Solutions for Problem Set for Ch. 23

(compiled by Nate Bode)
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A-E

23.3 Geodesic Equation in an arbitrary coordinate system [by Keith Matthews '05]

$$\nabla_{\vec{p}}\vec{p} = p^{\beta}\nabla_{\beta}p^{\alpha} = p^{\beta}(\partial_{\beta}p^{\alpha} + \Gamma^{\alpha}{}_{\beta\rho}p^{\rho}) = \frac{d}{d\zeta}p^{\alpha} + \Gamma^{\alpha}{}_{\beta\rho}p^{\beta}p^{\rho} = 0$$

$$\frac{d^2x^{\alpha}}{d\zeta^2} = -\Gamma^{\alpha}{}_{\beta\rho}\frac{dx^{\beta}}{d\zeta}\frac{dx^{\rho}}{d\zeta}$$

23.4 Constant of geodesic motion in a spacetime with symmetry [by Alexander Putilin '99]

(a) Geodesic equation $\nabla_{\vec{p}}\vec{p} = 0$,i.e.

$$p^{\beta}p_{\alpha;\beta} = 0 \tag{1}$$

$$(p_{\alpha,\beta} - \Gamma^{\mu}{}_{\alpha\beta}p_{\mu})p^{\beta} = \frac{dx^{\beta}}{d\zeta}\frac{\partial p_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}{}_{\alpha\beta}p_{\mu}p^{\beta} \tag{2}$$

$$= \frac{dp_{\alpha}}{d\zeta} - \Gamma_{\mu\alpha\beta}p^{\mu}p^{\beta} = 0 \tag{3}$$

which gives

$$\frac{dp_{\alpha}}{d\zeta} = \frac{1}{2}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})p^{\mu}p^{\beta} \tag{4}$$

where in the brackets the first and the third terms are antisymmetric over $(\beta\mu)$ so their contraction with the symmetric tensor $p^{\beta}p^{\mu}$ is zero. Thus

$$\frac{dp_{\alpha}}{d\zeta} = \frac{1}{2}g_{\mu\beta,\alpha}p^{\mu}p^{\beta} \tag{5}$$

Take α to be A and using $g_{\alpha\beta,A} = 0$, we find

$$\frac{dp_A}{d\zeta} = 0 \tag{6}$$

namely p_A is a constant of motion.

(b) Let $x^j(t)$ be the trajectory of a particle. Its proper time

$$d\tau^2 = -ds^2 = dt^2 \left[1 + 2\Phi - (\delta_{jk} + h_{jk})v^j v^k\right] \tag{7}$$

$$= dt^2(1 + 2\Phi - \delta_{jk}v^j v^k + O(\frac{v^4}{c^4})) \tag{8}$$

thus

$$d\tau = dt\sqrt{1 + 2\Phi - \mathbf{v}^2} = dt(1 + \Phi - \frac{1}{2}\mathbf{v}^2) \tag{9}$$

where we have omitted terms of order v^4/c^4(i.e. $|\Phi|^2$). The 4-velocity is given by

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt(1 + \Phi - \frac{1}{2}\mathbf{v}^2)} \tag{10}$$

$$= \frac{dx^\alpha}{dt}(1 - \Phi + \frac{1}{2}\mathbf{v}^2) \tag{11}$$

thus in particular $u^0 = 1 - \Phi + \frac{1}{2}\mathbf{v}^2$.

4-momentum: $p^\alpha = mu^\alpha$, and in particular $p^0 = mu^0 = m(1 - \Phi + \frac{1}{2}\mathbf{v}^2)$. And the conserved quantity is then given by

$$p_t = g_{0\alpha}p^\alpha = g_{00}p^0 = -(1 + 2\Phi)m(1 - \Phi + \frac{1}{2}\mathbf{v}^2) \tag{12}$$

$$= -m - (m\Phi + \frac{1}{2}m\mathbf{v}^2) \tag{13}$$

we see that p_t is indeed the non-relativistic energy of a particle aside from an additive constant $-m$ and an overall minus sign.

23.5 Action Principle for Geodesic Motion [by Xinkai Wu '00]
The action is given by:

$$S[x^\alpha(\lambda)] = \int_0^1 (-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2} d\lambda \tag{14}$$

$$\delta S = \int_0^1 \delta(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}d\lambda \tag{15}$$

$$= \int_0^1 \frac{1}{2}(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\delta(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})d\lambda \tag{16}$$

$$= -\int_0^1 \frac{1}{2}(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\Big\{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d\delta x^\nu}{d\lambda}\Big\}d\lambda \tag{17}$$

by renaming $\mu \leftrightarrow \nu$, and noticing $g_{\mu\nu} = g_{\nu\mu}$, we get:

$$\delta S = -\int_0^1 \frac{1}{2}(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\{\frac{\partial g_{\mu\nu}}{\partial x^\rho}\delta x^\rho\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\}d\lambda \tag{18}$$

Integrating the 2nd term in {...} by parts, we find, after renaming some indices:

$$\delta S = \int_0^1 (-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{-1/2}\Big\{g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{\partial g_{\rho\nu}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}g_{\mu\nu}\frac{dx^\nu}{d\lambda}\Big\}\delta x^\mu d\lambda \tag{19}$$

Noting that this is true for all variations δx^μ we know $\delta S = 0$ if and only if

$$0 = g_{\mu\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{1}{2}\frac{\partial g_{\rho\nu}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}fracdx^\beta d\lambda)^{1/2}}{d\lambda}g_{\mu\nu}\frac{dx^\nu}{d\lambda} \tag{20}$$

Contracting both sides with $g^{\pi\mu}$, we get

$$0 = \frac{d^2x^\pi}{d\lambda^2} + \frac{1}{2}g^{\pi\mu}\left\{2\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{\partial g_{\rho\nu}}{\partial x^\mu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda}\right\} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{dx^\pi}{d\lambda} \tag{21}$$

By renaming $\rho \leftrightarrow \nu$ for the first term in {...}, the above equation becomes

$$0 = \frac{d^2x^\pi}{d\lambda^2} + \frac{1}{2}g^{\pi\mu}\{\frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\mu}\}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{dx^\pi}{d\lambda} \tag{22}$$

which is just, using the expression for the Christoffel symbols,

$$0 = \frac{d^2x^\pi}{d\lambda^2} + \Gamma^\pi_{\rho\nu}\frac{dx^\rho}{d\lambda}\frac{dx^\nu}{d\lambda} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{dx^\pi}{d\lambda} \tag{23}$$

Now let's reparametrize the world line, $\lambda - > s(\lambda)$, then the equation becomes,

$$0 = (\frac{d^2x^\pi}{ds^2} + \Gamma^\pi_{\rho\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds})(\frac{ds}{d\lambda})^2 + \frac{dx^\pi}{ds}[\frac{d^2s}{d\lambda^2} - \frac{d\ln(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}}{d\lambda}\frac{ds}{d\lambda}] \tag{24}$$

Integrating [...] twice we readily find that [...] vanishes for

$$s = \int A(-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda})^{1/2}d\lambda + B \ , \tag{25}$$

where A and B are arbitrary constants.

After this reparametrization, we get the familiar geodesic equation:

$$0 = \frac{d^2x^\pi}{ds^2} + \Gamma^\pi_{\rho\nu}\frac{dx^\rho}{ds}\frac{dx^\nu}{ds} \tag{26}$$

23.7 Orders of magnitude of the radius of curvature [by Alexander Putilin '99]

Eq. (23.43) tells us that, if a system has characteristic mass M and characteristic length R, order of magnitude estimate gives,

$$\frac{1}{\mathcal{R}^2} \sim \frac{GM}{R^3} \tag{27}$$

where $\mathcal{R}$ is the radius of curvature

$$\mathcal{R} \sim \sqrt{\frac{R^3}{M}} \quad \text{in units } G = c = 1 \tag{28}$$

(a) near earth's surfae: $R \sim R_\oplus \sim 6.4 \times 10^6$m (earth's radius), $M \sim M_\oplus \sim$ 4.4mm (earth's mass), and $\mathcal{R} \sim 2.4 \times 10^{11}$m $\sim$ 1 astronomical unit.

(b) near sun's surface: $R \sim R_{\rm sun} \sim 7 \times 10^8$m, $M \sim M_{\rm sun} \sim$ 1.5km, and $\mathcal{R} \sim 5 \times 10^{11}$m $\sim$ 1AU.

(c) near the surface of a white-dwarf star: $R \sim$ 5000km, $M \sim M_{\rm sun} \sim$ 1.5km, and $\mathcal{R} \sim 3 \times 10^8$m $\sim \frac{1}{2}$(sun radius).

(d) near the surface of a neutron star: $R \sim$ 10km, $M \sim M_{\rm sun} \sim$ 3km, and $\mathcal{R} \sim$ 20km.

(e) near the surface of a one-solar-mass black hole: $M \sim M_{\rm sun} \sim$ 1.5km, $R \sim 2M \sim$ 3km, and $\mathcal{R} \sim$ 4km.

(f) in intergalactic space: $R \sim 10\times$(galaxy diameter) $\sim 10^6$ light-year, $M \sim$(galaxy mass)~ 0.03 light-year (for Milky way), and $\mathcal{R} \sim 6\times 10^9$ light-years $\sim$ Hubble Distance.

23.8 Components of Riemann in an arbitrary basis [by Xinkai Wu '02]

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = -R^\alpha{}_{\beta\gamma\delta}p^\beta \tag{29}$$

we have

$$p^\alpha{}_{;\gamma\delta} = (p^\alpha{}_{;\gamma})_{;\delta} = (p^\alpha{}_{,\gamma} + p^\mu\Gamma^\alpha{}_{\mu\gamma})_{;\delta} \tag{30}$$
$$= (p^\alpha{}_{,\gamma} + p^\mu\Gamma^\alpha{}_{\mu\gamma})_{,\delta} + \Gamma^\alpha{}_{\mu\delta}(p^\mu{}_{,\gamma} + p^\nu\Gamma^\mu{}_{\nu\gamma}) - \Gamma^\mu{}_{\gamma\delta}(p^\alpha{}_{,\mu} + p^\nu\Gamma^\alpha{}_{\nu\mu}) \tag{31}$$

interchaging γ and δ in the above expression and then taking the difference, we get

$$p^\alpha{}_{;\gamma\delta} - p^\alpha{}_{;\delta\gamma} = (\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta})p^\beta + \tag{32}$$
$$+(\Gamma^\mu{}_{\delta\gamma} - \Gamma^\mu{}_{\gamma\delta})\Gamma^\alpha{}_{\beta\mu}p^\beta + (p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma}) + (\Gamma^\mu{}_{\delta\gamma} - \Gamma^\mu{}_{\gamma\delta})p^\alpha{}_{,\mu} \tag{33}$$
$$= (\Gamma^\alpha{}_{\beta\gamma,\delta} - \Gamma^\alpha{}_{\beta\delta,\gamma} + \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta})p^\beta + \tag{34}$$
$$+c_{\gamma\delta}{}^\mu\Gamma^\alpha{}_{\beta\mu}p^\beta + (p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma}) + c_{\gamma\delta}{}^\mu p^\alpha{}_{,\mu} \tag{35}$$

where in the last step we've used $c_{\gamma\delta}{}^\mu = \Gamma^\mu{}_{\delta\gamma} - \Gamma^\mu{}_{\gamma\delta}$ (eq. (23.44)). We can see that the last two terms cancel, because

$$p^\alpha{}_{,\gamma\delta} - p^\alpha{}_{,\delta\gamma} = \nabla_{\vec{e}_\delta}\nabla_{\vec{e}_\gamma}p^\alpha - \nabla_{\vec{e}_\gamma}\nabla_{\vec{e}_\delta}p^\alpha \tag{36}$$
$$= \nabla_{[\vec{e}_\delta,\vec{e}_\gamma]}p^\alpha = c_{\delta\gamma}{}^\mu\nabla_{\vec{e}_\mu}p^\alpha \tag{37}$$
$$= c_{\delta\gamma}{}^\mu p^\alpha{}_{,\mu} = -c_{\gamma\delta}{}^\mu p^\alpha{}_{,\mu} \tag{38}$$

where to get to the second line, we've used the fact that for any scalar f,

$$\nabla_{\vec{A}}\nabla_{\vec{B}}f - \nabla_{\vec{B}}\nabla_{\vec{A}}f = A^\alpha(B^\beta f_{;\beta})_{;\alpha} - B^\beta(A^\alpha f_{;\alpha})_{;\beta} \tag{39}$$
$$= A^\alpha B^\beta f_{;\beta\alpha} + A^\alpha B^\beta{}_{;\alpha}f_{;\beta} - B^\beta A^\alpha f_{;\alpha\beta} - B^\beta A^\alpha{}_{;\beta}f_{;\alpha} \tag{40}$$
$$= (A^\alpha B^\beta{}_{;\alpha} - B^\alpha A^\beta{}_{;\alpha})f_{;\beta} \tag{41}$$
$$= [\vec{A},\vec{B}]^\beta f_{;\beta} \tag{42}$$
$$= \nabla_{[\vec{A},\vec{B}]}f\ . \tag{43}$$

(note $f_{;\alpha\beta} = f_{;\beta\alpha}$ by the "torsion free" condition).

Thus we finally conclude that

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\mu}c_{\gamma\delta}{}^\mu \tag{44}$$

23.9 Curvature of the surface of a sphere [by Alexander Putilin '99]

(a) We read off the metric components from the line element:

$$g_{\theta\theta} = a^2,\ g_{\phi\phi} = a^2 \sin^2\theta,\ g_{\theta\phi} = 0 \tag{45}$$

$$g^{\theta\theta} = \frac{1}{a^2},\ g^{\phi\phi} = \frac{1}{a^2 \sin^2\theta},\ g^{\theta\phi} = 0 \tag{46}$$

There are six independent connection coefficients

$$\Gamma^{\theta}{}_{\theta\theta} = g^{\theta\theta}\Gamma_{\theta\theta\theta} = g^{\theta\theta}\frac{1}{2}g_{\theta\theta,\theta} = 0 \tag{47}$$

$$\Gamma^{\theta}{}_{\theta\phi} = \Gamma^{\theta}{}_{\phi\theta} = g^{\theta\theta}\Gamma_{\theta\theta\phi} = \frac{1}{a^2}\ \frac{1}{2}(g_{\theta\theta,\phi} + g_{\theta\phi,\theta} - g_{\theta\phi,\theta}) = 0 \tag{48}$$

$$\Gamma^{\theta}{}_{\phi\phi} = g^{\theta\theta}\frac{1}{2}(2g_{\theta\phi,\phi} - g_{\phi\phi,\theta}) = -\frac{1}{2a^2}(a^2\sin^2\theta)_{,\theta} = -\sin\theta\cos\theta \tag{49}$$

$$\Gamma^{\phi}{}_{\theta\theta} = g^{\phi\phi}\frac{1}{2}(2g_{\phi\theta,\theta} - g_{\theta\theta,\phi}) = 0 \tag{50}$$

$$\begin{aligned}\Gamma^{\phi}{}_{\theta\phi} &= \Gamma^{\phi}_{\phi\theta} \\ &= g^{\phi\phi}\frac{1}{2}(g_{\phi\phi,\theta} + g_{\phi\theta,\phi} - g_{\phi\theta,\phi}) = \frac{1}{2a^2\sin^2\theta}(a^2\sin^2\theta)_{,\theta} = \cot\theta\end{aligned} \tag{51}$$

$$\Gamma^{\phi}{}_{\phi\phi} = g^{\phi\phi}\frac{1}{2}g_{\phi\phi,\phi} = 0 \tag{52}$$

(b) We can think of the Riemann tensor as a symmetric matrix $R_{[ij][kl]}$ with indices $[ij]$ and $[kl]$. Since R_{ijkl} is antisymmetric in the first and the second pairs of indices, the only nontrivial component is $[ij] = [\theta\phi]$, $[kl] = [\theta\phi]$

$$R_{\theta\phi\theta\phi} = -R_{\phi\theta\theta\phi} = -R_{\theta\phi\phi\theta} = R_{\phi\theta\phi\theta} \tag{53}$$

(c) Using eq. (23.57) and the fact that in a coordinate basis the $c_{\gamma\delta}{}^{\mu}$'s all vanish, we get

$$\begin{aligned}R^{\theta}{}_{\phi\theta\phi} &= \Gamma^{\theta}{}_{\phi\phi,\theta} - \Gamma^{\theta}{}_{\phi\theta,\phi} + \Gamma^{\theta}{}_{\mu\theta}\Gamma^{\mu}{}_{\phi\phi} - \Gamma^{\theta}{}_{\mu\phi}\Gamma^{\mu}{}_{\phi\theta} & (54)\\ &= -\frac{1}{2}(\sin 2\theta)_{,\theta} - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta} & (55)\\ &= -\cos(2\theta) - (-\sin\theta\cos\theta)\cot\theta & (56)\\ &= \sin^2\theta & (57)\end{aligned}$$

and thus

$$R_{\theta\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta \tag{58}$$

(d) The new basis is related to the old by $\vec{e}_{\hat{\theta}} = \frac{1}{a}\vec{e}_{\theta}$, $\vec{e}_{\hat{\phi}} = \frac{1}{a\sin\theta}\vec{e}_{\phi}$. Thus by the multilinearity of tensors in their slots, we have

$$g_{\hat{\theta}\hat{\theta}} = \frac{1}{a^2} g_{\theta\theta} = 1, \; g_{\hat{\phi}\hat{\phi}} = \frac{1}{a^2 \sin^2\theta} g_{\phi\phi} = 1, \; g_{\hat{\theta}\hat{\phi}} = \frac{1}{a^2 \sin\theta} g_{\theta\phi} = 0 \; . \tag{59}$$

i.e. $g_{\hat{j}\hat{k}} = \delta_{\hat{j}\hat{k}}$. We also have:

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^4 \sin^2\theta} R_{\theta\phi\theta\phi} = \frac{1}{a^2} \tag{60}$$

$$R_{\hat{j}\hat{k}} = g^{\hat{m}\hat{n}} R_{\hat{m}\hat{j}\hat{n}\hat{k}} = \delta^{\hat{m}\hat{n}} R_{\hat{m}\hat{j}\hat{n}\hat{k}} \tag{61}$$

thus

$$R_{\hat{\theta}\hat{\theta}} \; = \; R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\theta}} + R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\theta}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^2} \tag{62}$$

$$R_{\hat{\phi}\hat{\phi}} \; = \; R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} + R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{a^2} \tag{63}$$

$$R_{\hat{\theta}\hat{\phi}} \; = \; R_{\hat{\theta}\hat{\theta}\hat{\theta}\hat{\phi}} + R_{\hat{\phi}\hat{\theta}\hat{\phi}\hat{\phi}} = 0 \tag{64}$$

namely, $R_{\hat{k}\hat{k}} = \frac{1}{a^2} g_{\hat{j}\hat{k}}$.

$$R = R_{\hat{k}\hat{k}} g^{\hat{j}\hat{k}} = \frac{1}{a^2} g^{\hat{j}}_{\hat{j}} = \frac{2}{a^2} \tag{65}$$

23.10 Geodesic deviation on a sphere [by Alexander Putilin '99]

(a) The line element is given in problem 23.9: $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$. Then, on the equator, $\theta = \frac{\pi}{2}$, $dl^2 = a^2 d\phi^2$, $l = a\phi$ is the proper distance.

(b) Geodesic deviation eqn: $\nabla_{\vec{p}} \nabla_{\vec{p}} \vec{\xi} = -\mathbf{R}(..., \vec{p}, \vec{\xi}, \vec{p})$, with

$$\vec{p} = \frac{d}{dl} = \frac{1}{a} \; \frac{\partial}{\partial\phi}, \; p^\theta = 0, \; p^\phi = \frac{1}{a} \tag{66}$$

At $\theta = \frac{\pi}{2}$, connection coefficients vanish (see Ex. 23.9)

$$\nabla_{\vec{p}} \nabla_{\vec{p}} \xi^\alpha = \frac{1}{a^2} \left(\xi^\alpha{}_{;\phi} \right)_{;\phi} = \frac{1}{a^2} \left(\xi^\alpha{}_{;\phi} \right)_{,\phi} \tag{67}$$

$$\xi^\theta{}_{;\phi} \; = \; \xi^\theta{}_{,\phi} + \Gamma^\theta{}_{\mu\phi} \xi^\mu = \xi^\theta{}_{,\phi} - \sin\theta\cos\theta\xi^\phi \tag{68}$$

$$\xi^\phi{}_{;\phi} \; = \; \xi^\phi{}_{,\phi} + \Gamma^\phi{}_{\mu\phi} \xi^\mu = \xi^\phi{}_{,\phi} + cot\theta\xi^\theta \tag{69}$$

thus

$$\nabla_{\vec{p}} \nabla_{\vec{p}} \xi^\theta \; = \; \frac{1}{a^2} \left(\xi^\theta{}_{,\phi} - \sin\theta\cos\theta\xi^\phi \right)_{,\phi} |_{\theta=\frac{\pi}{2}} = \frac{1}{a^2} \xi^\theta{}_{,\phi\phi} \tag{70}$$

$$\nabla_{\vec{p}} \nabla_{\vec{p}} \xi^\phi \; = \; \frac{1}{a^2} \left(\xi^\phi{}_{,\phi} + cot\theta\xi^\theta \right)_{,\phi} |_{\theta=\frac{\pi}{2}} = \frac{1}{a^2} \xi^\phi{}_{,\phi\phi} \tag{71}$$

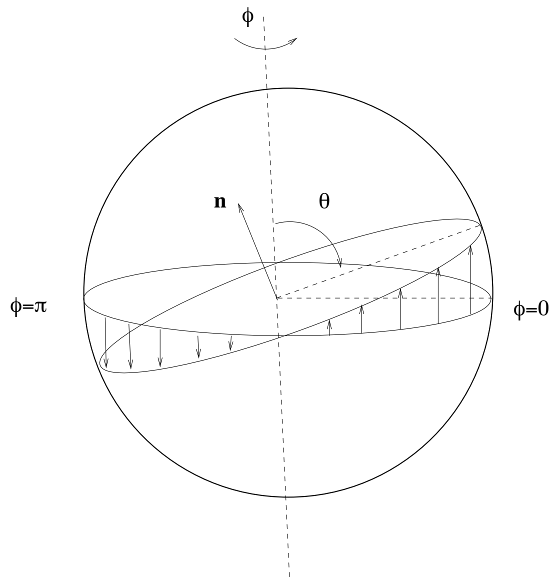

Figure 1: geodesic deviation on a sphere

On the other hand

$$\nabla_{\vec{p}}\nabla_{\vec{p}}\xi^\theta = -R^\theta{}_{\alpha\beta\gamma}p^\alpha\xi^\beta p^\gamma = -\frac{1}{a^2}R^\theta{}_{\phi\beta\phi}\xi^\beta = -\frac{1}{a^2}R^\theta{}_{\phi\theta\phi}\xi^\theta \tag{72}$$

$$= -\frac{\sin^2\theta}{a^2}\xi^\theta|_{\theta=\frac{\pi}{2}} = -\frac{1}{a^2}\xi^\theta \tag{73}$$

thus

$$\frac{1}{a^2}\xi^\theta{}_{,\phi\phi} = -\frac{1}{a^2}\xi^\theta \Rightarrow \frac{d^2\xi^\theta}{d\phi^2} = -\xi^\theta \tag{74}$$

$$\nabla_{\vec{p}}\nabla_{\vec{p}}\xi^\phi = -\frac{1}{a^2}R^\phi{}_{\phi\mu\phi}\xi^\mu = 0 \Rightarrow \frac{d^2\xi^\phi}{d\phi^2} = 0 \tag{75}$$

(c) Initial conditions (note that the geodesics are parallel at $\phi = 0$):

$$\xi^\theta(0) = b,\ \dot{\xi}^\theta(0) = 0;\ \xi^\phi(0) = 0,\ \dot{\xi}^\phi(0) = 0 \tag{76}$$

This gives $\xi^\phi = A\phi + B = 0$. And

$$\xi^\theta(\phi) = A'\cos\phi + B'\sin\phi = b\cos\phi \tag{77}$$

Let $\theta = \theta(\phi)$ be the eqn. for a "tilted" great circle. It's given by $\mathbf{n}\cdot\mathbf{x} = 0$, where $\mathbf{n} = (-\sin\Delta\theta, 0, \cos\Delta\theta) \approx (-\Delta\theta, 0, 1)$ is the orthogonal vector and $\Delta\theta = \frac{b}{a}$, while $\mathbf{x} = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$. $\mathbf{n}\cdot\mathbf{x} = a(-\sin\theta\cos\phi\cdot\Delta\theta + \cos\theta) = 0$ then gives: $cot\theta = \Delta\theta\cos\phi = tan(\frac{\pi}{2} - \theta) \approx \frac{\pi}{2} - \theta$, i.e. $\theta = \frac{\pi}{2} - \Delta\theta\cos\phi$.

From Fig. (1) we see that the separation vectors points along θ-direction (i.e. $\xi^\phi = 0$), and its magnitude is $\xi^\theta = a(\frac{\pi}{2} - \theta) = a\Delta\theta\cos\phi = b\cos\phi$, which is precisely what we got before.

23.12 Newtonian limit of general relativity [by Alexander Putilin '99]

(a) We are given that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and $|h_{\alpha\beta}| \ll 1$. Proper time:

$$d\tau^2 = -g_{\alpha\beta}dx^\alpha dx^\beta \approx -\eta_{\alpha\beta}dx^\alpha dx^\beta \approx dt^2 - d\mathbf{x}^2 \approx dt^2 \ . \tag{78}$$

where the last approximation is because in the non-relativistic limit, $|dx|/|dt| \sim |v/c| \ll 1$). Thus $d\tau \approx dt$, and

$$u^\alpha = \frac{dx^\alpha}{d\tau} \approx \frac{dx^\alpha}{dt} : u^0 = \frac{dt}{d\tau} \approx 1,\ u^j = \frac{dx^j}{d\tau} \approx \frac{dx^j}{dt} = v^j \ . \tag{79}$$

(b) Geodesics eqn: $\frac{du^\alpha}{d\tau} = -\Gamma^\alpha{}_{\beta\gamma}u^\beta u^\gamma$.

$$\frac{du^j}{d\tau} \approx \frac{dv^j}{dt} \approx -\Gamma^j{}_{00} = -\Gamma_{j00} = -\frac{1}{2}(2g_{j0,0} - g_{00,j}) \tag{80}$$

$$= -h_{j0,0} + \frac{1}{2}h_{00,j} \approx \frac{1}{2}h_{00,j} \tag{81}$$

where in the last step we've used $|h_{\alpha\beta,t}| \ll |h_{\alpha\beta,j}|$.

$$\frac{dv^j}{dt} = u^\alpha v^j{}_{,\alpha} \approx \frac{\partial v^j}{\partial t} + v^k\frac{\partial v^j}{\partial x^k} \text{ i.e. } \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} \tag{82}$$

$\frac{dv^j}{dt} = -\Phi_{,j} \Rightarrow h_{00} = -2\Phi$.

(c) We can write:

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(g_{\mu\beta,\gamma}+g_{\mu\gamma,\beta}-g_{\beta\gamma,\mu}) = \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\gamma}+h_{\mu\gamma,\beta}-h_{\beta\gamma,\mu})+O(h^2) \ . \tag{83}$$

And the Riemann tensor is:

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + O(\Gamma^2) \tag{84}$$

$$= \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\delta} + h_{\mu\delta,\beta} - h_{\beta\delta,\mu})_{,\gamma} - \frac{1}{2}\eta^{\alpha\mu}(h_{\mu\beta,\gamma} + h_{\mu\gamma,\beta} - h_{\beta\gamma,\mu})_{,\delta} + O(h^2)$$

$$= \frac{1}{2}(h^\alpha{}_{\beta,\gamma\delta} + h^\alpha{}_{\delta,\beta\gamma} - h_{\beta\delta}{}^\alpha{}_{,\gamma} - h^\alpha{}_{\beta,\delta\gamma} - h^\alpha{}_{\gamma,\beta\delta} + h_{\beta\gamma}{}^\alpha{}_{,\delta}) + O(h^2) \tag{85}$$

Notice that in the last line the first and fourth terms cancel. Thus we get

$$R_{\alpha\beta\gamma\delta} \approx \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) \tag{86}$$

(d) $R_{j0k0} = \frac{1}{2}(h_{j0,k0} + h_{k0,j0} - h_{jk,00} - h_{00,jk})$. Recall that in non-relativistic limit, time derivatives are small compared to spatial ones, thus the last term in the brackets dominates. And we get

$$R_{j0k0} \approx -\frac{1}{2}h_{00,jk} = \Phi_{,jk} \tag{87}$$

23.13 Gauge transformation in linearized theory [by Alexander Putilin '99]

(a) $x^\alpha_{\text{new}} = x^\alpha_{old} + \xi^\alpha$,

$$g^{\text{new}}_{\alpha\beta}(x_{\text{new}}) = \frac{\partial x^\mu_{old}}{\partial x^\alpha_{\text{new}}}\frac{\partial x^\nu_{old}}{\partial x^\beta_{\text{new}}} g_{\mu\nu}(x_{old}) \tag{88}$$

Evaluate l.h.s. and r.h.s. up to linear order in ξ^α and $h_{\alpha\beta}$:

$$\text{l.h.s.} = \eta_{\alpha\beta} + h^{\text{new}}_{\alpha\beta}(x_{old} + \xi) \approx \eta_{\alpha\beta} + h^{\text{new}}_{\alpha\beta}(x_{old}) \tag{89}$$

$$\begin{aligned}
\text{r.h.s.} &= (\delta^\mu{}_\alpha - \xi^\mu{}_{,\alpha})(\delta^\nu{}_\beta - \xi^\nu{}_{,\beta}) g_{\mu\nu}(x_{\text{old}}) && (90)\\
&= g_{\alpha\beta}(x_{\text{old}}) - g_{\mu\beta}(x_{\text{old}})\xi^\mu{}_{,\alpha} - g_{\alpha\nu}(x_{\text{old}})\xi^\nu{}_{,\beta} && (91)\\
&\approx \eta_{\alpha\beta} + h^{\text{old}}_{\alpha\beta} - \eta_{\mu\beta}\xi^\mu{}_{,\alpha} - \eta_{\alpha\nu}\xi^\nu{}_{,\beta} && (92)\\
&\approx \eta_{\alpha\beta} + h^{\text{old}}_{\alpha\beta}(x_{\text{old}}) - \xi_{\alpha,\beta}(x_{\text{old}}) - \xi_{\beta,\alpha}(x_{\text{old}}) && (93)\\
&\Rightarrow h^{\text{new}}_{\alpha\beta} = h^{\text{old}}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} && (94)
\end{aligned}$$

(b)

$$\bar{h}^{\text{new}}_{\mu\nu} = h^{\text{new}}_{\mu\nu} - \frac{1}{2}h^{\text{new}}\eta_{\mu\nu} = \bar{h}^{\text{old}}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha{}_{,\alpha} \tag{95}$$

Lorentz gauge: $\bar{h}^{\text{new},\nu}_{\mu\nu} = 0$.

$$\bar{h}^{\text{new},\nu}_{\mu\nu} = \bar{h}^{\text{old},\nu}_{\mu\nu} - \xi_{\mu,\nu}{}^\nu - \xi_{\nu,\mu}{}^\nu + \xi_{\alpha,\mu}{}^\alpha = 0 \tag{96}$$

thus we need

$$\Box\xi_\mu \equiv \xi_{\mu,\nu}{}^\nu = \bar{h}^{\text{old},\nu}_{\mu\nu} \tag{97}$$

(c) In Lorentz gauge, all terms on the l.h.s. of eq. (23.102) vanish except the first one, thus it reduces to

$$-\bar{h}_{\mu\nu,\alpha}{}^\alpha = 16\pi T_{\mu\nu} \tag{98}$$

23.14 External Field of a Stationary, Linearized Source [by Keith Matthews '05]

We start by examining the role of Gauss's law. Because $\frac{\partial x^j}{\partial x^k} = (x^j)_{,\,k} = \delta^j{}_k$ and $T^{\mu\nu}{}_{,\,\nu} = T^{\mu k}{}_{,\,k} = 0$ we find $(T^{\mu j} x^k)_{,\,j} = T^{\mu k}$. Then

$$\int_V T^{\mu k}\, d^3x = \int_V (T^{\mu j} x^k)_{,\,j}\, d^3x = \int_S T^{\mu j} x^k\, d\Sigma_j = 0 \tag{99}$$

where the third equality comes from choosing a surface of integration entirely outside of the source where $T = 0$. Similarly $(T^{\mu l} x^j x^k)_{,\,l} = T^{\mu j} x^k + T^{\mu k} x^j$, so

$$\int_V (T^{\mu j} x^k + T^{\mu k} x^j)\, d^3x = \int_S T^{\mu l} x^j x^k\, d\Sigma_l = 0. \tag{100}$$

(a) There is a typo in (23.107),the $x]$ is meant to be an x'. We make use of the standard formula from E-M (23.109)

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{\vec{x} \cdot \vec{x}'}{r^3} + \ldots = \frac{1}{r} + \frac{x^k x'_k}{r^3} + \ldots$$

By inserting (23.109) into (23.107) for $\mu\,\nu = 0\,0$

$$\begin{aligned}
\overline{h}_{00}(x) &= 4\int \frac{T_{00}(x')}{r} d^3x' + 4\int T_{00}(x') \frac{x^k x'_k}{r^3} d^3x' \\
&= \frac{4}{r}\int T_{00} d^3x' + \frac{4x^k}{r^3}\int T_{00}\, x'_k d^3x' \\
&= \frac{4M}{r} + \frac{4D_k x^k}{r^3}
\end{aligned}$$

We have the freedom to choose the origin of our coordinates to coincide with the center of mass of our source. $\vec{x}_{com} = \frac{1}{M}\int \rho\, \vec{x}\, d^3x = 0$ where $\rho = T^{00} = T_{00}$. Thus $\vec{D} = M\vec{x}_{com} = 0$. This gives us the desired result.

(b) We have

$$\overline{h}_{0i} = \frac{4}{r}\int T_{0i}\, d^3x' + \frac{4x^k}{r^3}\int T_{0i}\, x'_k\, d^3x'$$

The first term is $\frac{4}{r}P_i$. Our gauge condition taken at order $O(\frac{1}{r^2})$ gives

$$\begin{aligned}
\overline{h}^{0\mu}{}_{,\,\mu} &= \overline{h}^{00}{}_{,\,0} + \overline{h}^{0k}{}_{,\,k} = 0 \\
&= \frac{4x^i}{r^3} D_{i,\,0} + 4P^k \partial_k(\frac{1}{r}) \\
&= \frac{4}{r^2} D_{k,\,0} \hat{x}^k - \frac{4}{r^2} P_k \hat{x}^k \\
&= D_{k,\,0} - P_k = 0
\end{aligned}$$

So $P_k = D_{k,0} = 0$ makes the first term go to zero.

We can insert (100) into the second term

$$\begin{aligned}
&\frac{4x^k}{r^3}\int T_{0i}{x'}_k - \frac{1}{2}(T_{0i}{x'}_k + T_{0k}{x'}_i)d^3x' \\
&= \frac{2x^k}{r^3}\int T_{0i}{x'}_k - T_{0k}{x'}_i d^3x' \\
&= \frac{2x^m}{r^3}\int \delta_i^j\delta_m^n(T_{0j}{x'}_n - T_{0n}{x'}_j)d^3x' \\
&= \frac{2x^m}{r^3}\int (\delta_i^j\delta_m^n - \delta_i^n\delta_m^j)T_{0j}{x'}_n d^3x' \\
&= \frac{2x^m}{r^3}\int \epsilon_{imr}\epsilon^{jnr}T_{0j}{x'}_n d^3x' \\
&= -\frac{2}{r^3}\epsilon_{ijk}S^j x^k
\end{aligned}$$

(c) Using (99) we find that the $O(\frac{1}{r})$ term is zero. We can construct a combination of divergences that will allow us to evaluate the second integral.

$$(T^{il}x^jx^k)_{,\,l} + (T^{lj}x^ix^k)_{,\,l} - (T^{lk}x^ix^j)_{,\,l} = 2T^{ij}x^k$$

Then by applying Gauss's law we find that the second integral goes to zero also.