Solution for Problem Set 22

(compiled by Dan Grin and Nate Bode)
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A

22.1 Invariance of a Null Interval [by Jeff Atwell]

Let $\vec{e_\alpha}$ be an orthonormal basis, so $(\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\beta) = \eta_{\alpha\beta}$. Let $-c$ be the value of $\tilde{g}_{00} = \tilde{\mathbf{g}}(\vec{e}_0, \vec{e}_0)$. Our goal is to show that $\tilde{g}_{\alpha\beta} = c\eta_{\alpha\beta}$ for all α and β.
Choose some arbitrary spatial basis vector $\vec{e}_j$. The vectors $\vec{e}_0 + \vec{e}_j$ and $\vec{e}_0 - \vec{e}_j$ are both null, so

$$0 = \tilde{\mathbf{g}}(\vec{e}_0 + \vec{e}_j, \vec{e}_0 + \vec{e}_j) - \tilde{\mathbf{g}}(\vec{e}_0 - \vec{e}_j, \vec{e}_0 - \vec{e}_j) = 4\tilde{\mathbf{g}}(\vec{e}_0, \vec{e}_j) = 4\tilde{g_{0j}},$$

which means that $\tilde{g}_{0j} = 0$. Similarly,

$$0 = \tilde{\mathbf{g}}(\vec{e}_0 + \vec{e}_j, \vec{e}_0 + \vec{e}_j) = \tilde{g_{00}} + 2\tilde{g}_{0j} + \tilde{g}_{jj} = \tilde{g}_{00} + \tilde{g}_{jj},$$

and since $\tilde{g}_{00} = -c$, this means that $\tilde{g}_{jj} = +c$ [where there is no summation on j] for any $j = 1, 2, 3$.

The remaining components we must compute are $\tilde{g}_{jk}$ for $j \neq k$. Consider the two different basis vectors $\vec{e}_j$ and $\vec{e}_k$. The vector $\sqrt{2}\vec{e}_0 + \vec{e}_j + \vec{e}_k$ is null. Therefore

$$\begin{aligned} 0 &= \tilde{\mathbf{g}}(\sqrt{2}\vec{e}_0 + \vec{e}_j + \vec{e}_k, \sqrt{2}\vec{e}_0 + \vec{e}_j + \vec{e}_k) \\ &= 2\tilde{g}_{00} + \sqrt{2}\tilde{g}_{0j} + \sqrt{2}\tilde{g}_{0k} + \tilde{g}_{jj} + \tilde{g}_{kk} + 2\tilde{g}_{jk} \\ &= -2c + 0 + 0 + c + c + 2\tilde{g}_{jk} = 2\tilde{g}_{jk}, \end{aligned}$$

which means that $\tilde{g}_{jk}$ vanishes when $j \neq k$. Combining all our components, we conclude that $\tilde{g}_{\alpha\beta} = c\eta_{\alpha\beta} = cg_{\alpha\beta}$.

22.2 Causality [by Alexei Dvoretskii 2000]

Consider two different reference frames - primed and unprimed. Assume without loss of generality that event P_1 occurs at a point $(0, 0)$ in spacetime in both frames and event P_2 at a point $(t, 0)$ in the unprimed frame (i.e. at the same spacial point) and at a point $(t', \mathbf{x}')$ in the primed frame. Now recall the invariance of the interval:

$$s^2 = s'^2 = -t^2 = -t'^2 + x'^2,$$

and

$$t' = \pm\sqrt{t^2 + x'^2}$$

. The transformations from one frame to another are continuous and in the limit of very small transformations $t' \approx t$. Therefore,

$$t' = \sqrt{t^2 + x'^2} > 0,$$

and so the temporal order of events is the same in all inertial frames. Of course, were this not true, causality would be violated.

It's also obvious that $t' \geq t$ and that apart from that there are no limits on the values of the spacial and temporal separation of the two events.

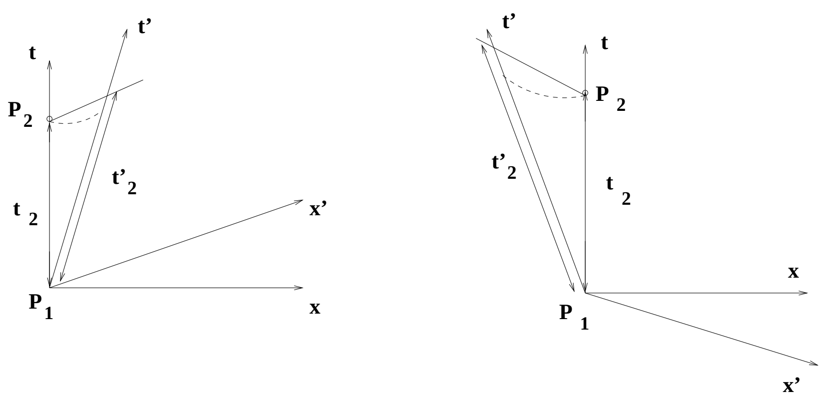


Figure 1: Causality

See Fig. 1 for the spacetime diagrams. As the velocity of the primed frame increases, the time of event P_2, t'_2 moves up the t' axis. Clearly $t'_2 > t_2$ (c.f. dashed hyperbola of events all at same interval from P_1). Similiar diagram can be drawn for velocity in the opposite direction. The diagram shows that for t'_2, $t_2 \leq t'_2$ and t'_2 can be made arbitrarily large.

B

22.4 Index manipulation rules from duality [by Alexei Dvoretskii 2000]

(a) Let's expand a given dual basis vector in the original basis:

$$\vec{e}^{\mu} = f^{\mu\beta}\vec{e}_{\beta}.$$

Now to find the coefficients $f^{\mu\beta}$ multiply both sides of the equation by $\vec{e}^{\alpha}$ and use the duality relation to obtain

$$\vec{e}^{\mu} \cdot \vec{e}^{\alpha} = f^{\mu\beta}\delta^{\alpha}_{\beta}$$

$$\mathbf{g}(\vec{e}^{\mu}, \vec{e}^{\alpha}) = g^{\mu\alpha} = f^{\mu\alpha}.$$

This proves the first relation:

$$\vec{e}^{\mu} = g^{\mu\beta}\vec{e}_{\beta}$$

The proof of the second relation is similar.

(b) Using the result above,

$$F^{\mu\nu} = \mathbf{F}(\vec{e}^{\mu}, \vec{e}^{\nu}) = \mathbf{F}(g^{\mu\alpha}\vec{e}_{\alpha}, \vec{e}^{\nu}).$$

Now use the linearity of the tensor to write

$$F^{\mu\nu} = g^{\mu\alpha}\mathbf{F}(\vec{e}_{\alpha}, \vec{e}^{\nu}) = g^{\mu\alpha}F_{\alpha}{}^{\nu}.$$

The proof of the second relation is similar.

(c) Consider for example the first identity:

$$\mathbf{F} = F^{\mu\nu}\vec{e}_{\mu} \otimes \vec{e}_{\nu}.$$

Now consider

$$\mathbf{F}(\vec{e}^{\alpha}, \vec{e}^{\beta}) = F^{\mu\nu}(\vec{e}_{\mu} \cdot \vec{e}^{\alpha})(\vec{e}_{\nu} \cdot \vec{e}^{\beta}) = F^{\mu\nu}\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} = F^{\alpha\beta}.$$

So all the components of the tensors on the left-hand side and the right hand side are equal which proves the identity. Similar proofs can be given in the other cases.

C

22.6(a-b),22.5b Connection coefficients/Transformation Matrices for circular polar coordinates [by Alexei Dvoretskii 2000 and Xinkai Wu]
(22.6a) First let's consider the Coordinate basis

$$\vec{e}_{\varpi} = \partial_{\varpi}, \vec{e}_{\phi} = \partial_{\phi}$$

$$[\vec{e}_\varpi, \vec{e}_\varpi] = [\vec{e}_\varpi, \vec{e}_\phi] = [\vec{e}_\phi, \vec{e}_\phi] = 0.$$

Hence

$$c_{\alpha\beta\gamma} = 0.$$

The metric tensor is given by

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^2 \end{pmatrix}.$$

The only non-zero Christoffel symbols are then

$$\Gamma_{\varpi\phi\phi} = (-1/2)g_{\phi\phi,\varpi} = -\varpi$$

$$\Gamma_{\phi\phi\varpi} = \Gamma_{\phi\varpi\phi} = \varpi,$$

and the connection coefficients are

$$\Gamma^{\varpi}{}_{\phi\phi} = -\varpi, \quad \Gamma^{\phi}{}_{\varpi\phi} = \Gamma^{\phi}{}_{\phi\varpi} = 1/\varpi.$$

(22.6b) Orthonormal basis:

$$\vec{e}_{\hat{\varpi}} = \partial_\varpi, \vec{e}_{\hat{\phi}} = 1/\varpi\partial_\phi$$

. In this basis the metric tensor is of course just

$$\mathbf{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The commutation coefficients are readily computed from

$$[\vec{e}_{\hat{\varpi}}, \vec{e}_{\hat{\phi}}] = [\partial_\varpi, 1/\varpi\partial_\phi] = -1/\varpi^2\partial_\phi = -1/\varpi\vec{e}_{\hat{\phi}}.$$

Hence, the only non-zero commutation coefficients are

$$c_{\hat{\varpi}\hat{\phi}}{}^{\hat{\phi}} = -1/\varpi, \quad c_{\hat{\phi}\hat{\varpi}}{}^{\hat{\phi}} = 1/\varpi.$$

And the Christoffel symbols and the connection coefficients are

$$\Gamma_{\hat{\varpi}\hat{\phi}\hat{\phi}} = \Gamma^{\hat{\varpi}}{}_{\hat{\phi}\hat{\phi}} = -1/\varpi,$$

$$\Gamma_{\hat{\phi}\hat{\varpi}\hat{\phi}} = \Gamma^{\hat{\phi}}{}_{\hat{\varpi}\hat{\phi}} = 1/\varpi.$$

(22.5b)

By the chain rule, we have

$$\begin{aligned} \frac{\partial f}{\partial \varpi} &= \cos\phi\frac{\partial f}{\partial x} + \sin\phi\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \phi} &= -\varpi\sin\phi\frac{\partial f}{\partial x} + \varpi\cos\phi\frac{\partial f}{\partial y}. \end{aligned} \qquad (1)$$

for any function f. Combining this with

$$\vec{e}_{\hat{\varpi}} = \frac{\partial}{\partial \varpi}, \quad \vec{e}_{\hat{\phi}} = \frac{1}{\varpi}\frac{\partial}{\partial \phi}$$
$$\vec{e}_x = \frac{\partial}{\partial x}, \quad \vec{e}_y = \frac{\partial}{\partial y}, \tag{2}$$

we get the transformation matrix

$$L^x{}_{\hat{\varpi}} = \cos\phi,\ L^y{}_{\hat{\varpi}} = \sin\phi,\ L^x{}_{\hat{\phi}} = -\sin\phi,\ L^y{}_{\hat{\phi}} = \cos\phi, \tag{3}$$

whose inverse is

$$L^{\hat{\varpi}}{}_x = \cos\phi,\ L^{\hat{\phi}}{}_x = -\sin\phi,\ L^{\hat{\varpi}}{}_y = \sin\phi,\ L^{\hat{\phi}}{}_y = \cos\phi. \tag{4}$$

D

22.9 Index gymnastics [by Xinkai Wu 2002]
(a) First notice that $P_{\alpha\beta}u^\beta = u_\alpha + u_\alpha u_\beta u^\beta = 0$, using the fact that $\vec{u}^2 = -1$. Thus

$$P_{\alpha\beta}P^\beta{}_\gamma = P_{\alpha\beta}(g^\beta{}_\gamma + u^\beta u_\gamma) = P_{\alpha\gamma}. \tag{5}$$

(b) $P_{\alpha\beta}A^\beta u^\alpha = 0$ because $P_{\alpha\beta}u^\alpha = 0$, thus $P_{\alpha\beta}A^\beta$ is orthogonal ot $\vec{u}$. $P_{\alpha\beta}A^\beta = A_\alpha + u_\alpha u_\beta A^\beta = A_\alpha$ if $u_\beta A^\beta = 0$.

(c) In the fluid's local rest frame, $g_{\alpha\beta} = \eta_{\alpha\beta}$, and $u^\alpha = \delta^{\alpha 0}$. Thus $P_{\alpha\beta}$ is diagonal in this frame, with $P_{00} = 0$, $P_{jj} = 1, j = 1, 2, 3$.

(d)

$$(\nabla_{\vec{u}}\vec{u})_\alpha = u^\beta u_{\alpha;\beta} = -a_\alpha \vec{u}^2 = a_\alpha, \tag{6}$$

where we've used Eq. (22.53) and the fact that $P_{\alpha\beta}$, $\sigma_{\alpha\beta}$, and $\omega_{\alpha\beta}$ are all orthogonal to $\vec{u}$. Thus we see $\nabla_{\vec{u}}\vec{u} = \vec{a}$. Also

$$\begin{aligned}\vec{a}\cdot\vec{u} &= a_\alpha u^\alpha = u^\beta u_{\alpha;\beta}u^\alpha \\ &= u^\beta[(u_\alpha u^\alpha)_{;\beta} - u^\alpha u_{\alpha;\beta}] = -u^\beta u^\alpha u_{\alpha;\beta} = -\vec{a}\cdot\vec{u}.\end{aligned} \tag{7}$$

Thus $\vec{a}\cdot\vec{u} = 0$.

(e) Contracting eq. (22.53) with $g^{\alpha\beta}$, using $\vec{a}\cdot\vec{u}=0$, tracelessness of $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$, and $g^{\alpha\beta}P_{\alpha\beta}=g^{\alpha\beta}g_{\alpha\beta}+\vec{u}^2=3$, we immediately get $\nabla\cdot\vec{u}=\theta$.

(f) Symmetrizing eq. (22.53) and using the expressions of $\vec{a}$ and θ worked out in the previous parts, one gets

$$\sigma_{\alpha\beta}=\frac{1}{2}\left(u_{\alpha;\beta}+u_{\beta;\alpha}+u^\gamma u_{\alpha;\gamma}u_\beta+u^\gamma u_{\beta;\gamma}u_\alpha-\frac{2}{3}u^\gamma{}_{;\gamma}P_{\alpha\beta}\right). \tag{8}$$

Antisymmetrizing eq. (22.53), one gets

$$\omega_{\alpha\beta}=\frac{1}{2}\left(u_{\alpha;\beta}-u_{\beta;\alpha}+u^\gamma u_{\alpha;\gamma}u_\beta-u^\gamma u_{\beta;\gamma}u_\alpha\right). \tag{9}$$

(g) (i) The four velocity is given by $\vec{u}=(\gamma,\gamma v^j)$, where $\gamma=1/\sqrt{1-v^jv^j}$. So to first order in v^j, we have $u^0=1$, and $u^j=v^j$. (ii) $\theta=u^\alpha{}_{;\alpha}=u^\alpha{}_{,\alpha}$, to first order in v^j, this becomes $\theta=u^j{}_{,j}=v^j{}_{,j}$. (iii) The 3rd and 4th terms in the expression of σ_{jk} worked out in part (f) are of higher order in v^j and can be ignored, so we get $\sigma_{jk}=\frac{1}{2}(v_{j,k}+v_{k,j})-\frac{1}{3}\theta g_{jk}$,which is the fluid's nonrelativistic shear. (iv) In the expression for ω_{jk} worked out in part (f), the 3rd and 4th terms are of higher order, and we get $\omega_{jk}=\frac{1}{2}(v_{j,k}-v_{k,j})$, which is the nonrelativstic rotation.

E

22.10 Gauss's Law [by Xinkai Wu 2002]

$\mathbf{E}\cdot d\mathbf{\Sigma}=E^jd\Sigma_j$, where $d\Sigma_j=\epsilon(\vec{e}_j,d\theta\partial/\partial\theta,d\phi\partial/\partial\phi)$. By the antisymmetry of ϵ, only $d\Sigma_r$ doesn't vanish, and is given by $\epsilon_{r\theta\phi}d\theta d\phi=R^2\sin\theta d\theta d\phi$. On the r.h.s. of Eq. (22.55), $d\Sigma$ was already worked out in the text: $d\Sigma=r^2\sin\theta drd\theta d\phi$. Thus Eq. (22.55) becomes

$$\int_{r=R}E^rR^2\sin\theta d\theta d\phi=\int_{r<R}\frac{\rho_e}{\epsilon_0}r^2\sin\theta drd\theta d\phi. \tag{10}$$

F

22.11 Stress energy tensor for a perfect fluid [by Alexei Dvoretskii 2000]

(a) If the components of two tensors are equal in a given frame, then they will be equal in any frame, so we just need to verify that

$$\mathbf{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\mathbf{g}$$

reduces to

$$T^{00} = \rho, \quad T^{ij} = P\delta^{jk}$$

in the rest frame. It's a trivial exercise given the simple form of $\vec{u}$ in the rest frame:

$$\vec{u} = (1, 0, 0, 0).$$

(b) If observer is moving with a speed v much smaller than the speed of light with respect to the rest frame of the fluid, then the momentum density in this frame can be written as

$$T^{0j} = \rho^{ij}_{\text{inertial}} v_i,$$

which is the definition for the inertial mass density. In the limit of small v the momentum density can be written as

$$T^{0j} = (\rho + P)v_j$$

and so

$$\rho^{ij}_{\text{inertial}} = (\rho + P)\delta^{ij}.$$

(c) For non-relativistic velocities v the 4-velocity can be written as

$$\vec{u} = (1 + v^2/2, \mathbf{v}).$$

Also

$$\rho = \rho_0(1 + u).$$

Calculating to order v^2 get

$$T^{0j} = (\rho_0 + \rho_0 u + P)(1 + v^2/2)v^j = (\rho_0 + \rho_0 v^2/2 + \rho_0 u + P)v^j,$$

$$T^{ij} = (\rho_0 + \rho_0 u + P)v^i v^j + P\delta^{ij} = \rho_0 v^i v^j + P\delta^{ij},$$

$$T^{00} = (\rho_0 + \rho_0 u + P)(1 + v^2/2)^2 - P = \rho_0 + \rho_0 u + \rho_0 v^2.$$

(d) As in b,

$$T^{0j} = \rho^{ij}_{\text{inertial}} v_i = (\rho + P)v_j.$$

G

22.14 Stress-energy tensor for a point particle [by Alexei Dvoretskii 2000]

We want to prove that

$$p^\alpha(\zeta_0) = \int p^\alpha(\zeta)p^\beta(\zeta)\delta(Q, P(\zeta))d\Sigma_\beta d\zeta. \tag{11}$$

Because of the δ-function the argument of 4-momenta is fixed: $\zeta = \zeta_0$ and the right hand side reduces to

$$RHS = p^\alpha(\zeta_0)p^\beta(\zeta_0)\int \delta(Q, P(\zeta))d\Sigma_\beta d\zeta. \tag{12}$$

Let x^μ be the coordinates of Q (not necessarily Lorentzian) and $y^\mu(\zeta)$ - coordinates of P in the same coordinate system. Since the expression is Lorentz invariant we can choose the coordinate system any way we like. To simplify the calculation we make it satisfy the following requirements:

- The surface Σ is given by the eqn. $x^0 = 0$.
- The world line of the particle intersects Σ at $x^j = 0$.
- The coordinate system is local Lorentz one at the point of intersection $x^\mu = 0$.

Then the surface element $d\Sigma_\beta$ has only one non-vanishing component:

$$d\Sigma_0 = d^3\mathbf{x} = dx^1dx^2dx^3, \qquad d\Sigma_j = 0,\ j = 1, 2, 3, \tag{13}$$

and the δ-function can be written as a product

$$\delta(Q, P(\zeta)) = \delta(x_0 - y_0(\zeta))\delta^3(\mathbf{x} - \mathbf{y}(\zeta)) = \delta(y_0(\zeta))\delta^3(\mathbf{x} - \mathbf{y}(\zeta)). \tag{14}$$

The resulting integral can be easily calculated.

$$\begin{aligned} RHS &= p^\alpha(\zeta_0)p^0(\zeta_0)\int \delta(y^0(\zeta))\delta^3(\mathbf{x} - \mathbf{y}(\zeta))d^3\mathbf{x}d\zeta = \\ &= p^\alpha(\zeta_0)p^0(\zeta_0)\int \delta(y^0(\zeta))d\zeta = p^\alpha(\zeta_0)p^0(\zeta_0)\frac{1}{\frac{dy^0}{d\zeta}|_{\zeta_0}}. \end{aligned}$$

But $\frac{dy^0}{d\zeta}|_{\zeta_0} = p^0(\zeta_0)$ by definition of momentum, so

$$RHS = p^\alpha(\zeta_0). \tag{15}$$

H

22.15 Proper Reference Frame [by Alexei Dvoretskii 2000]
(a) It's fairly straightforward to obtain the transformation law in differential form (the hats on the right-hand side are dropped to simplify notation):

$$d\mathbf{x} = d\mathbf{x} + \mathbf{a}x^0 dx^0 + (\Omega \times d\mathbf{x})x^0 + (\Omega \times \mathbf{x})dx^0$$

$$dx^0 = dx^0(1 + \mathbf{a}\cdot\mathbf{x}) + x^0\mathbf{a}\cdot d\mathbf{x}.$$

Squaring and only keeping terms linear in $\mathbf{x}$ we get

$$d\mathbf{x}^2 = d\mathbf{x}^2 + 2\mathbf{a}\cdot d\mathbf{x}x^0 dx^0 + 2(\Omega \times \mathbf{x})\cdot d\mathbf{x}dx^0,$$

$$(dx^0)^2 = (dx^0)^2(1 + 2\mathbf{a}\cdot\mathbf{x}) + 2\mathbf{a}\cdot d\mathbf{x}x^0 dx^0.$$

Given this it's easy to see that the new metric is indeed

$$ds^2 = -(1 + 2\mathbf{a}\cdot\mathbf{x})(dx^0)^2 + 2(\Omega \times \mathbf{x})\cdot d\mathbf{x}dx^0 + d\mathbf{x}^2.$$

(b) Recall that the components of the metric in the proper frame are (again dropping the hats for simplicity of notation)

$$g_{00} = -(1 + 2a_j x^j),\ g_{0i} = \epsilon_{ijk}\Omega^j x^k,\ g_{jk} = \delta_{jk}, \tag{16}$$

and to linear order in x^j, the inverse metric $g^{\mu\nu}$ is given by taking $x^j \to -x^j$ in $g_{\mu\nu}$. To compute the connection coefficients along the world line, we only need the inverse metric at $x^j = 0$, which is just $\eta^{\mu\nu}$. We have

$$\Gamma^\mu{}_{\alpha 0} = \eta^{\mu\nu}\frac{1}{2}(g_{\nu 0,\alpha} - g_{\alpha 0,\nu}), \tag{17}$$

which gives

$$\Gamma^0{}_{00} = 0,\ \Gamma^0{}_{j0} = a_j;\quad \Gamma^j{}_{00} = a_j,\ \Gamma^i{}_{j0} = \epsilon_{ikj}\Omega^k. \tag{18}$$

Also it's not hard to see that $\Gamma^\mu{}_{ij}$ all vanish. The above results can be verified by, say, GRtensor, which is straightforward and we omit here.

(c) Using the connection coefficients we obtained the previous part, we find

$$\nabla_{\vec{U}}\vec{e}_0 = \Gamma^\mu{}_{00}\vec{e}_\mu = \Gamma^i{}_{00}\vec{e}_i = a^i\vec{e}_i = \vec{a} \tag{19}$$

and

$$\nabla_{\vec{U}}\vec{e}_j = \Gamma^\mu{}_{j0}\vec{e}_\mu = \Gamma^0{}_{j0}\vec{e}_0 + \Gamma^i{}_{j0}\vec{e}_i \tag{20}$$

$$= a_j\vec{e}_0 + \epsilon_{ikj}\Omega^k\vec{e}_i = (\vec{a}\cdot\vec{e}_j)\vec{U} + \epsilon(\vec{U},\vec{\Omega},\vec{e}_j,...). \tag{21}$$

(d) Now we are away from the world line, $x^j \neq 0$. However, we see that at our order of approximation , $\Gamma^i{}_{00}$ and $\Gamma^i{}_{j0}$ are still given by the expressions worked out in part (b). Plugging them into Eq. (22.95), we readily get

$$\frac{d^2\mathbf{x}}{(dx^0)^2} = -\mathbf{a} - 2\boldsymbol{\Omega} \times \mathbf{v}. \tag{22}$$

I

22.16 Uniformly accelerated observer [by Jeff Atwell]
(a) The proper time is the time measured by a clock that the observer carries. A small change in proper time is equal to the lapse in coordinate time of its momentary rest frame. If one decides to work in a different Lorentz coodinate system (such as the one we are given in this problem), then a small change in proper time may be computed from the invariant interval. So, consider two nearby events:

$$t_1 = \frac{1}{a}\sinh{(a\tau)}, \;\; x_1 = \frac{1}{a}\cosh{(a\tau)},$$

and

$$t_2 = \frac{1}{a}\sinh{(a(\tau+\Delta\tau))}, \;\; x_2 = \frac{1}{a}\cosh{(a(\tau+\Delta\tau))}.$$

Working to first order in $\Delta\tau$,

$$\Delta t = \frac{1}{a}\sinh{(a(\tau+\Delta\tau))} - \frac{1}{a}\sinh{(a\tau)} \approx \cosh{(a\tau)}\Delta\tau,$$

and

$$\Delta x = \frac{1}{a}\cosh{(a(\tau+\Delta\tau))} - \frac{1}{a}\cosh{(a\tau)} \approx \sinh{(a\tau)}\Delta\tau.$$

So

$$\sqrt{-\Delta s^2} = \sqrt{\Delta t^2 - \Delta x^2} = \Delta\tau\sqrt{\cosh^2{(a\tau)} - \sinh^2{(a\tau)}} = \Delta\tau.$$

This shows that τ is the proper time.

First compute the observer's 4-velocity: $u^0 = \frac{dt}{d\tau} = \cosh{(a\tau)}$, $u^1 = \frac{dx}{d\tau} = \sinh{(a\tau)}$, $u^2 = \frac{dy}{d\tau} = 0$, and $u^3 = \frac{dz}{d\tau} = 0$. Now for the 4-acceleration: $a^0 = \frac{du^0}{d\tau} = a\sinh{(a\tau)}$, $a^1 = \frac{du^1}{d\tau} = a\cosh{(a\tau)}$, $a^2 = \frac{du^2}{d\tau} = 0$, and $a^3 = \frac{du^3}{d\tau} = 0$. So

$$\vec{a}\cdot\vec{a} = -(a^0)^2 + (a^1)^2 = -a^2\sinh^2{(a\tau)} + a^2\cosh^2{(a\tau)} = a^2.$$

This shows that the constant parameter a in Eq. (22.97) is the magnitude of the observer's 4-acceleration.

(b) Notice that $x^2 - t^2 = a^{-2}$. So the world line is a hyperbola in a spacetime diagram. Equation (22.83) in the notes tells us that $\vec{e}_{\hat{0}} = \vec{u}$, where the components of $\vec{u}$ are given in part (a). Also notice that the 4-acceleration from part (a) is orthogonal to the 4-velocity: $\vec{a} \cdot \vec{u} = 0$. It follows that $\vec{e}_{\hat{1}} = \frac{1}{a}\vec{a}$. Also $\vec{e}_{\hat{2}}$ and $\vec{e}_{\hat{3}}$ are the same as the Lorentz coordinate basis vectors $\vec{e}_2$ and $\vec{e}_3$. The resulting spacetime diagram is:

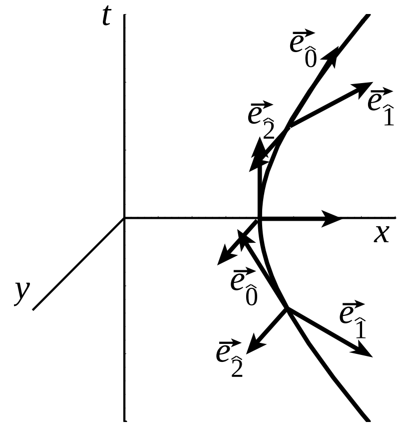
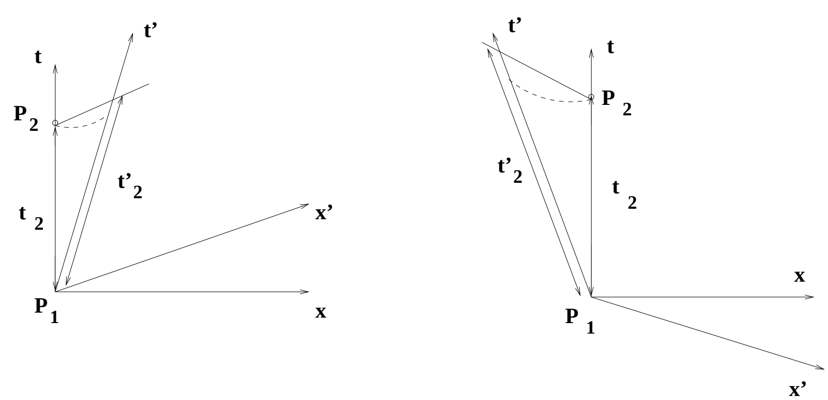


Figure 2:



(c) $\vec{\Omega} = 0$ for this proper reference frame. The basis vectors in the 3-plane orthogonal to the observer's world line do not rotate.

(d) Let the displacement vector from the origin of the original inertial frame to the position of the observer at proper time τ be $\vec{z}(\tau)$. At that point the observer has three spacelike basis vectors $\vec{e}_{\hat{1}}(\tau)$, $\vec{e}_{\hat{2}}(\tau)$, $\vec{e}_{\hat{3}}(\tau)$. A typical point $\vec{x}$

near the observer in the hyperplane spanned by these three basis vectors can be represented in the form

$$\vec{x} = x^{\hat{1}}\vec{e}_{\hat{1}}(\tau) + x^{\hat{2}}\vec{e}_{\hat{2}}(\tau) + x^{\hat{3}}\vec{e}_{\hat{3}}(\tau) + \vec{z}(\tau).$$

Now use the basis vectors described in part (b):

$$\begin{aligned}
\vec{e}_{\hat{0}} &= (\cosh(a\tau), \sinh(a\tau), 0, 0)\,, \\
\vec{e}_{\hat{1}} &= (\sinh(a\tau), \cosh(a\tau), 0, 0)\,, \\
\vec{e}_{\hat{2}} &= (0, 0, 1, 0)\,, \\
\vec{e}_{\hat{3}} &= (0, 0, 0, 1)\,,
\end{aligned}$$

and eq. (23.97):

$$\vec{z}(\tau) = \left(\frac{1}{a}\sinh(a\tau), \frac{1}{a}\cosh(a\tau), 0, 0\right),$$

in the above $\vec{x}$ equation to get

$$\begin{aligned}
x^0 &= \left(\frac{1}{a} + x^{\hat{1}}\right)\sinh(a\tau), \\
x^1 &= \left(\frac{1}{a} + x^{\hat{1}}\right)\cosh(a\tau), \\
x^2 &= x^{\hat{2}}, \\
x^3 &= x^{\hat{3}}.
\end{aligned}$$

This is the coordinate transformation. Now compute the metric:

$$\begin{aligned}
&-\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \\
&= -\left[dx^{\hat{1}}\sinh(a\tau) + \left(1 + ax^{\hat{1}}\right)\cosh(a\tau)d\tau\right]^2 + \\
&\left[dx^{\hat{1}}\cosh(a\tau) + \left(1 + ax^{\hat{1}}\right)\sinh(a\tau)d\tau\right]^2 + \left(dx^{\hat{2}}\right)^2 + \left(dx^{\hat{3}}\right)^2 \\
&= -\left(1 + ax^{\hat{1}}\right)^2 d\tau^2 + \left(dx^{\hat{1}}\right)^2 + \left(dx^{\hat{2}}\right)^2 + \left(dx^{\hat{3}}\right)^2.
\end{aligned}$$

To make this look like eq. (23.86), expand $\left(1 + ax^{\hat{1}}\right)^2$ and only keep the first order term to get

$$= -\left(1 + 2ax^{\hat{1}}\right)d\tau^2 + \left(dx^{\hat{1}}\right)^2 + \left(x^{\hat{2}}\right)^2 + \left(dx^{\hat{3}}\right)^2.$$

This agrees with Eq. (22.86) since in this frame $\mathbf{a}\cdot\mathbf{x} = ax^{\hat{1}}$.

Solution 22.

23.6 (a) In the local Lorentz frame, Connetion Coeffients must be zero and first derivative of g must be zero

$$\Rightarrow \nabla g \rightarrow g_{\mu\nu,\rho} - \Gamma_{\nu\mu\rho} - \Gamma_{\mu\nu\rho} = 0$$

As it's zero in one frame, it must vanish in all frames.

(b) $\nabla_{\vec{A}}\vec{B} - \nabla_{\vec{B}}\vec{A} = A^\alpha \nabla_\alpha \vec{B} - B^\alpha \nabla_\alpha \vec{A}$

$$= A^\alpha (B^\mu{}_{,\alpha} + B^\gamma \Gamma^\mu{}_{\gamma\alpha}) \vec{e}_\mu - B^\alpha (A^\mu{}_{,\alpha} + A^\gamma \Gamma^\mu{}_{\gamma\alpha}) \vec{e}_\mu$$

In local Lorentz frame, $\Gamma = 0$

$$\Rightarrow \nabla_{\vec{A}}\vec{B} - \nabla_{\vec{B}}\vec{A} = A^\alpha \frac{\partial B^\mu}{\partial x^\alpha} \frac{\partial}{\partial x^\mu} - B^\alpha \frac{\partial A^\mu}{\partial x^\alpha} \frac{\partial}{\partial x^\mu}$$

$$= \left(A^\alpha \frac{\partial B^\mu}{\partial x^\alpha} - B^\alpha \frac{\partial A^\mu}{\partial x^\alpha} \right) \frac{\partial}{\partial x^\mu} = [\vec{A}, \vec{B}]$$

~~Since~~ It must be frame independent $\Rightarrow \nabla_{\vec{A}}\vec{B} - \nabla_{\vec{B}}\vec{A} = [\vec{A}, \vec{B}]$

Solution 22

23.7 (i) $\nabla g = 0 = g_{\mu\nu,\rho} - \Gamma_{\mu\rho\nu} - \Gamma_{\nu\rho\mu}$

$\Rightarrow g_{\mu\nu,\rho} = \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho}$ (symmetric part)

# of independent components $= n \times \frac{1}{2} n(n+1) = \frac{n^2}{2}(n+1)$

(ii) $[\vec{e}_\alpha, \vec{e}_\beta] = c^{\mu}{}_{\alpha\beta}\, \vec{e}_\mu = \nabla_\alpha \vec{e}_\beta - \nabla_\beta \vec{e}_\alpha = \Gamma^{\mu}{}_{\beta\alpha}\, \vec{e}_\mu - \Gamma^{\mu}{}_{\alpha\beta}\, \vec{e}_\mu$

$\Rightarrow C_{\alpha\beta\gamma} = \Gamma_{\gamma\beta\alpha} - \Gamma_{\gamma\alpha\beta}$ (asymmetric part)

# of independent components $= \frac{n^2}{2}(n-1)$

(iii) $\frac{1}{2}(g_{\mu\beta,\sigma} + g_{\mu\sigma,\beta} - g_{\beta\sigma,\mu}) = \Gamma_{(\mu\beta)\sigma} + \Gamma_{(\mu\sigma)\beta} - \Gamma_{(\beta\sigma)\mu}$

$= \Gamma_{\mu\beta\sigma} + (-\Gamma_{\mu[\beta\sigma]} + \Gamma_{\beta[\mu\sigma]} + \Gamma_{\sigma[\mu\beta]})$

Also $C_{\mu\nu}{}^{\rho}\, \vec{e}_\rho = [\vec{e}_\mu, \vec{e}_\nu] = 2\Gamma^{\rho}{}_{[\mu\nu]}\, \vec{e}_\rho$

$\Rightarrow \Gamma^{\rho}{}_{[\mu\nu]} = -\frac{1}{2} C_{\mu\nu}{}^{\rho}$ or $\Gamma_{\rho[\mu\nu]} = -\frac{1}{2} C_{\mu\nu\rho}$

$\Rightarrow \Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} + C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha})$