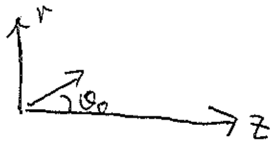
Ph136 Solution 7

7.8 (a) $$\frac{d}{ds}\left(n\frac{d\vec{x}}{ds}\right) = \nabla n$$



$$\Rightarrow \begin{cases} \frac{d}{ds}\left(n\frac{dr}{ds}\right) = \frac{\partial n}{\partial r} \\ \frac{d}{ds}\left(n\frac{dz}{ds}\right) = 0 \Rightarrow n\cos\theta = \text{Const} \Rightarrow \cos\theta = \frac{\cos\theta_0}{\sqrt{1-\alpha^2 r^2}} \end{cases}$$

$$\Rightarrow \frac{dz}{dr} = \cot\theta = \frac{\cos\theta_0}{\sqrt{\sin^2\theta_0 - \alpha^2 r^2}}$$

$$\xrightarrow{\text{integrating}} \quad r = \frac{\sin\theta_0}{\alpha}\left|\sin\left(\frac{\alpha z}{\cos\theta_0}\right)\right|$$

(b) $$dt = \frac{ds}{v} = \frac{ds\, n}{c} = \frac{dz\, n}{c\cos\theta}$$

$$= \frac{dz\, n_0\left[1 - \frac{\sin^2\theta_0}{2}\left(1 - \cos\frac{2\alpha z}{\cos\theta_0}\right)\right]}{c\cos\theta_0}$$

$$\Rightarrow T = \int dt \approx \int_0^L \frac{n_0}{c}dz\, \frac{1-\theta_0^2/2}{1-\frac{\theta_0^2}{2}+O(\theta_0^4)}$$

$$= \frac{n_0}{c}L\left(1+O(\theta_0^4)\right)$$

Solution 7 Ph136

7.11 (a) Let $A \rightarrow \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ free propagation over distance d

$B \rightarrow \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}$ Spherical mirror

Note $ABABABA - 2bABA - I = \begin{pmatrix} 0 & 0 \\ -\frac{2}{R} & 0 \end{pmatrix}$ with $b = 1 - \frac{4d}{R} + \frac{2d^2}{R^2}$

So $\vec{X}_{k+2} - 2b\vec{X}_{k+1} + \vec{X}_k = \left(ABABABA - 2bABA - I\right)\vec{X}_k = \begin{pmatrix} 0 \\ -\frac{2}{R}X_k \end{pmatrix}$

$\Rightarrow X_{k+2} - 2bX_{k+1} + X_k = 0$

(b) $X_{k+2} - 2bX_{k+1} + X_k = 0$ can be solved by assuming

$X_k \sim Ce^{ik\alpha} + De^{-ik\alpha}$ and it's easy to verify

$\alpha = \cos^{-1} b$

(c) $d = k \cdot 2R$ and $b = 1 - 8k + 8k^2$

for $k = 1$, $b = 1$ and $X_k = A$

$k < 1$ or $k > 0$, $b < 1$, all rays oscillate about the optic axis

$k > 1$, $b > 1$, α is imaginary and the system is unstable

(d) $\cos\theta = b = 1 - \frac{4d}{R} + \frac{2d^2}{R}$ $\Rightarrow$ $\frac{d}{R} = 1 \pm \cos\frac{\theta}{2}$

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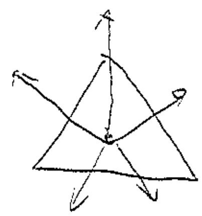
4. (a) Since $N_{saddle} = N_-$ and $N_{max} + N_{min} - 1 = N_+$

$\Rightarrow N_+ - N_- = 1 \iff N_{max} + N_{min} - N_{saddle} = 2$

(b) Now the triangles are like

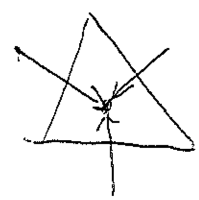
None

$\alpha = -2\pi$



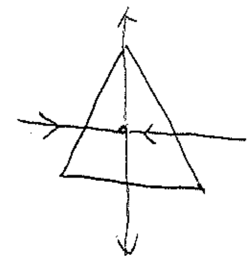
max

$\alpha = 0$



min

$\alpha = 0$

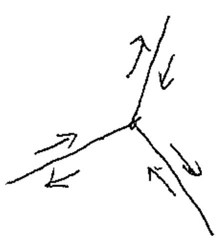


Saddle

$\alpha = -4\pi$

$$\Rightarrow \alpha = -2\pi\left(N_{max} - N_{saddle} + N_{min} - F\right)$$

(c)(d) It's obvious that edge contributions vanish, so we only need to consider the vortex. For each vortex, j edges contribute $-j\pi$ angles plus the total rotation 2π



$$\Rightarrow \alpha = -\sum_{j=1}^{V} E_j \pi - 2\pi = 2\pi(V - E)$$

(e) $\alpha = 2\pi(V - E) = -2\pi\left(N_{max} - N_{saddle} + N_{min} - F\right)$

$\Rightarrow N_{max} + N_{min} - N_{saddle} = F - E + V = 2$ (for 2-D sphere)