega = \omega_p$ represents a cut-off. There is no turn-on.

The unmagnetized condition places us on the x-axis, and the lower limit of the frequency places us to the left of unity. These waves correspond to the diagram in the lower left corner with the modification that with no magnetic field the distinction between X and O dissolves, and R and L modes have the same phase velocity, so the are represented by the same circle.

(b) Denote
REM: Right hand polarized electromagnetic waves
LEM: Left hand polarized electromagnetic waves
RW: Right hand polarized Whistler waves
LW: Left hand polarized Whistler waves.

From Fig. 19.3 we can deduce the following relation between $V_\phi = \frac{c}{n}$ and ω.

As we move from the lower left to the upper right on the CMA diagram, ω decreases with B and n fixed. So we sequentially encounter

$$\begin{pmatrix} LEM \\ REM \end{pmatrix} \longrightarrow (LEM) \longrightarrow (RW) \longrightarrow \begin{pmatrix} LW \\ RW \end{pmatrix}.$$

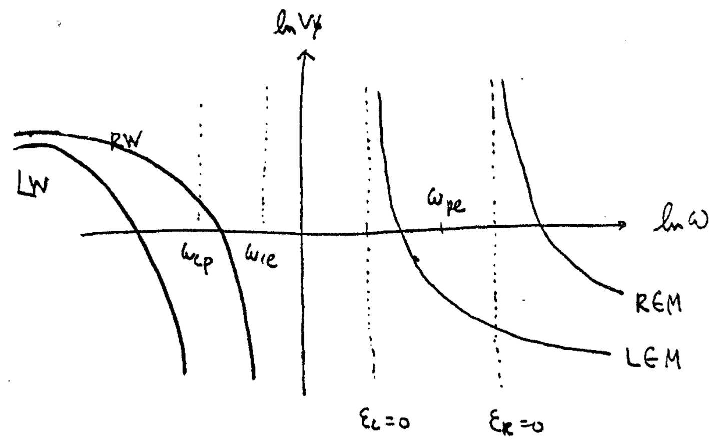

Figure 1: phase velocity vs. frequency for parallel propagating modes

The boundaries corresponding to $\epsilon_L = 0$ and $\epsilon_R = 0$ are specified in the diagram and correspond to the mode changes above.

(c) Denote:
O: ordinary mode
XL: extraordinary lower mode
XH: extraordinary hybrid mode
XU: extraordinary upper mode.

From Fig. 19.5 we can extract the following relation between the phase velocity and the frequency .

As ω decreases with B and n fixed, we observe the following pattern

$$\begin{pmatrix} XL \\ O \end{pmatrix} \longrightarrow \begin{pmatrix} XH \\ O \end{pmatrix} \longrightarrow (XH) \longrightarrow (XL).$$

The boundary for these regions are described by $\epsilon_1 = 0$, $\epsilon_3 = 0$, $\epsilon_1 = 0$ respectively. Combined with the previous result b), we obtain the pattern shown in the CMA diagram.

On the upper right in CMA one can see that RW becomes XL as we move from parallel to perpendicular mode and LW ceases to exist at some $\theta < \pi/2$. Similar phenomena can be read off from the CMA diagram.

C

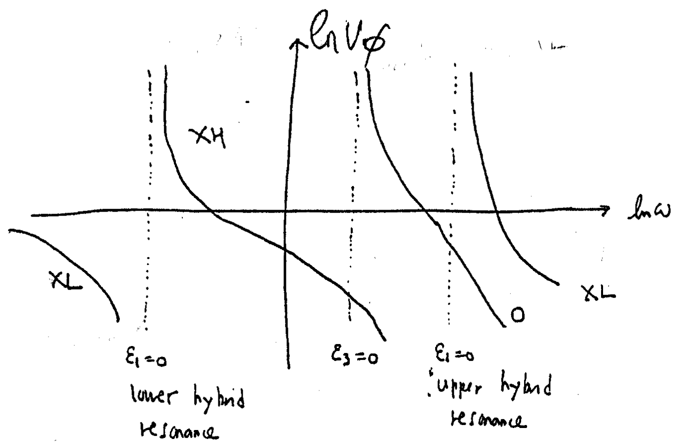

Figure 2: phase velocity vs. frequency for perpendicular propagating modes

19.8 Reflection of Short Waves by the Ionosphere [by Keith Matthews 2005]

The brute force approach:

For adiabatic spatial variations in the index of refraction, Fermat's principle chooses rays of extremal time. This gives simple differential equations (Eq. (6.42)) which we express as

$$\begin{aligned} &\text{r component:} \quad \frac{d^2r}{ds^2} + \left(\frac{d\tilde{n}/dr}{\tilde{n}}\right)\left[\left(\frac{dr}{ds}\right)^2 - 1\right] = 0 \\ &\theta \text{ component:} \quad \tilde{n}\frac{d\theta}{ds} = C \quad \text{a constant.} \end{aligned} \tag{24}$$

We choose polar coordinates because, as we shall see, the maximum range is of the order of the radius of the earth r_e. s parametrizes the path length.

We are told that the electron density is exponential in altitude:

$$n_e = n_0 e^{y/y_0}.$$

where y is the altitude $r - r_e$. From the values given in the problem: $y_0 = 21.71\,\text{km}$ and $n_0 = 10^7/\text{m}^3$. From Eq. (19.74): $\tilde{n}^2 = 1 - \frac{{\omega_{pe}}^2}{\omega^2}$ which we write as $\tilde{n}^2 \equiv 1 - \eta n_e$ where $\eta \equiv \frac{e^2}{m_e \epsilon_0 \omega^2}$. ($\eta n_0 = 8.04 \times 10^{-6}$ and $r_e = 6.4 \times 10^3$ km.) Note that $\tilde{n}' = \frac{1}{y_0}(\tilde{n} - 1)$.

I define ψ_0 at the point of transmission as the angle between the vertical and the outgoing ray. As a result the initial conditions are $\cos\psi_0 = (\frac{dr}{ds})_0$, and because $\sin\psi_0 = r_0(\frac{d\theta}{ds})_0$, $C = \left(\frac{\tilde{n}_0}{r_0}\right)\sin\psi_0$. $r_0 = r_e$ and I set $\theta_0 = 0$.

Numerically integrate with s as the independent variable until $r(s_f) = r_e$. (I used Mathematica.) The results give r and θ as a function of s, so you have to calculate the range $R = r_e\,\theta(s_f)$. By searching around I found the maximum range to be 6677 km for $\psi_0 = 90$ deg. It is vital that this ray does not pass through the maximum altitude of $y_{max} = 200$ km. I found that $y_{\rm top} = 25.7$ km.

20.1 Two-fluid Equation of Motion [by Xinkai Wu 2002]

We'll first write things in components and in the end convert back to vector/tensor notation. The Vlasov equation reads

$$\frac{\partial f_s}{\partial t} + v^j\frac{\partial f_s}{\partial x^j} + a^j\frac{\partial f_s}{\partial v^j} = 0. \tag{25}$$

We'll make use of the following definitions

$$\begin{aligned} n_s &= \int f_s dV_v \\ n_s u_s^i &= \int f_s v^i dV_v \\ P_s^{ik} &= m_s\int f_s(v^i - u_s^i)(v^k - u_s^k)dV_v \\ &= m_s\int f_s v^i v^k dV_v - m_s n_s u_s^i u_s^k. \end{aligned} \tag{26}$$

Multiplying the Vlasov eq. by v^k, integrating over velocity space, using integration by parts at various places and the fact that $\frac{\partial a^j}{\partial v^j} = 0$, also using the explicit expression Eq. (20.3) for the acceleration due to external EM field, we get

$$\frac{\partial(n_s u_s^k)}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{P_s^{jk}}{m} + n_s u_s^j u_s^k\right) - \frac{n_s q_s}{m_s}(E^k + (\mathbf{u}_s\times\mathbf{B})^k) = 0. \tag{27}$$

Now using the continuity equation,

$$\frac{\partial n_s}{\partial t} = -\frac{\partial}{\partial x^j}(n_s u_s^j), \tag{28}$$

one immediately sees that Eq. (27) reduces to Eq. (20.11) after converting to vector notation.

D

20.4 Landau Contour Deduced Using Laplace Transforms [by Xinkai Wu 2000]
(a) This part is nothing but a definition of Laplace transformation.
(b) The z-dependence is e^{ikz}, giving $\partial/\partial z \to ik$. Also integration by parts gives

$$\int_0^\infty dte^{-pt}\partial F_{s1}/\partial t = -F_{s1}(v,0) + p\int_0^\infty dte^{-pt}F_{s1}(v,t) \tag{29}$$

Noticing the above facts, we get by Laplace transforming the Vlasov equation:

$$0 = -F_{s1}(v,0) + p\tilde{F}_{s1}(v,p) + vik\tilde{F}_{s1}(v,p) + (q_s/m_s)F'_{s0}\tilde{E}(p) \tag{30}$$

where $s = p, e$. Laplace transforming $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives us a second equation:

$$ik\tilde{E}(p) = \sum_s (q_s/\epsilon_0)\int_{-\infty}^{\infty} dv[F_{s0}(v)/p + \tilde{F}_{s1}(v,p)] = \sum_s (q_s/\epsilon_0)\int_{-\infty}^{\infty} dv\tilde{F}_{s1}(v,p). \tag{31}$$

Where to get the last equality we've used the fact that the unperturbed charge density is zero, i.e. the contribution from $F_{s0}(v)$ vanishes. Combining these two equations we easily get Eq. (20.41).
(c) Setting $ip = \omega$, and plugging Eq. (20.41) into Eq. (20.42), we immediately get Eq. (20.26) without that overall minus sign.

E

20.5 Ion Acoustic Dispersion Relation [by Xinkai Wu 2000] Recall the definitions of Debye length and plasma frequency for species s:

$$\begin{aligned} \omega_{ps} &= \left(\frac{ne^2}{\epsilon_0 m_s}\right)^{1/2} \\ \lambda_{Ds} &= \left(\frac{\epsilon_0 k_B T_s}{ne^2}\right)^{1/2}, \end{aligned} \tag{32}$$

and the Maxwellian distribution is

$$F_s(v) = n\left(\frac{m_s}{2\pi k_B T_s}\right)^{1/2} exp\left[-\frac{m_s v^2}{2k_B T_s}\right]. \tag{33}$$

Now consider the integral

$$I_s(\frac{\omega_r}{k}) \equiv \int_P \frac{F_s'(v)}{v - \omega_r/k} dv. \tag{34}$$

For $\frac{\omega_r}{k} >> \sqrt{\frac{k_B T_s}{m_s}}$, $I_s \approx \frac{nk^2}{\omega_r^2}$(see eq. (21.37)), and this is the formula we are going to use for I_p. For $\frac{\omega_r}{k} << \sqrt{\frac{k_B T_s}{m_s}}$, $I_s \approx -\frac{nm_s}{k_B T_s}$(as can be easily seen by ignoring the ω_r/k in the denominator in the integral), and this is the formula we'll use for I_e.

In our problem, the total $F(v)$ is given by $F = F_e + \frac{m_e}{m_p} F_p$, and thus the total $I(\omega_r/k)$ is given by $I = I_e + \frac{m_e}{m_p} I_p$.

Now that we have the explicit (approximate) expressions for F and I, substitution into Eqs. (20.34) and (20.35) yields the desired expressions for ω_r and ω_i, after approximating $F_e(\omega_r/k)$ as $n\left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2}$ in the numerator of (20.35), which is justified by our assumption $\frac{\omega_r}{k} << \sqrt{\frac{k_B T_e}{m_e}}$).

20.8 Range of Unstable Wave Numbers [by Jeff Atwell]

For instability, we need a ζ with $\zeta_i > 0$ such that $k^2 = Z(\zeta)$ is real and positive. The corresponding $\omega = k\zeta$ specifies the mode.

ζ with $\zeta_i > 0$ correspond to points in the interior of the closed curve in the Z plane (see Fig. 20.4). So the intersection of the positive Z_r axis and the interior of the closed curve gives the range of wave numbers that we are looking for.

The shape of this closed curve depends on the distribution function $F(v)$. We are told our distribution function has two maxima, v_1 and v_2, and one minimum, $v_{\min}$. This means that the closed curve in the Z plane crosses the Z_r axis three times, twice moving downward, and once moving upward. (The curve in Fig. 20.4 happens to be an example of this situation.)

Suppose $v_1 < v_2$. Then in the case of the closed curve shown in Fig 20.4, we may write the range of wave numbers which have at least one unstable mode as $\sqrt{Z(v_2)} < k < \sqrt{Z(v_{\min})}$ (where Z is evaluated using Eq. (0.47)). For different shapes of the closed curve (i.e. for different combinations of $Z(v_1)$, $Z(v_{\min})$, and $Z(v_2)$ being positive and negative) the wave number range will be a bit different, but similar.

The steps in going from Eq. (20.47) to Eq. (20.49) work for maxima of $F(v)$, in addition to minima. This means that in the case shown in Fig. 20.4 we can instead write the minimum wave number with an unstable mode as

$$k_{\min}^2 = Z(v_2) = \frac{e^2}{m_e \epsilon_0} \int_{-\infty}^{+\infty} \frac{[F(v) - F(v_2)]}{(v - v_2)^2} dv,$$

for example.

Solution for Problem Set 19-20

(compiled by Dan Grin and Nate Bode)
April 16, 2009

A

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$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial u}{\partial z} = 0 \tag{1}$$

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Plugging the plane-wave solution where δn, u, and Φ are all of the form $\propto exp[i(kz - \omega t)]$ into the above linearized equations, we find

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$$-k^2\Phi = -\frac{e}{\epsilon_0}\left(\delta n - \frac{n_0 e}{k_B T_e}\Phi\right). \tag{6}$$

For the above algebraic equation to possess a solution, the determinant of its coefficient matrix must vanish, which gives

$$\omega^2\left(-k^2 - \frac{n_0 e^2}{\epsilon_0 k_B T_e}\right) + \frac{n_0 e^2}{\epsilon_0 m_p}k^2 = 0. \tag{7}$$

Solving this yields the dispersion relation

$$\omega = \omega_{pp}(1 + 1/k^2\lambda_D^2)^{-1/2}, \tag{8}$$

with $\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_0 e^2}\right)^{1/2}$ and $\omega_{pp} = \left(\frac{n_0 e^2}{\epsilon_0 m_p}\right)^{1/2}$.

For long-wavelength, $k\lambda_D << 1$, one finds

$$\omega \approx \omega_{pp} k\lambda_D = k\left(\frac{k_B T_e}{m_p}\right)^{1/2}, \tag{9}$$

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The difference in propagation time is given by

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Simplifying and assuming $\omega_H \gg \omega_L \gg \omega_p$ and constant, we get

$$\triangle t = \frac{L}{c}\frac{\omega_p^2}{2\omega_L^2}.$$

Using $n = 3 \times 10^4 \mathrm{m}^{-3}$ we get $\omega_p = 10^4 \mathrm{s}^{-1}$, so indeed $\omega_p \ll \omega_L \ll \omega_H$. The distance to the pulsar is then $L \approx 3 \times 10^{17}$ m.

(b) Consider the Faraday rotation

$$\frac{d\chi}{dx} = \frac{\omega_p^2 \omega_c}{2\omega^2 c}.$$

The overall rotation angle is then given by

$$\triangle\chi = \frac{1}{2\omega^2 c}\int \omega_p^2 \omega_c dx$$

and

$$\frac{\triangle\chi}{\triangle t} = \frac{e}{m}\frac{\int \omega_p^2 B_\parallel}{\int \omega_p^2 dx} = \frac{e}{m}\frac{\int n B_\parallel}{\int n dx} = \frac{e}{m} < B_\parallel > . \tag{11}$$

From here the interstellar magnetic field can be estimated to be
$< B_\parallel > \approx 1.7 \times 10^{-6}$ G.

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$$\begin{aligned} u &= v_c(\epsilon u_1 + \epsilon^2 u_2 + ...) \\ \Phi &= \alpha(\epsilon \Phi_1 + \epsilon^2 \Phi_2 + ...). \end{aligned} \tag{12}$$

For an arbitrary field ψ,

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= -\alpha \frac{\sqrt{2}}{\lambda_D} \epsilon^{1/2} \frac{\partial \psi}{\partial \eta} + \sqrt{2}\omega_{pp} \epsilon^{3/2} \frac{\partial \psi}{\partial \tau} \\ \frac{\partial \psi}{\partial z} &= \frac{2\omega_{pp}}{\lambda_D} \epsilon^{1/2} \frac{\partial \psi}{\partial \eta}. \end{aligned} \tag{13}$$

i) From the continuity eqn at leading order $\epsilon^{3/2}$ we find

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$$\frac{\partial u_1}{\partial \eta} - \frac{e\alpha}{m_p {v_c}^2} \frac{\partial \Phi_1}{\partial \eta} = 0.$$

However $\frac{e\alpha}{m_p {v_c}^2} = 1$ so we have

$$\frac{\partial u_1}{\partial \eta} - \frac{\partial \Phi_1}{\partial \eta} = 0. \tag{16}$$

We note that $\lambda_D \omega_p p = v_c$ so to $\epsilon^{5/2}$ we have

$$-\frac{\partial u_2}{\partial \eta} + \frac{\partial u_1}{\partial \tau} + \frac{1}{2} \frac{\partial (u_1)^2}{\partial \eta} = -\frac{\partial \Phi_2}{\partial . \eta} \tag{17}$$

iii) $\frac{e n_0 {\lambda_D}^2}{\alpha \epsilon_0 2} = \frac{1}{2}$ so at order ϵ and ϵ^2 Poisson's eqn gives

$$n_1 = \Phi_1 \tag{18}$$

$$\frac{\partial^2 \Phi_1}{\partial \eta^2} = -\frac{1}{2}(n_2 - \Phi_2 - \frac{1}{2}{\Phi_1}^2). \tag{19}$$

iv) From Eqs. (14), (16) and (18) we find

$$n_1 = u_1 = \Phi_1. \tag{20}$$

Taking $\frac{\partial}{\partial \eta}$ of Eq. (19) and invoking Eq. (20) gives

$$\frac{\partial^3 n_1}{\partial \eta^3} = -\frac{1}{2}\left(\frac{\partial n_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \eta} - \frac{1}{2}\frac{\partial (n_1)^2}{\partial \eta}\right). \tag{21}$$

We find that we can take care of the $\frac{\partial n_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \eta}$ term by adding Eqs. (15) and (17) and again invoking Eq. (20).

$$\frac{\partial n_2}{\partial \eta} - \frac{\partial \Phi_2}{\partial \eta} = 2\frac{\partial n_1}{\partial \tau} + \frac{3}{2}\frac{\partial (n_1)^2}{\partial \eta}. \tag{22}$$

Substituting Eq. (22) into (21) gives us

$$\frac{\partial^3 n_1}{\partial \eta^3} = -\frac{1}{2}\left(2\frac{\partial n_1}{\partial \tau} + n_1\frac{\partial n_1}{\partial \eta}\right). \tag{23}$$

which is just what the doctor ordered. Since $n_1 = u_1 = \Phi_1$ any of the three can solve this KdV equation.

19.9 Exploration of Modes in CMA Diagram [by Kip Thorne 2005]
(a) EM waves in a cold unmagnetized plasma can only propagate when their frequency exceeds the plasma frequency ω_p, as can be seen from their dispersion relation

$$\omega = \sqrt{\omega_p^2 + c^2k^2}.$$

Thus $\omega = \omega_p$ represents a cut-off. There is no turn-on.

The unmagnetized condition places us on the x-axis, and the lower limit of the frequency places us to the left of unity. These waves correspond to the diagram in the lower left corner with the modification that with no magnetic field the distinction between X and O dissolves, and R and L modes have the same phase velocity, so the are represented by the same circle.

(b) Denote
REM: Right hand polarized electromagnetic waves
LEM: Left hand polarized electromagnetic waves
RW: Right hand polarized Whistler waves
LW: Left hand polarized Whistler waves.

From Fig. 19.3 we can deduce the following relation between $V_\phi = \frac{c}{n}$ and ω.

As we move from the lower left to the upper right on the CMA diagram, ω decreases with B and n fixed. So we sequentially encounter

$$\begin{pmatrix} LEM \\ REM \end{pmatrix} \longrightarrow (LEM) \longrightarrow (RW) \longrightarrow \begin{pmatrix} LW \\ RW \end{pmatrix}.$$

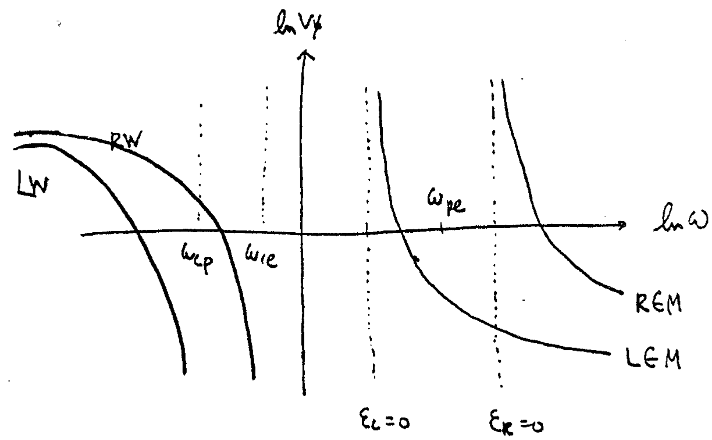

Figure 1: phase velocity vs. frequency for parallel propagating modes

The boundaries corresponding to $\epsilon_L = 0$ and $\epsilon_R = 0$ are specified in the diagram and correspond to the mode changes above.

(c) Denote:
O: ordinary mode
XL: extraordinary lower mode
XH: extraordinary hybrid mode
XU: extraordinary upper mode.

From Fig. 19.5 we can extract the following relation between the phase velocity and the frequency .

As ω decreases with B and n fixed, we observe the following pattern

$$\begin{pmatrix} XL \\ O \end{pmatrix} \longrightarrow \begin{pmatrix} XH \\ O \end{pmatrix} \longrightarrow (XH) \longrightarrow (XL).$$

The boundary for these regions are described by $\epsilon_1 = 0$, $\epsilon_3 = 0$, $\epsilon_1 = 0$ respectively. Combined with the previous result b), we obtain the pattern shown in the CMA diagram.

On the upper right in CMA one can see that RW becomes XL as we move from parallel to perpendicular mode and LW ceases to exist at some $\theta < \pi/2$. Similar phenomena can be read off from the CMA diagram.

C

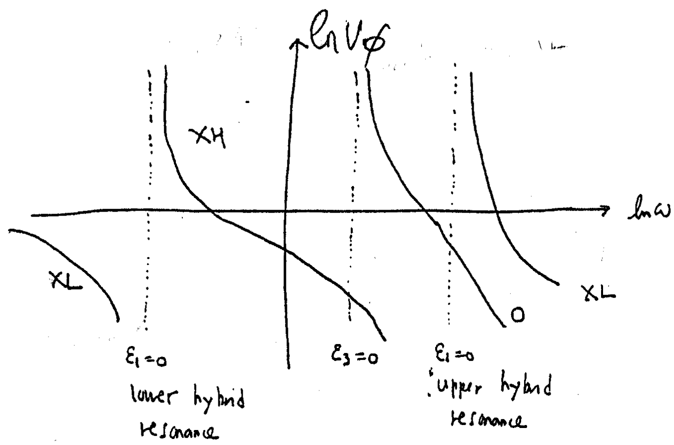

Figure 2: phase velocity vs. frequency for perpendicular propagating modes

19.8 Reflection of Short Waves by the Ionosphere [by Keith Matthews 2005]

The brute force approach:

For adiabatic spatial variations in the index of refraction, Fermat's principle chooses rays of extremal time. This gives simple differential equations (Eq. (6.42)) which we express as

$$\begin{aligned} &\text{r component:} && \tfrac{d^2r}{ds^2} + \left(\tfrac{d\tilde{n}/dr}{\tilde{n}}\right)\left[\left(\tfrac{dr}{ds}\right)^2 - 1\right] = 0 \\ &\theta \text{ component:} && \tilde{n}\tfrac{d\theta}{ds} = C \quad \text{a constant.} \end{aligned} \tag{24}$$

We choose polar coordinates because, as we shall see, the maximum range is of the order of the radius of the earth r_e. s parametrizes the path length.

We are told that the electron density is exponential in altitude:

$$n_e = n_0 e^{y/y_0}.$$

where y is the altitude $r - r_e$. From the values given in the problem: $y_0 = 21.71$ km and $n_0 = 10^7/\text{m}^3$. From Eq. (19.74): $\tilde{n}^2 = 1 - \frac{{\omega_{pe}}^2}{\omega^2}$ which we write as $\tilde{n}^2 \equiv 1 - \eta n_e$ where $\eta \equiv \frac{e^2}{m_e \epsilon_0 \omega^2}$. ($\eta n_0 = 8.04 \times 10^{-6}$ and $r_e = 6.4 \times 10^3$ km.) Note that $\tilde{n}' = \frac{1}{y_0}(\tilde{n} - 1)$.

I define ψ_0 at the point of transmission as the angle between the vertical and the outgoing ray. As a result the initial conditions are $\cos\psi_0 = (\frac{dr}{ds})_0$, and because $\sin\psi_0 = r_0(\frac{d\theta}{ds})_0$, $C = \left(\frac{\tilde{n}_0}{r_0}\right)\sin\psi_0$. $r_0 = r_e$ and I set $\theta_0 = 0$.

Numerically integrate with s as the independent variable until $r(s_f) = r_e$. (I used Mathematica.) The results give r and θ as a function of s, so you have to calculate the range $R = r_e\,\theta(s_f)$. By searching around I found the maximum range to be 6677 km for $\psi_0 = 90$ deg. It is vital that this ray does not pass through the maximum altitude of $y_{max} = 200$ km. I found that $y_{\rm top} = 25.7$ km.

20.1 Two-fluid Equation of Motion [by Xinkai Wu 2002]

We'll first write things in components and in the end convert back to vector/tensor notation. The Vlasov equation reads

$$\frac{\partial f_s}{\partial t} + v^j\frac{\partial f_s}{\partial x^j} + a^j\frac{\partial f_s}{\partial v^j} = 0. \tag{25}$$

We'll make use of the following definitions

$$\begin{aligned} n_s &= \int f_s dV_v \\ n_s u_s^i &= \int f_s v^i dV_v \\ P_s^{ik} &= m_s\int f_s(v^i - u_s^i)(v^k - u_s^k)dV_v \\ &= m_s\int f_s v^i v^k dV_v - m_s n_s u_s^i u_s^k. \end{aligned} \tag{26}$$

Multiplying the Vlasov eq. by v^k, integrating over velocity space, using integration by parts at various places and the fact that $\frac{\partial a^j}{\partial v^j} = 0$, also using the explicit expression Eq. (20.3) for the acceleration due to external EM field, we get

$$\frac{\partial(n_s u_s^k)}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{P_s^{jk}}{m} + n_s u_s^j u_s^k\right) - \frac{n_s q_s}{m_s}(E^k + (\mathbf{u}_s\times\mathbf{B})^k) = 0. \tag{27}$$

Now using the continuity equation,

$$\frac{\partial n_s}{\partial t} = -\frac{\partial}{\partial x^j}(n_s u_s^j), \tag{28}$$

one immediately sees that Eq. (27) reduces to Eq. (20.11) after converting to vector notation.

D

20.4 Landau Contour Deduced Using Laplace Transforms [by Xinkai Wu 2000]
(a) This part is nothing but a definition of Laplace transformation.
(b) The z-dependence is e^{ikz}, giving $\partial/\partial z \to ik$. Also integration by parts gives

$$\int_0^\infty dte^{-pt}\partial F_{s1}/\partial t = -F_{s1}(v,0) + p\int_0^\infty dte^{-pt}F_{s1}(v,t) \tag{29}$$

Noticing the above facts, we get by Laplace transforming the Vlasov equation:

$$0 = -F_{s1}(v,0) + p\tilde{F}_{s1}(v,p) + vik\tilde{F}_{s1}(v,p) + (q_s/m_s)F'_{s0}\tilde{E}(p) \tag{30}$$

where $s = p, e$. Laplace transforming $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ gives us a second equation:

$$ik\tilde{E}(p) = \sum_s (q_s/\epsilon_0)\int_{-\infty}^{\infty} dv[F_{s0}(v)/p + \tilde{F}_{s1}(v,p)] = \sum_s (q_s/\epsilon_0)\int_{-\infty}^{\infty} dv\tilde{F}_{s1}(v,p). \tag{31}$$

Where to get the last equality we've used the fact that the unperturbed charge density is zero, i.e. the contribution from $F_{s0}(v)$ vanishes. Combining these two equations we easily get Eq. (20.41).
(c) Setting $ip = \omega$, and plugging Eq. (20.41) into Eq. (20.42), we immediately get Eq. (20.26) without that overall minus sign.

E

20.5 Ion Acoustic Dispersion Relation [by Xinkai Wu 2000] Recall the definitions of Debye length and plasma frequency for species s:

$$\begin{aligned} \omega_{ps} &= \left(\frac{ne^2}{\epsilon_0 m_s}\right)^{1/2} \\ \lambda_{Ds} &= \left(\frac{\epsilon_0 k_B T_s}{ne^2}\right)^{1/2}, \end{aligned} \tag{32}$$

and the Maxwellian distribution is

$$F_s(v) = n\left(\frac{m_s}{2\pi k_B T_s}\right)^{1/2} exp\left[-\frac{m_s v^2}{2k_B T_s}\right]. \tag{33}$$

Now consider the integral

$$I_s(\frac{\omega_r}{k}) \equiv \int_P \frac{F_s'(v)}{v - \omega_r/k} dv. \qquad (34)$$

For $\frac{\omega_r}{k} >> \sqrt{\frac{k_B T_s}{m_s}}$, $I_s \approx \frac{nk^2}{\omega_r^2}$(see eq. (21.37)), and this is the formula we are going to use for I_p. For $\frac{\omega_r}{k} << \sqrt{\frac{k_B T_s}{m_s}}$, $I_s \approx -\frac{nm_s}{k_B T_s}$(as can be easily seen by ignoring the ω_r/k in the denominator in the integral), and this is the formula we'll use for I_e.

In our problem, the total $F(v)$ is given by $F = F_e + \frac{m_e}{m_p}F_p$, and thus the total $I(\omega_r/k)$ is given by $I = I_e + \frac{m_e}{m_p}I_p$.

Now that we have the explicit (approximate) expressions for F and I, substitution into Eqs. (20.34) and (20.35) yields the desired expressions for ω_r and ω_i, after approximating $F_e(\omega_r/k)$ as $n\left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2}$ in the numerator of (20.35), which is justified by our assumption $\frac{\omega_r}{k} << \sqrt{\frac{k_B T_e}{m_e}}$).

20.8 Range of Unstable Wave Numbers [by Jeff Atwell]

For instability, we need a ζ with $\zeta_i > 0$ such that $k^2 = Z(\zeta)$ is real and positive. The corresponding $\omega = k\zeta$ specifies the mode.

ζ with $\zeta_i > 0$ correspond to points in the interior of the closed curve in the Z plane (see Fig. 20.4). So the intersection of the positive Z_r axis and the interior of the closed curve gives the range of wave numbers that we are looking for.

The shape of this closed curve depends on the distribution function $F(v)$. We are told our distribution function has two maxima, v_1 and v_2, and one minimum, $v_{\min}$. This means that the closed curve in the Z plane crosses the Z_r axis three times, twice moving downward, and once moving upward. (The curve in Fig. 20.4 happens to be an example of this situation.)

Suppose $v_1 < v_2$. Then in the case of the closed curve shown in Fig 20.4, we may write the range of wave numbers which have at least one unstable mode as $\sqrt{Z(v_2)} < k < \sqrt{Z(v_{\min})}$ (where Z is evaluated using Eq. (0.47)). For different shapes of the closed curve (i.e. for different combinations of $Z(v_1)$, $Z(v_{\min})$, and $Z(v_2)$ being positive and negative) the wave number range will be a bit different, but similar.

The steps in going from Eq. (20.47) to Eq. (20.49) work for maxima of $F(v)$, in addition to minima. This means that in the case shown in Fig. 20.4 we can instead write the minimum wave number with an unstable mode as

$$k_{\min}^2 = Z(v_2) = \frac{e^2}{m_e \epsilon_0} \int_{-\infty}^{+\infty} \frac{[F(v) - F(v_2)]}{(v - v_2)^2} dv,$$

for example.