Solution 21

Please check previous solution for Prob. 21.4 / 21.8

21.3 $\nabla \cdot P = \rho_e$

$$\rho_e = \sum_s q_s \int F_{s1} dv = \sum_s \frac{-iq_s^2}{m_s} \int \frac{F_{s0}' dv}{\omega - kv} E$$

$$\Rightarrow \begin{cases} \nabla \cdot P = E \cdot \left(\sum_s \frac{iq_s^2}{m_s} \int \frac{F_{s0}'}{\omega - kv} dv \right) \\ ik_z \cdot P_z = E_z \left(\sum_s \frac{iq_s}{m_s} \int \frac{F_{s0}'}{\omega - kv} dv \right) \end{cases}$$

$P_z = \varepsilon_0 \chi_{zz}$

$$\Rightarrow \quad \varepsilon_{zz} = 1 + \sum_s \frac{q_s^2}{m_s k_z \varepsilon_0} \int \frac{F_{s0}'}{\omega - kv} dv$$

$$L_{ij} = k_i k_j - k^2 \delta_{ij} + \frac{\omega^2}{c^2} \varepsilon_{ij}$$

$$\det L = \begin{vmatrix} -k^2 & & \\ & -k^2 & \\ & & \frac{\omega^2}{c^2} \varepsilon_{zz} \end{vmatrix} = 0$$

$$\Rightarrow \varepsilon_{zz} = 0 \quad \text{or} \quad 1 + \sum_s \frac{q_s^2}{m_s k \varepsilon_0} \int \frac{F_{s0}'}{\omega - kv} dv = 0$$

$\varepsilon_3(\omega, k) = 0$ which gives wave's dispersion relation

Solution 21

22.7
$$\begin{cases} \dfrac{\partial F_0}{\partial t} = \dfrac{\pi e^2}{m_e^2 \varepsilon_0} \dfrac{\partial}{\partial v}\left(\dfrac{\varepsilon_k}{v} \dfrac{\partial F_0}{\partial v} \right) \\[2ex] \dfrac{\partial \varepsilon_k}{\partial t} = \dfrac{\pi e^2}{m_e \varepsilon_0 \omega_p} v^2 \varepsilon_k \dfrac{\partial F_0}{\partial v} \end{cases}$$

$\Rightarrow \quad \dfrac{\partial F_0}{\partial t} = -3 \dfrac{\omega_p}{m_e} \dfrac{1}{v^4} \dfrac{\partial \varepsilon_k}{\partial t}$, So all the linear growth rates are negative.

22.9. $\Phi(\phi) = \dfrac{n_0 k_B T_e}{\varepsilon_0} \left\{ \left[1 - \left(1 - \dfrac{2e\phi}{m_p v^2}\right)^{1/2} \right] \dfrac{m_p v^2}{k_B T_e} - \left(e^{\frac{e\phi}{k_B T}} - 1 \right) \right\}$

Can be obtained by integrating the right hand side of

$$-\nabla \Phi(\phi) = \phi'' = -\frac{n_0 e}{\varepsilon_0} \left[\left(1 - \frac{2e\phi}{m_p v^2}\right)^{-1/2} - e^{e\phi/k_B T} \right]$$

$$\Phi\left(\frac{m_p v^2}{2e}\right) = \frac{n_0 k_B T_e}{\varepsilon_0} \left[\frac{m_p v^2}{k_B T_e} + 1 - e^{m_p v^2/2k_B T_e} \right]$$

$$= \frac{n_0 k_B T_e}{\varepsilon_0} \left[M^2 + 1 - e^{M^2/2} \right]$$

Numerically solve $M^2 + 1 = e^{M^2/2} \Rightarrow M \approx 1.5852$

Solution for Problem Set 21

(compiled by Nate Bode)
April 24, 2009

A

21.1 Non-resonant Particle Energy in Wave [by Keith Matthews adapted from Chris Hirata]

The rate of change of the non-resonant electron kinetic energy is given by

$$\frac{dU_e}{dt} = \frac{d}{dt}\int \frac{1}{2} m_e v^2 F_0 dv$$

Insert eq. (21.20)

$$\frac{d}{dt}F_0 = \frac{\partial}{\partial t}F_0 + v\frac{\partial}{\partial z}F_0 = \frac{\partial}{\partial v}\left(D\frac{\partial}{\partial v}F_0\right)$$

We note that $D^{non-res}$ is independent of v, integrate by parts twice, and then simply integrate to get

$$\frac{dU_e}{dt} = m_e D \int F_0 dv = m_e n D$$

Insert eq. (21.23) for $D = D^{non-res}$

$$\frac{dU_e}{dt} = \frac{1}{2}\int_0^{\infty} 2\omega_i \varepsilon_k dk$$

which is the desired eq. (21.24).

21.2 Energy Conservation [by Alexander Putilin/ '99]

The electron kinetic energy density and momentum density are given by

$$U^e = \int \frac{1}{2} m_e v^2 F_0 dv \tag{1}$$

$$S_z^e = \int \frac{1}{2} m_e v^3 F_0 dv \tag{2}$$

and we have

$$\frac{\partial U^e}{\partial t} + \frac{\partial S_z^e}{\partial z} = \int \frac{1}{2} m_e v^2 \left(\frac{\partial F_0}{\partial t} + v \frac{\partial F_0}{\partial z} \right) dv \tag{3}$$

$$= \int \frac{1}{2} m_e v^2 \frac{\partial}{\partial v} \left(D \frac{\partial F_0}{\partial v} \right) dv \tag{4}$$

$$= -\int D \frac{\partial F_0}{\partial v} m_e v dv \tag{5}$$

Now using eq. (21.22) for the resonant electrons

$$D = \frac{e^2 \pi}{\epsilon_0 m_e^2} \int dk \mathcal{E}_k \delta(\omega_r - kv) \tag{6}$$

we find

$$\frac{\partial U^e}{\partial t} + \frac{\partial S_z^e}{\partial z} = \frac{-e^2 \pi}{\epsilon_0 m_e} \int dk \frac{\mathcal{E}_k \omega_r}{k^2} F_0' \left(\frac{\omega_r}{k} \right) \tag{7}$$

which becomes, upon using eq. (21.12) $F_0' \left(\frac{\omega_r}{k} \right) = \frac{2\epsilon_0 m_e}{\pi e^2} \frac{k^2}{\omega_r} \omega_i$,

$$-\int dk\ 2\omega_i \mathcal{E}_k \tag{8}$$

$$= -\frac{\partial}{\partial t} \int dk \mathcal{E}_k - \frac{\partial}{\partial z} \int dk \mathcal{E}_k \frac{\omega_r}{k} \quad \text{using eq. (21.18)} \tag{9}$$

$$= -\frac{\partial U^w}{\partial t} - \frac{\partial S_z^w}{\partial z} \tag{10}$$

where we've used the facts that $\int dk \mathcal{E}_k$ is the wave's energy density U^w, and $\int dk \mathcal{E}_k \frac{\partial \omega_r}{\partial k}$ is the wave's energy flux S_z^w.

Thus we finally have

$$\frac{\partial U^e}{\partial t} + \frac{\partial S_z^e}{\partial z} = -\frac{\partial U^w}{\partial t} - \frac{\partial S_z^w}{\partial z} \tag{11}$$

which is the energy conservation law.

21.3 Cerenkov Power in Electrostatic Waves [by Alexander Putilin '99]

The emission rate of plasmons is given by (21.43)

$$W = \frac{\pi e^2 \omega_r}{\epsilon_0 k^2 \hbar} \delta(\omega_r - \mathbf{k} \cdot \mathbf{v}) \tag{12}$$

Each plasmon has energy $\hbar\omega_r$, so the radiated power per unit time is

$$P = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} W \hbar \omega_r = \frac{e^2}{8\pi^2 \epsilon_0} \int d^3\mathbf{k} \frac{\omega_r^2}{k^2} \delta(\omega_r - \mathbf{k} \cdot \mathbf{v}) \tag{13}$$

The integration is over the region $k < k_{max}$(outside this region waves are strongly Landau damped). A good estimate of k_{max} is the inverse Debye length: $k_{max} \sim 1/\lambda_D$(see the discussion at the end of Sec. 21.3.5). Since $k\lambda_D < 1$, we can approximate $\omega_r(k)$ by a constant ω_p.

Choosing $\mathbf{v}$ to point along z-axis, we have

$$P = \frac{e^2\omega_p^2}{8\pi^2\epsilon_0}\int_{k<k_{max}} d^3\mathbf{k}\frac{1}{k^2}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) \tag{14}$$

$$= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0}\int_{k<k_{max}} d^2k_\perp dk_z \frac{1}{k_\perp^2 + k_z^2}\delta(\omega_r - \mathbf{k}\cdot\mathbf{v}) \tag{15}$$

$$= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0 v}\int_{k_\perp^2<k_{max}^2-(\omega_p^2/v^2)} d^2k_\perp \frac{1}{k_\perp^2 + (\omega_p^2/v^2)} \tag{16}$$

$$= \frac{e^2\omega_p^2}{8\pi^2\epsilon_0 v}\int_0^{k_{max}^2-(\omega_p^2/v^2)} 2\pi k_\perp dk_\perp \frac{1}{k_\perp^2 + (\omega_p^2/v^2)} \tag{17}$$

$$= \frac{e^2\omega_p^2}{4\pi\epsilon_0 v}\ln\left(\frac{k_{max}v}{\omega_p}\right) \tag{18}$$

Note that P depends on k_{max} logarithmically. So if v is sufficiently large, it doesn't make much difference what particular definition we use for k_{max}.

21.4 Electron Fokker-Planck Equation [by Xinkai Wu '02]

It's very straightforward to carry out the Taylor expansion in $\hbar$ to second order, so we don't bother to write down all the formulae here. Rather we'll just point out a few things worth noticing in the expansion. Fristly you may find it's easier to use component notation rather than tensor notation. Secondly, you may want to group terms coming out of expanding the r.h.s. of eq. (21.47) into three categories: f terms, $\frac{\partial f}{\partial v^j}$ terms, and $\frac{\partial^2 f}{\partial v^j \partial v^n}$ terms. Thirdly, to get the final answer, notice that in the classical limit both η and W are large, i.e. of order $1/\hbar$. After all these, you can set $\hbar \to 0$ and recover eq. (21.48).

21.5 Three-Wave Mixing [by Xinkai Wu '02]

(a) Diagram (b) in Fig. 21.5 gives the rate of creation of B,

$$\left(\frac{d\eta_B}{dt}\right)_{creation} = \int W_{C\to AB}\eta_C(1+\eta_A)(1+\eta_B)\frac{dV_{k_A}}{(2\pi)^3}\frac{dV_{k_C}}{(2\pi)^3} \tag{19}$$

where in the above expression, the unity in $(1+\eta_A)$ corresponds to spontaneous emission and η_A corresponds to induced emission. Similarly for $(1+\eta_B)$.

Diagram (a) in Fig. 21.5 gives the rate of destruction of B,

$$\left(\frac{d\eta_B}{dt}\right)_{destruction} = -\int W_{AB\to C}(\eta_C+1)\eta_A\eta_B\frac{dV_{k_A}}{(2\pi)^3}\frac{dV_{k_C}}{(2\pi)^3} \tag{20}$$

The net rate of change is given by the sum of these two, and by the principle of detailed balance $W_{AB\to C} = W_{C\to AB}$. So we get

$$\frac{d\eta_B}{dt} = \int W_{AB\leftrightarrow C}\left[(1+\eta_A+\eta_B)\eta_C - \eta_A\eta_B\right]\frac{dV_{k_A}}{(2\pi)^3}\,\frac{dV_{k_C}}{(2\pi)^3} \tag{21}$$

(b) Under the approximation stated in the problem, we have

$$\eta_C(1+\eta_A+\eta_B) - \eta_A\eta_B \approx \eta_B(\eta_C - \eta_A) \tag{22}$$

Also let's change notation: $\mathbf{k}_B \equiv \mathbf{k}$, $\mathbf{k}_C \equiv \mathbf{k}'$, $\eta_B(\mathbf{k}_B) \equiv \eta_{ia}(\mathbf{k})$, $\eta_A(\mathbf{k}_A) \equiv \eta_L(\mathbf{k}_A)$, $\eta_C(\mathbf{k}_C) \equiv \eta_L(\mathbf{k}')$, also change ω_A and ω_C to ω_L. Then using the Taylor expansion

$$\eta_L(\mathbf{k}') - \eta_L(\mathbf{k}'-\mathbf{k}) \approx \mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}') \tag{23}$$

$$\omega_L(\mathbf{k}'-\mathbf{k}) - \omega_L(\mathbf{k}') \approx -\mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\omega_L(\mathbf{k}') = -\mathbf{k}\cdot\mathbf{V}_g(\mathbf{k}') \tag{24}$$

we easily get eq. (21.59).

(c) [Thorne/Matthews '05] Eq. (21.59) becomes almost identical to eq. (21.42) upon $\eta_{ia}(\mathbf{k}) \to \eta(\mathbf{k})$, and $\eta_L(\mathbf{k}') \to f(\mathbf{v})$.

Cerenkov radiation occurs when the emitting particle travels faster than the group velocity of the emitted wave (provided there exists a process that couples the two.) As a result the emitting particle generates a shockwave like that produced by a supersonic aircraft. This shockwave takes on a form almost independent of the type of source and the waves generated.

The formation of this shockwave depends upon the emitted plasmons having much lower energy than the source and following boson statistics. This causes them to have much higher number density than the emitting plasmons which facilitates stimulated emission.

In the same manner in which fast electrons generate a shockwave of Langmuir waves, Langmuir wave excitations are plasmons (i.e. particles) that travel faster than ion-acoustic modes and generate a shockwave in them.

These similarities, exceeding speed, low emitted plasmon energy and stimulated emission are the physical source of the similarity between the two results.

(d) Keith and Kip hope to provide a solution to this part by Monday of next week.

21.6 Three-Wave Mixing - Langmuir Evolution [by Xinkai Wu '02]

(a) See Fig. 1 for the four relevant diagrams. The desired rate of change for the Langmuir occupation number is given by the sum/difference of these four diagrams:

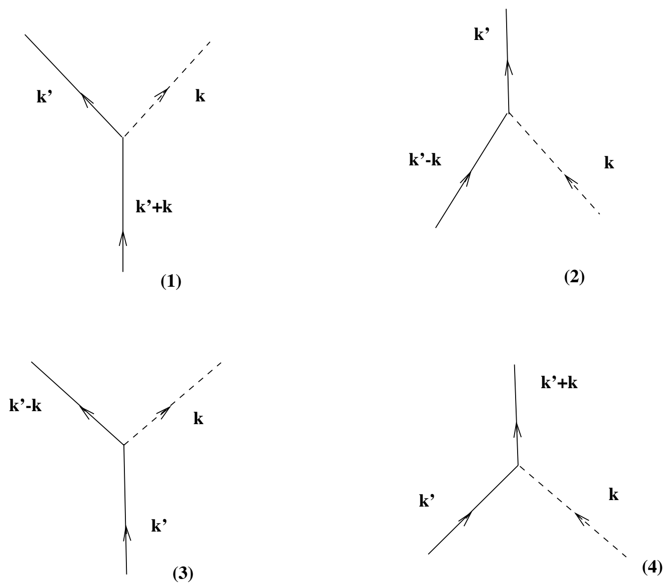

Figure 1: Diagrams for the change of Langmuir occupation number (we use solid lines to denote Langmuir plasmons, and dashed lines ion acoustic plasmons)

$$\frac{d\eta_L(\mathbf{k}')}{dt} = \int \frac{d^3\mathbf{k}}{(2\pi)^6}[(1)+(2)-(3)-(4)] \tag{25}$$

where

$$(1) = \eta_L(\mathbf{k}'+\mathbf{k})[1+\eta_L(\mathbf{k}')][1+\eta_{ia}(\mathbf{k})]R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}'+\mathbf{k})-\omega_L(\mathbf{k}')-\omega_{ia}(\mathbf{k})) \tag{26}$$

$$(2) = \eta_L(\mathbf{k}'-\mathbf{k})\eta_{ia}(\mathbf{k})[1+\eta_L(\mathbf{k}')]R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}')-\omega_L(\mathbf{k}'-\mathbf{k})-\omega_{ia}(\mathbf{k})) \tag{27}$$

$$(3) = \eta_L(\mathbf{k}')[1+\eta_L(\mathbf{k}'-\mathbf{k})][1+\eta_{ia}(\mathbf{k})]R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}')-\omega_L(\mathbf{k}'-\mathbf{k})-\omega_{ia}(\mathbf{k})) \tag{28}$$

$$(4) = \eta_L(\mathbf{k}')\eta_{ia}(\mathbf{k})[1+\eta_L(\mathbf{k}'+\mathbf{k})]R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\delta(\omega_L(\mathbf{k}'+\mathbf{k})-\omega_L(\mathbf{k}')-\omega_{ia}(\mathbf{k})) \tag{29}$$

(b) Now under the approximation stated in ex. 21.5 part b, all the δ-functions in the previous part reduce to the same expression $\delta(\omega_{ia}(\mathbf{k})-\mathbf{k}\cdot\mathbf{V}_g(\mathbf{k}'))$. We also have(dropping this δ-function for the moment being)

$$\begin{aligned}(1)-(4) &= \{\eta_L(\mathbf{k}'+\mathbf{k})[1+\eta_L(\mathbf{k}')+\eta_{ia}(\mathbf{k})]-\eta_L(\mathbf{k}')\eta_{ia}(\mathbf{k})\}\,R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}') && (30)\\ &\approx [\eta_L(\mathbf{k}'+\mathbf{k})-\eta_L(\mathbf{k}')]R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\eta_{ia}(\mathbf{k}) && (31)\\ &\approx \mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}'+\mathbf{k})R(\mathbf{k}'+\mathbf{k},\mathbf{k},\mathbf{k}')\eta_{ia}(\mathbf{k}) && (32)\end{aligned}$$

and

$$
\begin{aligned}
(2)-(3) &= \{\eta_L(\mathbf{k}'-\mathbf{k})\eta_{ia}(\mathbf{k}) - \eta_{\mathbf{L}}(\mathbf{k}')[\mathbf{1}+\eta_{\mathbf{L}}(\mathbf{k}'-\mathbf{k})+\eta_{\mathbf{ia}}(\mathbf{k})]\}\, R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}') && (33)\\
&\approx [\eta_L(\mathbf{k}'-\mathbf{k}) - \eta_L(\mathbf{k}')] R(\mathbf{k}'-\mathbf{k},\mathbf{k},\mathbf{k}')\eta_{ia}(\mathbf{k}) && (34)\\
&\approx -\mathbf{k}\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}')R(\mathbf{k}',\mathbf{k},\mathbf{k}'-\mathbf{k})\eta_{ia}(\mathbf{k}) && (35)
\end{aligned}
$$

where to reach the last expression in the above equation, we've used the fact that R is symmetric w.r.t. the two Langmuir wave momenta.

It's not hard to see that $(1)-(4)$ and $(2)-(3)$ are the same expression of $\mathbf{K}$ evaluated at $\mathbf{K}=\mathbf{k}'+\mathbf{k}$ and $\mathbf{k}'$ respectively So integrating their sum over phase space we immediately get the desired answer

$$
\frac{d\eta_L(\mathbf{k}')}{dt} = \nabla_{\mathbf{k}'}\cdot[\mathbf{D}(\mathbf{k}')\cdot\boldsymbol{\nabla}_{\mathbf{k}'}\eta_L(\mathbf{k}')] \tag{36}
$$

with

$$
\mathbf{D}(\mathbf{k}') \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^6}\eta_{ia}(\mathbf{k})\mathbf{k}\otimes\mathbf{k}\; R(\mathbf{k}',\mathbf{k},\mathbf{k}'-\mathbf{k})\delta(\omega_{ia}(\mathbf{k})-\mathbf{k}\cdot\mathbf{V}_g(\mathbf{k}')) \tag{37}
$$

(c) See above, ex. 21.5c. For the same reasons that the evolution equation for the emitted particles is characteristic, so is the evolution of the emitting particles.