# Solutions for Problem Set for Ch. 25

(compiled by Nate Bode)
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## A-D

**25.4 Behavior of $h_+$ and $h_\times$ under rotations and boosts** [by Xinkai Wu '02]

**(a)** Quantities with a tilde denote those in the new basis, and those without tilde in the old basis. As suggested in equation 25.51 we perform a change of basis: $\tilde{\mathbf{e}}_x + i\tilde{\mathbf{e}}_y = (\mathbf{e}_x + i\mathbf{e}_y)e^{i\psi}$. Then,

$$\tilde{\mathbf{e}}_x = \mathbf{e}_x \cos\psi - \mathbf{e}_y \sin\psi, \quad \tilde{\mathbf{e}}_y = \mathbf{e}_y \cos\psi + \mathbf{e}_x \sin\psi \ . \tag{1}$$

Plugging the above transformation matrix into equation 25.41 we find the components of Riemann in the new basis

$$R_{\tilde{x}0\tilde{x}0} = \cos^2\psi R_{x0x0} + \sin^2\psi R_{y0y0} - 2\cos\psi\sin\psi R_{x0y0} \tag{2}$$

$$= \cos 2\psi \left(-\frac{1}{2}\ddot{h}_+\right) - \sin 2\psi \left(-\frac{1}{2}\ddot{h}_\times\right) \tag{3}$$

on the other hand $R_{\tilde{x}0\tilde{x}0} = -\frac{1}{2}\ddot{\tilde{h}}_+$, thus we get

$$\tilde{h}_+ = (\cos 2\psi)h_+ - (\sin 2\psi)h_\times \tag{4}$$

Similarly, by looking at $R_{\tilde{x}0\tilde{y}0}$, we find

$$\tilde{h}_\times = (\cos 2\psi)h_\times + (\sin 2\psi)h_+ \tag{5}$$

Translated into complex numbers, this is just

$$\tilde{h}_+ + i\tilde{h}_\times = (h_+ + ih_\times)e^{2i\psi} \tag{6}$$

**(b)** The desired boost is a boost along the $z$ direction, which gives

$$\vec{\tilde{e}}_0 = \vec{e}_0 \cosh\beta + \vec{e}_z \sinh\beta, \quad \vec{\tilde{e}}_z = \vec{e}_0 \sinh\beta + \vec{e}_z \cosh\beta \tag{7}$$

with $\vec{e}_x$, $\vec{e}_y$ unchanged. And the corresponding transformation for the coordinates is

$$\tilde{t} = t\cosh\beta - z\sinh\beta, \quad \tilde{z} = -t\sinh\beta + z\cosh\beta \tag{8}$$

$$\text{which gives } \tilde{t} - \tilde{z} = (\cosh\beta + \sinh\beta)(t - z) \tag{9}$$

with $x, y$ unchanged.

Look at components of Riemann in the new basis using the above transformation matrix, we find

$$R_{x\tilde{0}x\tilde{0}} = (\cosh\beta - \sinh\beta)^2 R_{x0x0} = \left(\frac{-1}{2}\right)(\cosh\beta - \sinh\beta)^2 \ddot{h}_+(t - z) \tag{10}$$

on the other hand $R_{x\tilde{0}x\tilde{0}} = \left(\frac{-1}{2}\right)\ddot{\tilde{h}}_+(\tilde{t}-\tilde{z}) = (-1/2)\,\ddot{\tilde{h}}_+(t-z)/(\cosh\beta+\sinh\beta)^2$. Equating this with eq. (10) we find

$$\ddot{\tilde{h}}_+(t - z) = \ddot{h}_+(t - z) \Rightarrow \tilde{h}_+ = h_+ \tag{11}$$

By looking at $R_{x\tilde{0}y\tilde{0}}$ one can show the invariance of $h_\times$ in a similar manner.

**25.5 Energy-momentum conservation in geometric optics limit** [by Alexander Putilin '00]

Starting from equation 25.58:

$$T^{GW}_{\alpha\beta} = \frac{1}{16\pi}\langle h_{+,\alpha}h_{+,\beta} + h_{\times,\alpha}h_{\times,\beta}\rangle \tag{12}$$

In the geometric optics limit:

$$h_+ = \frac{Q_+(\tau_r;\theta,\phi)}{r}, \quad h_\times = \frac{Q_\times(\tau_r;\theta,\phi)}{r} \tag{13}$$

The wave vector $\vec{k} \equiv -\vec{\nabla}\tau_r$ is null, and we have $\nabla_{\vec{k}}\vec{k} = 0$, $\nabla_{\vec{k}} r = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})r$.

To show $T^{GW|\beta}_{\alpha\beta} = 0$, it's sufficient to prove that ${h_{+|\beta}}^{\beta} = {h_{\times|\beta}}^{\beta} = 0$.

We'll follow the pattern used in Sec 25.3.6,

$$h_{+,\beta} = h_{+,\tau_r}\frac{\partial\tau_r}{\partial x^\beta} + h_{+,\beta'} = -h_{+,\tau_r}k_\beta + h_{+,\beta'} \tag{14}$$

where prime denotes derivatives at fixed $\tau_r$ (i.e. $h_{+,\beta'} = h_{+,\theta}\frac{\partial\theta}{\partial x^\beta} + h_{+,\phi}\frac{\partial\phi}{\partial x^\beta} + h_{+,r}\frac{\partial r}{\partial x^\beta}$)

Differentiating the second time we get

$$\begin{aligned}
{h_{+|\beta}}^{\beta} &= -{h_{+,\tau_r}}^{,\beta}k_\beta - h_{+,\tau_r}({k_\beta}^{|\beta}) + {h_{+|\beta'}}^{\beta} &(15)\\
&= h_{+,\tau_r\tau_r}k^\beta k_\beta - h_{+,\tau_r|\beta'}k^\beta - (\vec{\nabla}\cdot\vec{k})h_{+,\tau_r} - h_{+|\beta',\tau_r}k^\beta + {h_{+|\beta'}}^{\beta'} &(16)\\
&= \vec{k}^2 h_{+,\tau_r\tau_r} - 2k^\beta h_{+,\tau_r|\beta'} - (\vec{\nabla}\cdot\vec{k})h_{+,\tau_r} + {h_{+|\beta'}}^{\beta'} &(17)
\end{aligned}$$

$\vec{k}^2 = 0$ so the first term vanishes. Notice that in geometric optics limit $h_+$ is a fast varying function of $\tau_r$ and slowly varying function of $\theta, \phi, r$. It means that derivatives of $h_+$ w.r.t. $\tau_r$ are much larger than derivatives at fixed $\tau_r$ and we can neglect the last term.

$$h_{+|\beta}{}^{\beta} \approx -2k^\beta h_{+,\tau_r|\beta'} - (\vec{\nabla}\cdot\vec{k})h_{+,\tau_r} \tag{18}$$

$k^\beta h_{+,\tau_r|\beta'}$ is the directional derivative along $\vec{k}$ (at fixed $\tau_r$). Since $\theta, \phi$ are constant along a ray, only $1/r$ factors can vary

$$k^\beta h_{+,\tau_r|\beta'} = k^\beta \left(\frac{1}{r}\frac{\partial Q_+}{\partial \tau_r}\right)_{,\beta'} = \frac{\partial Q_+}{\partial \tau_r}\left(-\frac{1}{r^2}\right)\nabla_{\vec{k}} r = -\frac{1}{r}\nabla_{\vec{k}} r\ h_{+,\tau_r} \tag{19}$$

$$\text{using } \nabla_{\vec{k}} r = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})r \tag{20}$$

$$= -\frac{1}{2}(\vec{\nabla}\cdot\vec{k})h_{+,\tau_r} \tag{21}$$

Finally

$$h_{+|\beta}{}^{\beta} = (\vec{\nabla}\cdot\vec{k})h_{+,\tau_r} - (\vec{\nabla}\cdot\vec{k})h_{+,\tau_r} = 0 \tag{22}$$

The equality $h_{\times|\beta}{}^{\beta} = 0$ can be derived in exactly the same way.

**25.6 Transformation to TT gauge** [by Alexander Putilin '99 and Keith Matthews '05]

**(a)** Consider the gauge transformation generated by $\xi_\alpha$:

$$h_{\alpha\beta} \to h'_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}\ , \tag{23}$$

or,

$$\bar{h}_{\alpha\beta} \to \bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi_\mu{}^{,\mu}\ . \tag{24}$$

Then

$$\bar{h}'_{\alpha\beta}{}^{,\beta} = \bar{h}_{\alpha\beta}{}^{,\beta} - \xi_{\alpha,\beta}{}^{\beta} - \xi_{\beta,\alpha}{}^{\beta} + \xi_{\mu,\alpha}{}^{\mu} \tag{25}$$

$$= \bar{h}_{\alpha\beta}{}^{,\beta} - \xi_{\alpha,\beta}{}^{\beta} \tag{26}$$

$$= -\xi_{\alpha,\beta}{}^{\beta} \tag{27}$$

where to get the last expression we've used the fact that $\bar{h}_{\alpha\beta}{}^{,\beta} = 0$, since $\bar{h}_{\alpha\beta}$ is in Lorentz gauge.

If we want $\bar{h}'_{\alpha\beta}$ to remain in Lorentz gauge, we see that the generators $\xi_\alpha$ should satisfy wave equation: $\xi_{\alpha,\beta}{}^{\beta} = 0$

The general solution of this equation can be written as a sum of plane waves:

$$\xi_\alpha(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[A_\alpha(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + B_\alpha(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}\right] \tag{28}$$

The first term describes the wave propagating in the $\mathbf{k}$ direction and second one in $-\mathbf{k}$ direction. In our cases we need only the first term (since we consider a gravitational wave propagating in some particular direction). So

$$\xi_\alpha(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} A_\alpha(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{29}$$

At time $t=0$: $\xi_\alpha(0,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} A_\alpha(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$, or $A_\alpha(\mathbf{k}) = \int d^3\mathbf{x}\xi_\alpha(0,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}$. We see that $\xi_\alpha(x)$ are completely determined by four functions of three spatial coordinates: $\xi_\alpha(0,\mathbf{x})$. These functions give initial conditions for the wave equation at $t=0$.

**(b)** Consider a plane gravitational wave propagating in the $z$-direction.

$$\bar{h}_{\alpha\beta} = \bar{h}_{\alpha\beta}(t-z) = \bar{h}_{\alpha\beta}(\tau), \quad \tau \equiv t-z \tag{30}$$

$\bar{h}_{\alpha\beta}$ is in Lorentz gauge, i.e.

$$\begin{aligned} \bar{h}_{\alpha\beta}{}^{,\beta} &= \bar{h}_{\alpha t}{}^{,t} + \bar{h}_{\alpha z}{}^{,z} = -\bar{h}_{\alpha t,t} + \bar{h}_{\alpha z,z} = -\bar{h}'_{\alpha t} - \bar{h}'_{\alpha z} & (31)\\ &= 0 & (32)\end{aligned}$$

Where prime denotes derivatives with respect to $\tau_r$ as defined in the previous problem. Integrating: $\bar{h}_{\alpha z} = -\bar{h}_{\alpha t} + \text{const}$. The constant is irrelevant and we can set it to zero, thus $\bar{h}_{\alpha z} = -\bar{h}_{\alpha t}$.

These four gauge conditions reduce the number of independent components of $\bar{h}_{\alpha\beta}$ from 10 to 6: $\bar{h}_{tt}, \bar{h}_{tx}, \bar{h}_{ty}, \bar{h}_{xx}, \bar{h}_{xy}, \bar{h}_{yy}$.

Now make additional gauge transformation with

$$\xi_\alpha = \xi_\alpha(\tau) = \xi_\alpha(t-z), \qquad \xi_{\alpha,\beta}{}^{\beta} = 0 \tag{33}$$

$$\bar{h}^{\text{new}}_{\alpha\beta} \to \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi_\mu{}^{,\mu} \tag{34}$$

We note that $\xi_\mu{}^{,\mu} = -\xi_{t,t} + \xi_{z,z} = -\xi'_t - \xi'_z$.

We want to choose $\xi_\alpha$ so that $\bar{h}^{\text{new}}_{\alpha\beta}$ satisfy additional constraints: $\bar{h}^{\text{new}}_{tt} = \bar{h}^{\text{new}}_{tx} = \bar{h}^{\text{new}}_{ty} = 0$, $\bar{h}^{\text{new}}_{xx} + \bar{h}^{\text{new}}_{yy} = 0$.

$$\begin{aligned}
\bar{h}^{\text{new}}_{tt} &= \bar{h}_{tt} - 2\xi_{t,t} + (\xi'_t + \xi'_z) = \bar{h}_{tt} + \xi'_z - \xi'_t & (35)\\
\bar{h}^{\text{new}}_{tx} &= \bar{h}_{tx} - \xi_{t,x} - \xi_{x,t} = \bar{h}_{tx} - \xi'_x & (36)\\
\bar{h}^{\text{new}}_{ty} &= \bar{h}_{ty} - \xi_{t,y} - \xi_{y,t} = \bar{h}_{ty} - \xi'_y & (37)\\
\bar{h}^{\text{new}}_{xx} &= \bar{h}_{xx} - 2\xi_{x,x} - \xi'_t - \xi'_z = \bar{h}_{xx} - \xi'_t - \xi'_z & (38)\\
\bar{h}^{\text{new}}_{yy} &= \bar{h}_{yy} - \xi'_t - \xi'_z & (39)\\
\bar{h}^{\text{new}}_{xy} &= \bar{h}_{xy} & (40)
\end{aligned}$$

This gives the system of equations:

$$\xi'_x = \bar{h}_{tx} \tag{41}$$
$$\xi'_y = \bar{h}_{ty} \tag{42}$$
$$\xi'_t = \frac{\bar{h}_{tt} + \frac{1}{2}(\bar{h}_{xx} + \bar{h}_{yy})}{2} \tag{43}$$
$$\xi'_z = \frac{-\bar{h}_{tt} + \frac{1}{2}(\bar{h}_{xx} + \bar{h}_{yy})}{2} \tag{44}$$

These equations have unique solutions (up to an additive constant) given by simple integrations.

**(c)** We apply eqns 25.94 and 25.95 where we use $z^k = e^k_{\hat{z}}$ for $n^k$. Then $P^l_j P^m_k h_{lm} = h_{jk} - z_j h_{kz} - z_k h_{jz} + z_j z_k h_{zz}$ and $P_{jk} P^{lm} h_{lm} = (\delta_{jk} - z_j z_k)(h_{xx} + h_{yy})$. Here are the results:

$$\begin{aligned}
h^{TT}_{xx} &= h_{xx} - 0 - 0 + 0 - \frac{1}{2}(h_{xx} + h_{yy}) = \frac{1}{2}(h_{xx} - h_{yy}) \\
h^{TT}_{yy} &= h_{yy} - 0 - 0 + 0 - \frac{1}{2}(h_{xx} + h_{yy}) = -\frac{1}{2}(h_{xx} - h_{yy}) \\
h^{TT}_{zz} &= h_{zz} - h_{zz} - h_{zz} + h_{zz} - 0 = 0 \\
h^{TT}_{xy} &= h_{xy} - \frac{1}{2}(0) = h_{xy} \\
h^{TT}_{xz} &= h_{xz} - 0 - h_{xz} + 0 - \frac{1}{2}(0) = 0 \\
h^{TT}_{yz} &= h_{yz} - h_{yz} = 0
\end{aligned}$$

We still have $h^{TT}_{\mu\nu,}{}^{\nu} = 0$ so $h^{TT}_{tt} = -h^{TT}_{tz} = h^{TT}_{zz} = 0$, $h^{TT}_{xt} = -h^{TT}_{xz} = 0$ and $h^{TT}_{ty} = -h^{TT}_{yz} = 0$.

$$h^{TT}_{jk} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(h_{xx} - h_{yy}) & h_{xy} & 0 \\ 0 & h_{xy} & -\frac{1}{2}(h_{xx} - h_{yy}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we find that $h^{TT}_{xx} = -h^{TT}_{yy} = h_+$, $h^{TT}_{xy} = h^{TT}_{yx} = h_\times$ and $tr(h^{TT}) = 0$ as they should.

**25.9 Energy removed by gravitational radiation reaction** [by Keith Matthews '05]

$$\frac{dM}{dt} = -\frac{1}{5}\int \rho \frac{dx^j}{dt}\left(\frac{\partial^5 \mathcal{I}_{lm}}{\partial t^5} x^l x^m\right)_{,j} d^3x = -\frac{2}{5}\mathcal{I}^5_{jl}\int \rho \frac{dx^{(j}}{dt} x^{l)} d^3x \tag{45}$$

Where $\mathcal{I}^5$ indicates $\frac{\partial^5 \mathcal{I}}{\partial t^5}$ and it comes out of the integral because it is not a function of $\vec{x}$. To evaluate the integral on the right we consider

$$\mathcal{I}^1_{jk} = \frac{\partial}{\partial t}\mathcal{I}_{jk} = \frac{d}{dt}\mathcal{I}_{jk} = 2\int \rho \frac{dx^{(j}}{dt} x^{l)} d^3x - \frac{1}{3}\delta^{jl}\int \rho \frac{d(r^2)}{dt} d^3x \qquad (46)$$

Because $\delta^{ij}x^i x^j = r^2$ and $\delta^{ij}\delta^{ij} = 3$, as applied below, we find that the second integral above doesn't contribute.

$$\begin{aligned}
\mathcal{I}^{ij}\mathcal{I}^{ij} &= \int\int \rho(\vec{x})\rho(\vec{x}')(x^i x^j - \frac{1}{3}\delta^{ij}r^2)(x'^i x'^j - \frac{1}{3}\delta^{ij}r'^2) d^3x\, d^3x' && (47)\\
&= \int\int \rho(\vec{x})\rho(\vec{x}')(x^i x^j x'^i x'^j - \frac{2}{3}r^2 r'^2 + \frac{1}{9}\delta^{ij}\delta^{ij}r^2 r'^2) d^3x\, d^3x' && (48)\\
&= \left(\int \rho(\vec{x})(x^i x^j - \frac{1}{3}\delta^{ij}r^2)d^3x\right)\left(\int \rho(\vec{x}')x'^i x'^j d^3x'\right) && (49)
\end{aligned}$$

So

$$\frac{dM}{dt} = \frac{d\langle M\rangle}{dt} = -\frac{1}{5}\left\langle \mathcal{I}^5_{jl}\mathcal{I}^1_{jl}\right\rangle \qquad (50)$$

which we integrate by parts twice to give

$$\frac{dM}{dt} = \frac{d\langle M\rangle}{dt} = -\frac{1}{5}\left\langle \mathcal{I}^3_{jl}\mathcal{I}^3_{jl}\right\rangle \qquad (51)$$

which is the desired result.

**25.10 Propagation of waves through an expanding universe** [by Alexander Putilin '00]

$$ds^2 = b^2[-d\eta^2 + d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2)], \text{ where } b = b_0\eta^2\ . \qquad (52)$$

**(a)** We can prove that curves of constant $\theta, \phi, \eta - \chi$ satisfy geodesic equation by explicit calculation of connection coefficients. But the easier way is to use symmetry. Spherical symmetry implies that a radial curve $\eta = \eta(\zeta)$, $\chi = \chi(\zeta)$, $\theta, \phi = const$ must be a geodesic for some parameter $\zeta$. Since a geodesic is null we have $-\left(\frac{d\eta}{d\zeta}\right)^2 + \left(\frac{d\chi}{d\zeta}\right)^2 = 0$, $\frac{d\eta}{d\zeta} = \frac{d\chi}{d\zeta}$, $\Rightarrow \eta - \chi =$ const along a geodesic.

**(b)** Symmetry also helps here. Spherical symmetry guarantees that $\nabla_{\vec{k}}\vec{e}_{\hat{\theta}}$ cannot point in $\chi$ or $\phi$ direction. So $\nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = a\vec{e}_{\hat{\theta}} + b\vec{k}$. $\vec{k} = k^\eta \vec{e}_{\hat{\eta}} + k^\chi \vec{e}_{\hat{\chi}} = k^\eta(\vec{e}_{\hat{\eta}} + \vec{e}_{\hat{\chi}})$, since $\vec{k}^2 = 0$ implies $k^\eta = k^\chi$. But $\vec{e}_{\hat{\theta}} \cdot \nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = a = \frac{1}{2}\nabla_{\vec{k}}(\vec{e}_{\hat{\theta}} \cdot \vec{e}_{\hat{\theta}}) = \frac{1}{2}\nabla_{\vec{k}}(1) = 0$ gives $a = 0$. and $\nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = b\vec{k} = bk^\eta(\vec{e}_{\hat{\eta}} + \vec{e}_{\hat{\chi}})$.

$$k^{\hat{\alpha}}\vec{e}_{\hat{\theta};\hat{\alpha}} = k^{\hat{\alpha}}\Gamma^{\hat{\mu}}{}_{\hat{\theta}\hat{\alpha}}\vec{e}_{\hat{\mu}} = bk^{\hat{\eta}}(\vec{e}_{\hat{\eta}} + \vec{e}_{\hat{\chi}}) \qquad (53)$$

Take a dot product of this eqn with $\vec{e}_{\hat{\chi}}$:

$bk^{\hat{\eta}} = k^{\hat{\alpha}}\Gamma^{\hat{\mu}}{}_{\hat{\theta}\hat{\alpha}}\eta_{\hat{\chi}\hat{\mu}} = k^{\hat{\alpha}}\Gamma_{\hat{\chi}\hat{\theta}\hat{\alpha}} = k^{\hat{\eta}}(\Gamma_{\hat{\chi}\hat{\theta}\hat{\eta}} + \Gamma_{\hat{\chi}\hat{\theta}\hat{\chi}})$, so $b = \Gamma_{\hat{\chi}\hat{\theta}\hat{\eta}} + \Gamma_{\hat{\chi}\hat{\theta}\hat{\chi}}$. Now we need only to calculate two connection coefficients to verify that $\Gamma_{\hat{\chi}\hat{\theta}\hat{\eta}} = \Gamma_{\hat{\chi}\hat{\theta}\hat{\chi}} = 0$, so that $b = 0 \quad \Rightarrow \nabla_{\vec{k}}\vec{e}_{\hat{\theta}} = 0$. The proof that $\nabla_{\vec{k}}\vec{e}_{\hat{\phi}} = 0$ is very similar.

**(c)** The general solutions are, in the geometric optics limit:

$$h_+ = \frac{Q_+(\tau_r,\theta,\phi)}{r}, \quad h_\times = \frac{Q_\times(\tau_r,\theta,\phi)}{r} \tag{54}$$

where $\vec{k} = -\vec{\nabla}\tau_r$ and $\nabla_{\vec{k}} r = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})r$.

To fix $\tau_r = \tau_r(\eta - \chi)$ notice that at $\chi = 0$ (or correspondingly $r = 0$) $\tau_r = t - r = t$, where $t$ is a proper time at $\chi = 0$.

$$dt^2 = -ds^2 = b^2 d\eta^2 = b_0^2\eta^4 d\eta^2$$

$$dt = b_0\eta^2 d\eta \Rightarrow t = \frac{1}{3}b_0\eta^3$$

$$\tau_r(\eta) = t = \frac{1}{3}b_0\eta^3$$

so

$$\tau_r(\eta-\chi) = \frac{1}{3}b_0(\eta-\chi)^3$$

$$\vec{k} = -\vec{\nabla}\tau_r \Rightarrow k^\eta = k^\chi = \frac{(\eta-\chi)^2}{b_0\eta^4} \tag{55}$$

$$(\vec{\nabla}\cdot\vec{k}) = \frac{1}{\sqrt{-g}}(\sqrt{-g}\;k^\alpha)_{,\alpha} = \frac{1}{\sqrt{-g}}\left[(\sqrt{-g}\;k^\eta)_{,\eta} + (\sqrt{-g}\;k^\chi)_{,\chi}\right] \tag{56}$$

$$= \frac{2(\eta-\chi)^2(\eta+2\chi)}{b_0\eta^5\chi} \quad \text{(after some calculations)} \tag{57}$$

Then

$$\nabla_{\vec{k}} r = \frac{1}{2}(\vec{\nabla}\cdot\vec{k})r = k^\eta(r_{,\eta} + r_{,\chi}) \tag{58}$$

reduces to

$$\left(\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\chi}\right) r = \left(\frac{1}{\chi} + \frac{2}{\eta}\right) r \tag{59}$$

changing variables: $a = \eta - \chi, b = \eta + \chi$, we get:

$$\frac{\partial}{\partial b}\; r = \left(\frac{1}{b-a} + \frac{2}{b+a}\right) r \tag{60}$$

$$r(a,b) = C(a)e^{\int db\left(\frac{1}{b-a}+\frac{2}{b+a}\right)} = C(a)(b-a)(b+a)^2 \tag{61}$$

$$\Rightarrow r(\chi,\eta) = C(\eta-\chi)\chi\eta^2 \tag{62}$$

where $C(\eta - \chi)$ is an arbitrary function.

Consider the region $\eta = \eta_0, \chi << \eta_0$. In this region we should have:

$$r(\chi, \eta) = r \tag{63}$$

where $dr^2 = ds^2 = b^2 d\chi^2 = b_0^2 \eta_0^4 dx^2$, $r = b_0 \eta_0^2 \chi$, $\Rightarrow$ $C(\eta_0)\chi\eta_0^2 = b_0\eta_0^2\chi$, $\Rightarrow$ $C(\eta_0) = b_0$. So finally we get

$$r = b_0 \eta^2 \chi \tag{64}$$

To determine $Q_+, Q_\times$, compare them to the solution of gravitational wave eqn. in the near zone: $\eta \approx \eta_0, \chi << \eta_0$ $(\tau_r = t - r)$

$$h_+ = \frac{2}{r}\left[\ddot{I}_{\hat{\theta}\hat{\theta}}(t-r)\right]^{TT} = \frac{2}{r}\left[\ddot{I}_{\hat{\theta}\hat{\theta}}(\tau_r)\right]^{TT} \tag{65}$$

$$h_\times = \frac{2}{r}\left[\ddot{I}_{\hat{\theta}\hat{\phi}}(t-r)\right]^{TT} = \frac{2}{r}\left[\ddot{I}_{\hat{\theta}\hat{\phi}}(\tau_r)\right]^{TT} \tag{66}$$

$\Rightarrow$

$$Q_+(\tau_r, \theta, \phi) = 2\left[\ddot{I}_{\hat{\theta}\hat{\theta}}(\tau_r)\right]^{TT} \tag{67}$$

$$Q_\times(\tau_r, \theta, \phi) = 2\left[\ddot{I}_{\hat{\theta}\hat{\phi}}(\tau_r)\right]^{TT} \tag{68}$$

**25.11 Gravitational waves emitted by a linear oscillator** [by Alexander Putilin '00]

Since the mass is moving along the $z$ direction the second moment of mass distribution has only a $zz$-component.

$$I_{zz}(t) = mz^2(t) = ma^2 \cos^2 \Omega t \tag{69}$$

or

$$I(t) = ma^2 \cos^2 \Omega t \; \vec{e}_z \otimes \vec{e}_z \tag{70}$$

and we have

$$h_{jk}^{TT} = 2\left[\frac{\ddot{I}_{jk}(t-r)}{r}\right]^{TT} \tag{71}$$

which gives

$$h^{TT} = \frac{2}{r} ma^2 \frac{-4\Omega^2 \cos 2\Omega(t-r)}{2} \left[\vec{e}_z \otimes \vec{e}_z\right]^{TT} \tag{72}$$

$$= -\frac{4m\Omega^2 a^2}{r} \cos\left(2\Omega(t-r)\right) \left[\vec{e}_z \otimes \vec{e}_z\right]^{TT} \tag{73}$$

To perform TT-projection notice that $\vec{e}_z = \cos\theta \vec{e}_{\hat{r}} - \sin\theta \vec{e}_{\hat{\theta}}$, and thus

$$\vec{e}_z \otimes \vec{e}_z = \cos^2\theta \vec{e}_{\hat{r}} \otimes \vec{e}_{\hat{r}} - \cos\theta\sin\theta(\vec{e}_{\hat{r}} \otimes \vec{e}_{\hat{\theta}} + \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{r}}) + \sin^2\theta \vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} \quad (74)$$

TT-projection on $(\vec{e}_{\hat{\theta}}, \vec{e}_{\hat{\phi}})$ plane gives:

$$[\vec{e}_z \otimes \vec{e}_z]^{TT} = \frac{1}{2}\sin^2\theta \left(\vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}}\right) \quad (75)$$

so

$$\begin{aligned} h^{TT} &= -\frac{2m\Omega^2 a^2}{r}\sin^2\theta\cos(2\Omega(t-r))\left(\vec{e}_{\hat{\theta}} \otimes \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\phi}} \otimes \vec{e}_{\hat{\phi}}\right) && (76) \\ &= -\frac{2m\Omega^2 a^2}{r}\sin^2\theta\cos(2\Omega(t-r))\ \mathbf{e}^+ && (77) \end{aligned}$$

It follows immediately from the result above that:

$$\begin{aligned} h_+(t,r,\theta,\phi) &= -\frac{2m\Omega^2 a^2}{r}\sin^2\theta\cos(2\Omega(t-r)) && (78) \\ h_\times(t,r,\theta,\phi) &= 0 && (79) \end{aligned}$$

In conventional units

$$h_+(t,r,\theta,\phi) = -\frac{2Gm\Omega^2 a^2}{rc^4}\sin^2\theta\cos(2\Omega(t-r)) \quad (80)$$

**25.12 Gravitational waves from waving arms** [by Xinkai Wu '02]

**(a)** Take the frequency to be $f \sim 2$Hz, then the wavelength is $\lambda \sim c/f \sim 1 \times 10^8$m. The major contribution to the gravitational wave comes from the mass quadrupole moment and is given by equation 25.112: $h_+ \sim h_\times \sim \frac{G}{c^4}\frac{Mv^2}{r}$, and we take $M \sim 10$kg, $v \sim Lf \sim 1\text{m} \times 2\text{Hz} \sim 2$m/s(where $L$ is the length of the arm), and $r \sim \lambda \sim 10^8$m, this gives $h_+ \sim h_\times \sim 10^{-51}$.

**(b)** The total power $\frac{dE}{dt}$ is given by equation 25.113. Restoring $G, c$, we get $\frac{dE}{dt} \sim \frac{G}{c^5}\frac{M^2v^6}{L^2} \sim 10^{-49}$J/s, and the number of gravitons emitted per second is $\frac{1}{\hbar 2\pi f}\frac{dE}{dt} \sim 10^{-16}$Hz, which means the gravitational waves emitted are so weak that they can't really be treated as classical waves.

**25.14 Light in an interferometric gravitational wave detector in TT gauge** [by Xinkai Wu '02]

**(a)** This expression for $\phi$ gives

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= -\omega_0\left[1 + \frac{1}{2}h(t-x) - \frac{1}{2}h(t)\right] && (81) \\ \frac{\partial\phi}{\partial x} &= -\omega_0\left[-1 - \frac{1}{2}h(t-x)\right] && (82) \end{aligned}$$

Ignoring terms quadratic (and higher) in $h$, we find

$$
\begin{aligned}
-\left(\frac{\partial\phi}{\partial t}\right)^2+[1-h(t)]\left(\frac{\partial\phi}{\partial x}\right)^2 &= -\omega_0^2[1+h(t-x)-h(t)] \\
&\qquad +[1-h(t)]\omega_0^2[1+h(t-x)] \quad (83)\\
&= 0 \quad (84)
\end{aligned}
$$

**(b)** Setting $x=0$, we get $\frac{\partial\phi}{\partial t}=-\omega_0$.

**(c)**

$$
\begin{aligned}
(\nabla_{\vec{k}}\vec{k})_\nu &= k^\mu\nabla_\mu k_\nu \quad (85)\\
&= k^\mu\nabla_\mu\nabla_\nu\phi \quad (86)\\
&= k^\mu\nabla_\nu\nabla_\mu\phi \quad (87)\\
&= k^\mu\nabla_\nu k_\mu \quad (88)\\
&= \frac{1}{2}\nabla_\nu(k^\mu k_\mu) \quad (89)\\
&= 0 \quad (90)
\end{aligned}
$$

**(d)** The null geodesic of the photon is given by $0=ds^2=-dt^2+[1+h(t)]dx^2$, which gives $\frac{dx}{dt}=1-\frac{1}{2}h(t)$. Now $p_x=-\partial\phi/\partial x=-\omega_0[1+\frac{1}{2}h(t-x)]$, and along the null geodesic we have

$$
\frac{dp_x}{dt}=-\omega_0\frac{1}{2}\dot{h}(t-x)\left(1-\frac{dx}{dt}\right)=-\omega_0\frac{1}{2}\dot{h}(t-x)\left(\frac{1}{2}h(t)\right)=0 \quad (91)
$$

up to linear order in $h$. Thus we see $p_x$ is indeed conserved along the geodesic.

**(e)** The observer at rest has 4-velocity $\vec{u}=(1,0,0,0)$, thus the photon's energy measured by him is $\omega=-\vec{k}\cdot\vec{u}=-k_\alpha u^\alpha=-k_t=-\partial\phi/\partial t=\omega_0[1+\frac{1}{2}h(t-x)-\frac{1}{2}h(t)]$. $\frac{d\omega}{dt}=\frac{\partial\omega}{\partial t}+\frac{\partial\omega}{\partial x}\frac{dx}{dt}$. Since $\frac{\partial\omega}{\partial x}$ is already of order $h$, we can approximate $\frac{dx}{dt}$ as unity, and thus getting $\frac{d\omega}{dt}\approx(\frac{\partial}{\partial t}+\frac{\partial}{\partial x})\omega$. And this is

$$
\begin{aligned}
\frac{d\omega}{dt} &\approx \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\omega \quad (92)\\
&= \omega_0\left[\frac{1}{2}\dot{h}(t-x)-\frac{1}{2}\dot{h}(t)-\frac{1}{2}\dot{h}(t-x)\right] \quad (93)\\
&= -\frac{1}{2}\omega_0\dot{h}(t) \quad (94)
\end{aligned}
$$

as desired.

# Solution 25

26.2 (a) $\tilde{E} = -p_t = -g_{tt}\, p^t = (1-\frac{2M}{r})\frac{dt}{d\tau}$

$\tilde{L} = p_\phi = g_{\phi\phi}\, p^\phi = r^2 \frac{d\phi}{d\tau}$

(b) $-d\tau^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)$

Here $\theta = \pi/2$ is constant. and $u = \frac{1}{r}$ , $\frac{dr}{d\tau} = \frac{dr/d\phi}{d\tau/d\phi}$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{E}^2}{\tilde{L}^2} - \left(u^2 + \frac{1}{\tilde{L}^2}\right)(1-2Mu)$$

(c) Differentiate the above equation with respect to $\phi$

$$\Rightarrow \frac{d^2u}{d\phi^2} + u - \frac{M}{\tilde{L}^2} = 3Mu^2 \text{ or } \frac{d^2u}{d\phi^2} + u - \frac{MG}{\tilde{L}^2} = \frac{3GM}{c^2}u^2$$

Newtonian obital equation : $\frac{d^2u}{d\phi^2} + u - \frac{GM}{\tilde{L}^2} = 0$ can be obtained by taking $c \to \infty$

(d) At zero order , $u = \frac{M}{\tilde{L}^2}(1 + e\cos\phi)$ is apparently a solution of $\frac{d^2u}{d\phi^2} + u - \frac{M}{\tilde{L}^2} = 0$

# Solution 25

26.2 $$\frac{d^2u_1}{d\phi^2} + u_1 = 3Mu_0^2 \quad \text{where} \quad u_0 = \frac{M}{L^2}(1+e\cos\phi)$$

$$= \frac{3M^3}{L^2}(1+2e\cos\phi + e^2\cos^2\phi)$$

$$= \frac{3M^3}{L^2}(1+\frac{e^2}{2} + 2e\cos\phi + \frac{e^2}{2}\cos 2\phi)$$

$$\Rightarrow u_1 = \frac{3M^3}{L^2}(1+\frac{e^2}{2} + \underbrace{e\phi\sin\phi}_{\text{relavant part}} + \cdots \cos 2\phi)$$

$$\frac{du_0}{d\phi} + \frac{du_1}{d\phi} = 0 \Rightarrow \Delta\phi = 2\pi \cdot \frac{3M^2}{L^2} = 6\pi M^2/L^2$$

It's easy to check for planet Mercury the rate is 43''/century.