Contents

Chapter 2

(T2) Special Relativity: Geometric Viewpoint

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Box 2.1
Reader's Guide

- Parts II (Statistical Physics), III (Optics), IV (Elasticity), V (Fluids), and VI (Plasmas) of this book deal almost entirely with Newtonian Physics; only a few sections and exercises are relativistic. Readers who are inclined to skip those relativistic items (which are all labeled Track Two) can skip this chapter and then return to it just before embarking on Part VII (General Relativity). Accordingly, this chapter is Track Two for readers of Parts II–VI, and Track One for readers of Part VII.
- More specifically, this chapter is a prerequisite for the following: sections on relativistic kinetic theory in Chap. 3, Sec. 13.8 on relativistic fluid dynamics, Ex. 17.11 on relativistic shocks in fluids, many comments in Parts II–VI about relativistic effects and connections between Newtonian physics and relativistic physics, and all of Part VII (General Relativity)
- We recommend that those readers, for whom relativity is relevant and who already have a strong understanding of special relativity, not skip this chapter entirely. Instead, we suggest they browse it, especially Secs. 2.2–2.4, 2.8, 2.11–2.13, to make sure they understand this book's geometric viewpoint and to ensure their familiarity with concepts such as the stress-energy tensor that they might not have met previously.

2.1 Overview

This chapter is a fairly complete introduction to special relativity, at an intermediate level. We extend the geometric viewpoint, developed in Chap. 1 for Newtonian physics, to the domain of special relativity; and we extend the tools of differential geometry, developed in Chap. 1 for Newtonian physics' arena, 3-dimensional Euclidean space, to special relativity's arena, 4-dimensional Minkowski spacetime.

We begin in Sec. 2.2 by defining inertial (Lorentz) reference frames, and then introducing fundamental, geometric, reference-frame-independent concepts: events, 4-vectors, and the invariant interval between events. Then in Sec. 2.3, we develop the basic concepts of tensor algebra in Minkowski spacetime (tensors, the metric tensor, the inner product and tensor product, and contraction), patterning our development on the corresponding concepts in Euclidean space. In Sec. 2.4, we illustrate our tensor-algebra tools by using them to describe — without any coordinate system or reference frame — the kinematics (world lines, 4-velocities, 4-momenta) of point particles that move through Minkowski spacetime. The particles are allowed to collide with each other and be accelerated by an electromagnetic field. In Sec. 2.5, we introduce components of vectors and tensors in an inertial reference frame and rewrite our frame-independent equations in slot-naming index notation; and then in Sec. 2.6, we use these extended tensorial tools to restudy the motions, collisions, and electromagnetic accelerations of particles. In Sec. 2.7, we discuss Lorentz transformations in Minkowski spacetime, and in Sec. 2.8, we develop spacetime diagrams and use them to study length contraction, time dilation, and simultaneity breakdown. In Sec. 2.9, we illustrate the tools we have developed by asking whether the laws of physics permit a highly advanced civilization to build time machines for traveling backward in time as well as forward. In Sec. 2.10, we introduce directional derivatives, gradients, and the Levi-Civita tensor in Minkowski spacetime, and in Sec. 2.11, we use these tools to discuss Maxwell's equations and the geometric nature of electric and magnetic fields. In Sec. 2.12, we develop our final set of geometric tools: volume elements and the integration of tensors over spacetime, and in Sec. 2.13, we use these tools to define the stress-energy tensor, and to formulate very general versions of the conservation of 4-momentum.

2.2 Foundational Concepts

2.2.1 Inertial frames, inertial coordinates, events, vectors, and spacetime diagrams

Because the nature and geometry of Minkowski spacetime are far less obvious intuitively than those of Euclidean 3-space, we shall need a crutch in our development of the geometric viewpoint for physics in spacetime. That crutch will be inertial reference frames.

An *inertial reference frame* is a three-dimensional latticework of measuring rods and clocks (Fig. 2.1) with the following properties: (*i*) The latticework is purely conceptual and has arbitrarily small mass so it does not gravitate. (*ii*) The latticework moves freely through spacetime (i.e., no forces act on it), and is attached to gyroscopes so it is inertially nonrotating. (*iii*) The measuring rods form an orthogonal lattice, and the length intervals

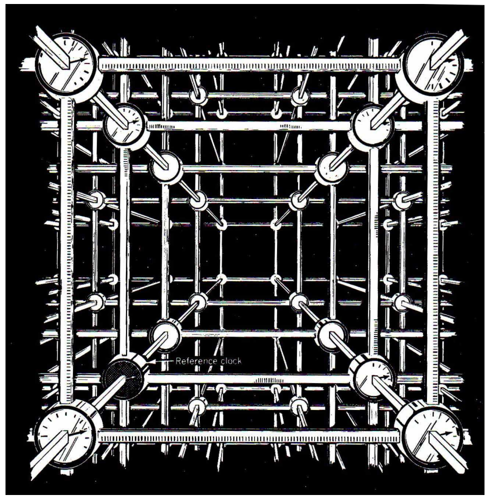


Fig. 2.1: An inertial reference frame. From Taylor and Wheeler (1992).

marked on them are uniform when compared to, e.g., the wavelength of light emitted by some standard type of atom or molecule. Therefore, the rods form an orthonormal, Cartesian coordinate system with the coordinate x measured along one axis, y along another, and z along the third. (*iv*) The clocks are densely packed throughout the latticework so that, ideally, there is a separate clock at every lattice point. (*v*) The clocks tick uniformly when compared to the period of the light emitted by some standard type of atom or molecule; i.e., they are *ideal clocks*. (*vi*) The clocks are synchronized by the Einstein synchronization process: If a pulse of light, emitted by one of the clocks, bounces off a mirror attached to another and then returns, the time of bounce t_b, as measured by the clock that does the bouncing, is the average of the times of emission and reception, as measured by the emitting and receiving clock: $t_b = \frac{1}{2}(t_e + t_r)$.[1]

(That inertial frames with these properties can exist, when gravity is unimportant, is an empiracle fact; and it tells us that, in the absence of gravity, spacetime is truly Minkowski.)

Our first fundamental, frame-independent relativistic concept is the *event*. An event is a precise location in space at a precise moment of time; i.e., a precise location (or "point") in 4-dimensional spacetime. We sometimes will denote events by capital script letters such as $\mathcal{P}$ and $\mathcal{Q}$ — the same notation as for points in Euclidean 3-space.

A *4-vector* (also often referred to as a *vector in spacetime* or just a *vector*) is a straight[2] arrow $\Delta\vec{x}$ reaching from one event $\mathcal{P}$ to another $\mathcal{Q}$. We often will deal with 4-vectors and ordinary (3-space) vectors simultaneously, so we shall use different notations for them: boldface Roman font for 3-vectors, $\Delta\mathbf{x}$, and arrowed italic font for 4-vectors, $\Delta\vec{x}$. Sometimes we shall identify an event $\mathcal{P}$ in spacetime by its vectorial separation $\vec{x}_{\mathcal{P}}$ from some arbitrarily chosen event in spacetime, the "origin" $\mathcal{O}$.

An inertial reference frame provides us with a coordinate system for spacetime. The

[1] For a deeper discussion of the nature of ideal clocks and ideal measuring rods see, e.g., pp. 23–29 and 395–399 of Misner, Thorne, and Wheeler (1973).

[2] By "straight" we mean that in any inertial reference frame the coordinates along $\Delta\vec{x}$ are linear functions of one another

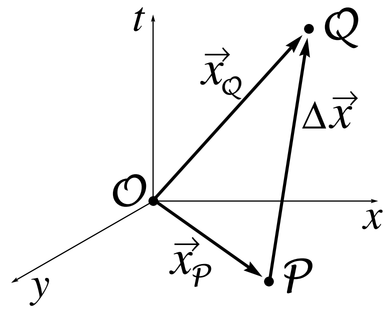


Fig. 2.2: A spacetime diagram depicting two events $\mathcal{P}$ and $\mathcal{Q}$, their vectorial separations $\vec{x}_{\mathcal{P}}$ and $\vec{x}_{\mathcal{Q}}$ from an (arbitrarily chosen) origin, and the vector $\Delta\vec{x} = \vec{x}_{\mathcal{Q}} - \vec{x}_{\mathcal{P}}$ connecting them. The laws of physics cannot involve the arbitrary origin $\mathcal{O}$; we introduce it only as a conceptual aid.

coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ which it associates with an event $\mathcal{P}$ are $\mathcal{P}$'s location (x, y, z) in the frame's latticework of measuring rods, and the time t of $\mathcal{P}$ *as measured by the clock that sits in the lattice at the event's location.* (Many apparent paradoxes in special relativity result from failing to remember that the time t of an event is always measured by a clock that resides at the event, and never by clocks that reside elsewhere in spacetime.)

It is useful to depict events on *spacetime diagrams*, in which the time coordinate $t = x^0$ of some inertial frame is plotted upward, two of the frame's three spatial coordinates, $x = x^1$ and $y = x^2$, are plotted horizontally, and the third coordinate $z = x^3$ is omitted. Figure 2.2 is an example. Two events $\mathcal{P}$ and $\mathcal{Q}$ are shown there, along with their vectorial separations $\vec{x}_{\mathcal{P}}$ and $\vec{x}_{\mathcal{Q}}$ from the origin and the vector $\Delta\vec{x} = \vec{x}_{\mathcal{Q}} - \vec{x}_{\mathcal{P}}$ that separates them from each other. The coordinates of $\mathcal{P}$ and $\mathcal{Q}$, which are the same as the components of $\vec{x}_{\mathcal{P}}$ and $\vec{x}_{\mathcal{Q}}$ in this coordinate system, are $(t_{\mathcal{P}}, x_{\mathcal{P}}, y_{\mathcal{P}}, z_{\mathcal{P}})$ and $(t_{\mathcal{Q}}, x_{\mathcal{Q}}, y_{\mathcal{Q}}, z_{\mathcal{Q}})$; and correspondingly, the components of $\Delta\vec{x}$ are

$$\begin{aligned} \Delta x^0 &= \Delta t = t_{\mathcal{Q}} - t_{\mathcal{P}}, \quad & \Delta x^1 &= \Delta x = x_{\mathcal{Q}} - x_{\mathcal{P}}, \\ \Delta x^2 &= \Delta y = y_{\mathcal{Q}} - y_{\mathcal{P}}, \quad & \Delta x^3 &= \Delta z = z_{\mathcal{Q}} - z_{\mathcal{P}}\ . \end{aligned} \tag{2.1}$$

We shall denote these components of $\Delta\vec{x}$ more compactly by Δx^α, where the index α and all other lower case Greek indexes range from 0 (for t) to 3 (for z).

When the physics or geometry of a situation being studied suggests some preferred inertial frame (e.g., the frame in which some piece of experimental apparatus is at rest), then we typically will use as axes for our spacetime diagrams the coordinates of that preferred frame. On the other hand, when our situation provides *no* preferred inertial frame, or when we wish to emphasize a frame-independent viewpoint, we shall use as axes the coordinates of a completely arbitrary inertial frame and we shall think of the spacetime diagram as depicting spacetime in a coordinate-independent, frame-independent way.

We shall use the terms *inertial coordinate system* and *Lorentz coordinate system* interchangeably[3] to mean the coordinate system (t, x, y, z) provided by an inertial frame; and we shall also use the term *Lorentz frame* interchangeably with *inertial frame*. A physicist or

[3]because it was Lorentz (1904) who first studied the relationship of one such coordinate system to another: the Lorentz transformation.

other intelligent being who resides in a Lorentz frame and makes measurements using its latticework of rods and clocks will be called an *observer*.

Although events are often described by their coordinates in a Lorentz reference frame, and 4-vectors by their components (coordinate differences), it should be obvious that the concepts of an event and a 4-vector need not rely on any coordinate system whatsoever for their definition. For example, the event $\mathcal{P}$ of the birth of Isaac Newton, and the event $\mathcal{Q}$ of the birth of Albert Einstein are readily identified without coordinates. They can be regarded as *points* in spacetime, and their separation vector is the straight arrow reaching through spacetime from $\mathcal{P}$ to $\mathcal{Q}$. Different observers in different inertial frames will attribute different coordinates to each birth and different components to the births' vectorial separation; but all observers can agree that they are talking about the same events $\mathcal{P}$ and $\mathcal{Q}$ in spacetime and the same separation vector $\Delta\vec{x}$. In this sense, $\mathcal{P}$, $\mathcal{Q}$, and $\Delta\vec{x}$ are *frame-independent, geometric objects* (points and arrows) that reside in spacetime.

2.2.2 The Principle of Relativity and Constancy of Light Speed

Einstein's Principle of Relativity, stated in modern form, says that *Every (special relativistic) law of physics must be expressible as a geometric, frame-independent relationship between geometric, frame-independent objects* (i.e. objects such as points in spacetime and 4-vectors and tensors, which represent physical quantities such as events and particle momenta and the electromagnetic field). This is nothing but our Geometric Principle for physical laws (Chap. 1), lifted from the Euclidean-space arena of Newtonian physics to the Minkowski-spacetime arena of Special Relativity.

Since the laws are all geometric (i.e., unrelated to any reference frame or coordinate system), there is no way that they can distinguish one inertial reference frame from any other. This leads to an alternative form of the Principle of Relativity (one commonly used in elementary textbooks and equivalent to the above): *All the (special relativistic) laws of physics are the same in every inertial reference frame, everywhere in spacetime.* This, in fact, is Einstein's own version of his Principle of Relativity; only in the half century since his death have we physicists reexpressed it in geometric language.

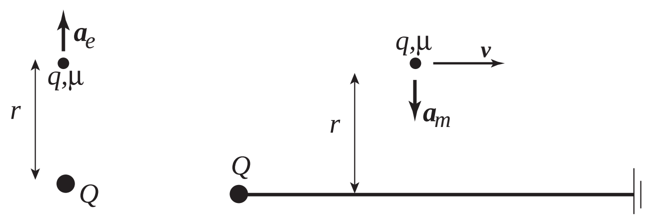
Because inertial reference frames are related to each other by Lorentz transformations (Sec. 2.7), we can restate Einstein's version of this Principle as *All the (special relativistic) laws of physics are Lorentz invariant.*

A more operational version of this Principle is the following: Give identical instructions for a specific physics experiment to two different observers in two different inertial reference frames at the same or different locations in Minkowski (i.e., gravity-free) spacetime. The experiment must be self contained, i.e., it must not involve observations of the external universe's properties (the "environment"). For example, an *unacceptable* experiment would be a measurement of the anisotropy of the Universe's cosmic microwave radiation and a computation therefrom of the observer's velocity relative to the radiation's mean rest frame; such an experiment studies the Universal environment, not the fundamental laws of physics. An *acceptable* experiment would be a measurement of the speed of light using the rods and clocks of the observer's own frame, or a measurement of cross sections for elementary particle reactions using particles accelerated in the reference frame's laboratory. The Principle of Relativity says that in these or any other similarly self-contained experiments, the two observers in their two different inertial frames must obtain identically the same experimental results—to within the accuracy of their experimental techniques. Since the experimental results are governed by the (nongravitational) laws of physics, this is equivalent to the statement that all physical laws are the same in the two inertial frames.

Perhaps the most central of special relativistic laws is the one stating that *the speed of light c in vacuum is frame-independent*, i.e., is a constant, independent of the inertial reference frame in which it is measured. In other words, there is no *aether* that supports light's vibrations and in the process influences its speed — a remarkable fact that came as a great experimental surprise to physicists at the end of the nineteenth century.

The constancy of the speed of light, in fact, is built into Maxwell's equations. In order for the Maxwell equations to be frame independent, the speed of light, which appears in them, must be frame independent. In this sense, the constancy of the speed of light follows from the Principle of Relativity; it is not an independent postulate. This is illustrated in Box 2.2.

Box 2.2
Measuring the Speed of Light Without Light



In some inertial reference frame, we perform two experiments using two particles, one with a large charge Q; the other, a test particle, with a much smaller charge q and mass μ. In the first experiment we place the two particles at rest, separated by a distance $|\Delta x| \equiv r$ and measure the electrical repulsive acceleration a_e of q (left diagram). In Gaussian cgs units (where the speed of light shows up explicitly instead of via $\epsilon_o \mu_o = 1/c^2$), the acceleration is $a_e = qQ/r^2\mu$. In the second experiment, we connect Q to ground by a long wire, and we place q at the distance $|\Delta x| = r$ from the wire and set it moving at speed v parallel to the wire. The charge Q flows down the wire with an e-folding time τ so the current is $I = dQ/d\tau = (Q/\tau)e^{-t/\tau}$. At early times $0 < t \ll \tau$, this current $I = Q/\tau$ produces a solenoidal magnetic field at q with field strength $B = (2/cr)(Q/\tau)$, and this field exerts a magnetic force on q, giving it an acceleration $a_m = q(v/c)B/\mu = 2vqQ/c^2\tau r/\mu$. The ratio of the electric acceleration in the first experiment to the magnetic acceleration in the second experiment is $a_e/a_m = c^2\tau/2rv$. Therefore, we can measure the speed of light c in our chosen inertial frame by performing this pair of experiments, carefully measuring the separation r, speed v, current Q/τ, and accelerations, and then simply computing $c = \sqrt{(2rv/\tau)(a_e/a_m)}$. The Principle of Relativity insists that the result of this pair of experiments should be independent of the inertial frame in which they are performed. Therefore, the speed of light c which appears in Maxwell's equations must be frame-independent. In this sense, the constancy of the speed of light follows from the Principle of Relativity as applied to Maxwell's equations.

What makes light so special? What about the propagation speeds of other types of waves? Are they or should they be the same as light's speed? For a digression on this topic, see Box 2.3.

Box 2.3

The Propagation Speeds of Other Waves

Electromagnetic radiation is not the only type of wave in nature. In this book, we shall encounter dispersive media, such as optical fibers and plasmas, where electromagnetic signals travel slower than c, and we shall analyze sound waves and seismic waves where the governing laws do not involve electromagnetism at all. How do these fit into our special relativistic framework? The answer is simple. Each of these waves involves an underlying medium that is at rest in one particular frame (not necessarily inertial), and the velocity at which the wave's information propagates (the group velocity) is most simply calculated in this frame *from the wave's and medium's fundamental laws*. We can then use the kinematic rules of Lorentz transformations to compute the velocity in another frame. However, if we had chosen to compute the wave speed in the second frame directly, *using the same fundamental laws*, we would have gotten the same answer, albeit perhaps with greater effort. All waves are in full compliance with the Principle of Relativity. What is special about vacuum electromagnetic waves and, by extension, photons, is that no medium (or "aether" as it used to be called) is needed for them to propagate. Their speed is therefore the same in all frames.

This raises an interesting question. What about other waves that do not require an underlying medium? What about electron de Broglie waves? Here the fundamental wave equation, Schrödinger's or Dirac's, is mathematically different from Maxwell's and contains an important parameter, the electron rest mass. This allows the fundamental laws of relativistic quantum mechanics to be written in a form that is the same in all inertial reference frames and at the same time allows an electron, considered as either a wave or a particle, to travel at a different speed when measured in a different frame.

What about non-electromagnetic waves whose quanta have vanishing rest mass? For a half century, we thought neutrinos were a good example, but we now know from experiment that their rest masses are non-zero. However, there are other particles that have not yet been detected, including photinos (the hypothesized, supersymmetric partners to photons) and gravitons (and their associated gravitational waves; Chap. 27), that are believed to exist without a rest mass (or an aether!), just like photons. Must these travel at the same speed as photons? The answer, according to the Principle of Relativity, is "yes". Why? Suppose there were two such waves or particles whose governing laws led to different speeds, c and $c' < c$, each claimed to be the same in all reference frames. Such a claim produces insurmountable conundrums. For example, if we move with speed c' in the direction of propagation of the second wave, we will bring it to rest, in conflict with our hypothesis that its speed is frame-independent. Therefore all signals, whose governing laws require them to travel with a speed that has no governing parameters (no rest mass and no underlying physical medium) must travel with a unique speed which we call "c". The speed of light is more fundamental to relativity than light itself!

The constancy of the speed of light underlies our ability to use the *geometrized units* introduced in Sec. 1.10. Any reader who has not studied that section should do so now. We shall use geometrized units throughout this chapter, and also throughout this book, when working with relativistic physics.

2.2.3 The Interval and its Invariance

We turn, next, to another fundamental concept, the *interval* $(\Delta s)^2$ between the two events $\mathcal{P}$ and $\mathcal{Q}$ whose separation vector is $\Delta\vec{x}$. In a specific but arbitrary inertial reference frame and in geometrized units, $(\Delta s)^2$ is given by

$$\boxed{(\Delta s)^2 \equiv -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = -(\Delta t)^2 + \sum_{i,j} \delta_{ij} \Delta x^i \Delta x^j \;;} \tag{2.2a}$$

cf. Eq. (2.1). If $(\Delta s)^2 > 0$, the events $\mathcal{P}$ and $\mathcal{Q}$ are said to have a *spacelike* separation; if $(\Delta s)^2 = 0$, their separation is *null* or *lightlike*; and if $(\Delta s)^2 < 0$, their separation is *timelike*. For timelike separations, $(\Delta s)^2 < 0$ implies that Δs is imaginary; to avoid dealing with imaginary numbers, we describe timelike intervals by

$$\boxed{(\Delta\tau)^2 \equiv -(\Delta s)^2 \,,} \tag{2.2b}$$

whose square root $\Delta\tau$ is real.

The coordinate separation between $\mathcal{P}$ and $\mathcal{Q}$ depends on one's reference frame; i.e., if $\Delta x^{\alpha'}$ and Δx^α are the coordinate separations in two different frames, then $\Delta x^{\alpha'} \neq \Delta x^\alpha$. Despite this frame dependence, the Principle of Relativity forces the interval $(\Delta s)^2$ to be the same in all frames:

$$\begin{aligned} (\Delta s)^2 &= -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= -(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \end{aligned} \tag{2.3}$$

In Box 2.4, we sketch a proof for the case of two events $\mathcal{P}$ and $\mathcal{Q}$ whose separation is timelike.

Because of its frame invariance, the interval $(\Delta s)^2$ can be regarded as a geometric property of the vector $\Delta\vec{x}$ that reaches from $\mathcal{P}$ to $\mathcal{Q}$; we shall call it the *squared length* $(\Delta\vec{x})^2$ of $\Delta\vec{x}$:

$$(\Delta\vec{x})^2 \equiv (\Delta s)^2 \,. \tag{2.4}$$

Note that this squared length, despite its name, can be negative (for timelike $\Delta\vec{x}$) or zero (for null $\Delta\vec{x}$) as well as positive (for spacelike $\Delta\vec{x}$).

The invariant interval $(\Delta s)^2$ between two events is as fundamental to Minkowski spacetime as the Euclidean distance between two points is to flat 3-space. Just as the Euclidean distance gives rise to the geometry of 3-space, as embodied, e.g., in Euclid's axioms, so the interval gives rise to the geometry of spacetime, which we shall be exploring. If this spacetime geometry were as intuitively obvious to humans as is Euclidean geometry, we would not need the crutch of inertial reference frames to arrive at it. Nature (presumably) has no need for such a crutch. To Nature (it seems evident), the geometry of Minkowski spacetime, as embodied in the invariant interval, is among the most fundamental aspects of physical law.

Box 2.4

Proof of Invariance of the Interval for a Timelike Separation

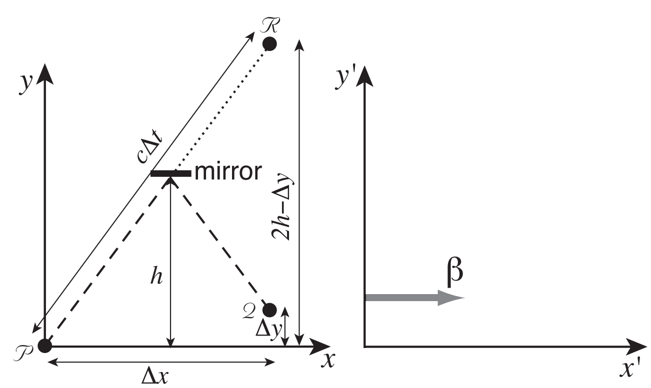
Let two reference frames, primed and unprimed, move with respect to each other. Choose the spatial coordinate systems of the two frames in such a way that (i) their relative motion (with speed β that will not enter into our analysis) is along the x direction and the x' direction, (ii) event $\mathcal{P}$ lies on the x and x' axes, and (iii) event $\mathcal{Q}$ lies in the x-y plane and in the x'-y' plane, as depicted below. Then evaluate the interval between $\mathcal{P}$ and $\mathcal{Q}$ in the unprimed frame by the following construction: Place a mirror parallel to the x-z plane at precisely the height h that permits a photon, emitted from $\mathcal{P}$, to travel along the dashed line to the mirror, then reflect off the mirror and continue along the dashed path, arriving at event $\mathcal{Q}$. If the mirror were placed lower, the photon would arrive at the spatial location of $\mathcal{Q}$ sooner than the time of $\mathcal{Q}$; if placed higher, it would arrive later. Then the distance the photon travels (the length of the two-segment dashed line) is equal to $c\Delta t = \Delta t$, where Δt is the time between events $\mathcal{P}$ and $\mathcal{Q}$ as measured in the unprimed frame. If the mirror had not been present, the photon would have arrived at event $\mathcal{R}$ after time Δt, so $c\Delta t$ is the distance between $\mathcal{P}$ and $\mathcal{R}$. From the diagram, it is easy to see that the height of $\mathcal{R}$ above the x axis is $2h - \Delta y$, and the Pythagorean theorem then implies that

$$(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 = -(2h - \Delta y)^2 + (\Delta y)^2 \, . \tag{1a}$$

The same construction in the primed frame must give the same formula, but with primes

$$(\Delta s')^2 = -(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 = -(2h' - \Delta y')^2 + (\Delta y')^2 \, . \tag{1b}$$

The proof that $(\Delta s')^2 = (\Delta s)^2$ then reduces to showing that the Principle of Relativity requires that distances perpendicular to the direction of relative motion of two frames be the same as measured in the two frames, $h' = h$, $\Delta y' = \Delta y$. We leave it to the reader to develop a careful argument for this [Ex. 2.2].



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EXERCISES

Exercise 2.1 *Practice: Geometrized Units*
Do exercise 1.15 in Chap. 1.

Exercise 2.2 *Derivation and Example: Invariance of the Interval*
Complete the derivation of the invariance of the interval given in the Box 2.4, using the Principle of Relativity in the form that the laws of physics must be the same in the primed and unprimed frames. Hints, if you need them:

(a) Having carried out the construction in the unprimed frame, depicted at the bottom left of Box 2.4, use the same mirror and photons for the analogous construction in the primed frame. Argue that, independently of the frame in which the mirror is at rest (unprimed or primed), the fact that the reflected photon has (angle of reflection) = (angle of incidence) in its rest frame implies that this is also true for this same photon in the other frame. Thereby conclude that the construction leads to Eq. (1b) of Box 2.4, as well as to (1a).

(b) Then argue that the perpendicular distance of an event from the common x and x' axis must be the same in the two reference frames, so $h' = h$ and $\Delta y' = \Delta y$; whence Eqs. (1b) and (1a) imply the invariance of the interval. [For a leisurely version of this argument, see Secs. 3.6 and 3.7 of Taylor and Wheeler (1992).]

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2.3 Tensor Algebra Without a Coordinate System

Having introduced points in spacetime (interpreted physically as events), the invariant interval $(\Delta s)^2$ between two events, 4-vectors (as arrows between two events), and the squared length of a vector (as the invariant interval between the vector's tail and tip), we can now introduce the remaining tools of tensor algebra for Minkowski spacetime in *precisely* the same way as we did for the Euclidean 3-space of Newtonian physics (Sec. 1.3), with the invariant interval between events playing the same role as the Euclidean squared length between Euclidean points. In particular:

A tensor $\boldsymbol{T}(\_,\_,\_)$ is a real-valued linear function of vectors in Minkowski spacetime. (We use slanted letters $\boldsymbol{T}$ for tensors in spacetime and unslanted letters $\mathbf{T}$ in Euclidean space.) A tensor's rank is equal to its number of slots. The inner product of two 4-vectors is

$$\vec{A} \cdot \vec{B} \equiv \frac{1}{4}\left[(\vec{A}+\vec{B})^2 - (\vec{A}-\vec{B})^2\right] , \qquad (2.5)$$

where $(\vec{A}+\vec{B})^2$ is the squared length of this vector, i.e. the invariant interval between its tail and its tip. The metric tensor of spacetime is that linear function of 4-vectors whose value is the inner product of the vectors

$$\boldsymbol{g}(\vec{A},\vec{B}) \equiv \vec{A}\cdot\vec{B}\ . \tag{2.6}$$

Using the inner product, we can regard any vector **A** as a rank-1 tensor: $\vec{A}(\vec{C}) \equiv \vec{A}\cdot\vec{C}$.

Similarly, the *tensor product* $\otimes$ is defined precisely as in the Euclidean domain, Eqs. (1.5), as is the *contraction* of two slots of a tensor against each other, Eqs. (1.6), which lowers the tensor's rank by two.

2.4 Particle Kinetics and Lorentz Force Without a Reference Frame

2.4.1 Relativistic Particle Kinetics: World Lines, 4-Velocity, 4-Momentum and its Conservation, 4-Force

In this section, we shall illustrate our geometric viewpoint by formulating the special relativistic laws of motion for particles.

An accelerated particle moving through spacetime carries an *ideal clock*. By "ideal" we mean that the clock is unaffected by accelerations: it ticks at a uniform rate when compared to unaccelerated atomic oscillators, which are momentarily at rest beside the clock and are well protected from their environments. The builders of inertial guidance systems for airplanes and missiles try to make their clocks as ideal as possible, in just this sense. We denote by τ the time ticked by the particle's ideal clock, and we call it the particle's *proper time*.

The particle moves through spacetime along a curve, called its *world line*, which we can denote equally well by $\mathcal{P}(\tau)$ (the particle's spacetime location $\mathcal{P}$ at proper time τ), or by $\vec{x}(\tau)$ (the particle's vector separation from some arbitrarily chosen origin at proper time τ).

We shall refer to the inertial frame in which the particle is momentarily at rest as its *momentarily comoving inertial frame* or *momentary rest frame*. Now, the particle's clock (which measures τ) is ideal and so are the inertial frame's clocks (which measure coordinate time t). Therefore, a tiny interval $\Delta\tau$ of the particle's proper time is equal to the lapse of coordinate time in the particle's momentary rest frame, $\Delta\tau = \Delta t$. Moreover, since the two events $\vec{x}(\tau)$ and $\vec{x}(\tau+\Delta\tau)$ on the clock's world line occur at the same spatial location in its momentary rest frame, $\Delta x^i = 0$ (where $i = 1,\ 2,\ 3$), to first order in $\Delta\tau$, the invariant interval between those events is $(\Delta s)^2 = -(\Delta t)^2 + \sum_{i,j}\Delta x^i\Delta x^j\delta_{ij} = -(\Delta t)^2 = -(\Delta\tau)^2$. This shows that *the particle's proper time τ is equal to the square root of the negative of the invariant interval, $\tau = \sqrt{-s^2}$, along its world line.*

Figure 2.3 shows the world line of the accelerated particle in a spacetime diagram where the axes are coordinates of an *arbitrary* Lorentz frame. This diagram is intended to emphasize the world line as a frame-independent, geometric object. Also shown in the figure is the particle's 4-*velocity* $\vec{u}$, which (by analogy with velocity in 3-space) is the time derivative of

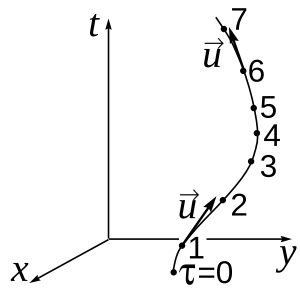

Fig. 2.3: Spacetime diagram showing the world line $\vec{x}(\tau)$ and 4-velocity $\vec{u}$ of an accelerated particle. Note that the 4-velocity is tangent to the world line.

its position:

$$\boxed{\vec{u} \equiv d\mathcal{P}/d\tau = d\vec{x}/d\tau\ .} \tag{2.7}$$

This derivative is defined by the usual limiting process

$$\frac{d\mathcal{P}}{d\tau} = \frac{d\vec{x}}{d\tau} \equiv \lim_{\Delta\tau\to 0} \frac{\mathcal{P}(\tau+\Delta\tau) - \mathcal{P}(\tau)}{\Delta\tau} = \lim_{\Delta\tau\to 0} \frac{\vec{x}(\tau+\Delta\tau) - \vec{x}(\tau)}{\Delta\tau}\ . \tag{2.8}$$

Here $\mathcal{P}(\tau+\Delta\tau) - \mathcal{P}(\tau)$ and $\vec{x}(\tau+\Delta\tau) - \vec{x}(\tau)$ are just two different ways to denote the same vector that reaches from one point on the world line to another.

The squared length of the particle's 4-velocity is easily seen to be -1:

$$\vec{u}^2 \equiv \boldsymbol{g}(\vec{u}, \vec{u}) = \frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} = \frac{d\vec{x} \cdot d\vec{x}}{(d\tau)^2} = -1\ . \tag{2.9}$$

The last equality follows from the fact that $d\vec{x} \cdot d\vec{x}$ is the squared length of $d\vec{x}$ which equals the invariant interval $(\Delta s)^2$ along it, and $(d\tau)^2$ is minus that invariant interval.

The particle's 4-momentum is the product of its 4-velocity and rest mass

$$\boxed{\vec{p} \equiv m\vec{u} = m d\vec{x}/d\tau \equiv d\vec{x}/d\zeta\ .} \tag{2.10}$$

Here the parameter ζ is a renormalized version of proper time,

$$\zeta \equiv \tau/m\ . \tag{2.11}$$

This ζ, and any other renormalized version of proper time with position-independent renormalization factor, are called *affine parameters* for the particle's world line. Expression (2.10), together with $\vec{u}^2 = -1$, implies that the squared length of the 4-momentum is

$$\boxed{\vec{p}^2 = -m^2}\ . \tag{2.12}$$

In quantum theory a particle is described by a relativistic wave function which, in the geometric optics limit (Chap. 7), has a wave vector $\vec{k}$ that is related to the classical particle's 4-momentum by

$$\vec{k} = \vec{p}/\hbar\ . \tag{2.13}$$

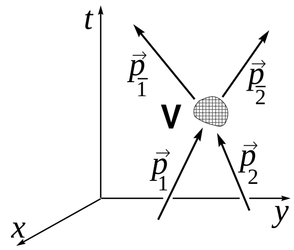

Fig. 2.4: Spacetime diagram depicting the law of 4-momentum conservation for a situation where two particles, numbered 1 and 2, enter an interaction region $\mathcal{V}$ in spacetime, there interact strongly, and produce two new particles, numbered $\bar{1}$ and $\bar{2}$. The sum of the final 4-momenta, $\vec{p}_{\bar{1}} + \vec{p}_{\bar{2}}$, must be equal to the sum of the initial 4-momenta, $\vec{p}_1 + \vec{p}_2$.

The above formalism is valid only for particles with nonzero rest mass, $m \neq 0$. The corresponding formalism for a *particle with zero rest mass* (e.g. a photon or a graviton) can be obtained from the above by taking the limit as $m \to 0$ and $d\tau \to 0$ with the quotient $d\zeta = d\tau/m$ held finite. More specifically, the 4-momentum of a zero-rest-mass particle is well defined (and participates in the conservation law to be discussed below), and it is expressible in terms of the particle's affine parameter ζ by Eq. (2.10)

$$\vec{p} = \frac{d\vec{x}}{d\zeta} \ . \tag{2.14}$$

The particle's 4-velocity $\vec{u} = \vec{p}/m$, by contrast, is infinite and thus undefined; and proper time $\tau = m\zeta$ ticks vanishingly slowly along its world line and thus is undefined. Because proper time is the square root of the invariant interval along the world line, the interval between two neighboring points on the world line vanishes. Therefore, *the world line of a zero-rest-mass particle is null*. (By contrast, since $d\tau^2 > 0$ and $ds^2 < 0$ along the world line of a particle with finite rest mass, *the world line of a finite-rest-mass particle is timelike*.)

The 4-momenta of particles are important because of the *law of conservation of* 4-*momentum* (which, as we shall see in Sec. 2.6, is equivalent to the conservation laws for energy and ordinary momentum): If a number of "initial" particles, named $A = 1, 2, 3, \ldots$ enter a restricted region of spacetime $\mathcal{V}$ and there interact strongly to produce a new set of "final" particles, named $\bar{A} = \bar{1}, \bar{2}, \bar{3}, \ldots$ (Fig. 2.4), then the total 4-momentum of the final particles must be the same as the total 4-momentum of the initial ones:

$$\boxed{\sum_{\bar{A}} \vec{p}_{\bar{A}} = \sum_{A} \ \vec{p}_A \ .} \tag{2.15}$$

Note that this law of 4-momentum conservation is expressed in frame-independent, geometric language—in accord with Einstein's insistence that all the laws of physics should be so expressible. As we shall see in Part VII, 4-momentum conservation is a consequence of the translation symmetry of flat, 4-dimensional spacetime. In general relativity's curved spacetime, where that translation symmetry is lost, we lose 4-momentum conservation except under special circumstances; see Sec. 25.9.4.

If a particle moves freely (no external forces and no collisions with other particles), then its 4-momentum $\vec{p}$ will be conserved along its world line, $d\vec{p}/d\zeta = 0$. Since $\vec{p}$ is tangent to the world line, this means that the direction of the world line in spacetime never changes; i.e., the free particle moves along a straight line through spacetime. To change the particle's 4-momentum, one must act on it with a 4-force $\vec{F}$,

$$d\vec{p}/d\tau = \vec{F} \; . \tag{2.16}$$

If the particle is a fundamental one (e.g., photon, electron, proton), then the 4-force must leave its rest mass unchanged,

$$0 = dm^2/d\tau = -d\vec{p}^2/d\tau = -2\vec{p}\cdot d\vec{p}/d\tau = -2\vec{p}\cdot\vec{F} \; ; \tag{2.17}$$

i.e., the 4-force must be orthogonal to the 4-momentum.

2.4.2 Geometric Derivation of the Lorentz Force Law

As an illustration of these physical concepts and mathematical tools, we shall use them to deduce the relativistic version of the Lorentz force law. From the outset, in accord with the Principle of Relativity, we insist that the law we seek be expressible in geometric, frame-independent language, i.e. in terms of vectors and tensors.

Consider a particle with charge q and rest mass $m \neq 0$, interacting with an electromagnetic field. It experiences an electromagnetic 4-force whose mathematical form we seek. The Newtonian version of the electromagnetic force $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ is proportional to q and contains one piece (electric) that is independent of velocity $\mathbf{v}$, and a second piece (magnetic) that is linear in $\mathbf{v}$. It is reasonable to expect that, in order to produce this Newtonian limit, the relativistic 4-force $\vec{F}$ will be proportional to q and will be linear in the 4-velocity $\vec{u}$. Linearity means there must exist some second-rank tensor $\boldsymbol{F}(\_\,,\_)$, the *electromagnetic field tensor,* such that

$$d\vec{p}/d\tau = \vec{F}(\_) = q\boldsymbol{F}(\_\,,\vec{u}) \; . \tag{2.18}$$

Because the 4-force $\vec{F}$ must be orthogonal to the particle's 4-momentum and thence also to its 4-velocity, $\vec{F}\cdot\vec{u} \equiv \vec{F}(\vec{u}) = 0$, expression (2.18) must vanish when $\vec{u}$ is inserted into its empty slot. In other words, for all timelike unit-length vectors $\vec{u}$,

$$\boldsymbol{F}(\vec{u},\vec{u}) = 0 \; . \tag{2.19}$$

It is an instructive exercise (Ex. 2.3) to show that this is possible only if $\boldsymbol{F}$ is *antisymmetric,* so the electromagnetic 4-force is

$$\boxed{d\vec{p}/d\tau = q\boldsymbol{F}(\_\,,\vec{u}) \; , \quad \text{where } \boldsymbol{F}(\vec{A},\vec{B}) = -\boldsymbol{F}(\vec{B},\vec{A}) \text{ for all } \vec{A} \text{ and } \vec{B} \; .} \tag{2.20}$$

This must be the relativistic form of the Lorentz force law. In Sec. 2.11 below, we shall deduce the relationship of the electromagnetic field tensor $\boldsymbol{F}$ to the more familiar electric and magnetic fields, and the relationship of this relativistic Lorentz force to its Newtonian form (1.7c).

This discussion of particle kinematics and the electromagnetic force is elegant, but perhaps unfamiliar. In Secs. 2.6 and 2.11 we shall see that it is equivalent to the more elementary (but more complex) formalism based on components of vectors in Euclidean 3-space.

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EXERCISES

Exercise 2.3 *Derivation and Example: Antisymmetry of Electromagnetic Field Tensor*
Show that Eq. (2.19) can be true for all timelike, unit-length vectors $\vec{u}$ if and only if $\boldsymbol{F}$ is antisymmetric. [Hints: (i) Show that the most general second-rank tensor $\boldsymbol{F}$ can be written as the sum of a symmetric tensor $\boldsymbol{S}$ and an antisymmetric tensor $\boldsymbol{A}$, and that the antisymmetric piece contributes nothing to Eq. (2.19). (ii) Let $\vec{B}$ and $\vec{C}$ be any two vectors such that $\vec{B}+\vec{C}$ and $\vec{B}-\vec{C}$ are both timelike; show that $\boldsymbol{S}(\vec{B},\vec{C})=0$. (iii) Convince yourself (if necessary using the component tools developed in the next section) that this result, together with the 4-dimensionality of spacetime and the large arbitrariness inherent in the choice of $\vec{A}$ and $\vec{B}$, implies $\boldsymbol{S}$ vanishes (i.e., it gives zero when *any* two vectors are inserted into its slots).]

Exercise 2.4 *Problem: Relativistic Gravitational Force Law*
In Newtonian theory the gravitational potential Φ exerts a force $\mathbf{F} = d\mathbf{p}/dt = -m\boldsymbol{\nabla}\Phi$ on a particle with mass m and momentum $\mathbf{p}$. Before Einstein formulated general relativity, some physicists constructed relativistic theories of gravity in which a Newtonian-like scalar gravitational field Φ exerted a 4-force $\vec{F} = d\vec{p}/d\tau$ on any particle with rest mass m, 4-velocity $\vec{u}$ and 4-momentum $\vec{p} = m\vec{u}$. What must that force law have been, in order to (i) obey the Principle of Relativity, (ii) reduce to Newton's law in the non-relativistic limit, and (iii) preserve the particle's rest mass as time passes?

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2.5 Component Representation of Tensor Algebra

In Minkowski spacetime, associated with any *inertial reference frame* (Fig. 2.1 and Sec. 2.2.1), there is a *Lorentz coordinate system* $\{t,x,y,z\} = \{x^0,x^1,x^2,x^3\}$ generated by the frame's rods and clocks. And associated with these coordinates there is a set of *basis vectors* $\{\vec{e}_t,\vec{e}_x,\vec{e}_y,\vec{e}_z\} = \{\vec{e}_0,\vec{e}_1,\vec{e}_2,\vec{e}_3\}$. (The reason for putting the indices up on the coordinates but down on the basis vectors will become clear below.) The basis vector $\vec{e}_\alpha$ points along the x^α coordinate direction, which is orthogonal to all the other coordinate directions, and it has squared length -1 for $\alpha=0$ (vector pointing in a timelike direction) and $+1$ for $\alpha=1,2,3$ (spacelike):

$$\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta} \ . \tag{2.21}$$

Here $\eta_{\alpha\beta}$ (a spacetime analog of the Kronecker delta) are defined by

$$\eta_{00} \equiv -1 \ , \quad \eta_{11} \equiv \eta_{22} \equiv \eta_{33} \equiv 1 \ , \quad \eta_{\alpha\beta} \equiv 0 \text{ if } \alpha \neq \beta \ . \tag{2.22}$$

Any basis in which $\vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$ is said to be *orthonormal* (by analogy with the Euclidean notion of orthonormality, $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$).

The fact that $\vec{e}_\alpha \cdot \vec{e}_\beta \neq \delta_{\alpha\beta}$ prevents many of the Euclidean-space component-manipulation formulas (1.9c)–(1.9h) from holding true in Minkowski spacetime. There are two approaches to recovering these formulas. One approach, used in many old textbooks (including the first and second editions of Goldstein's *Classical Mechanics* and Jackson's *Classical Electrodynamics*), is to set $x^0 = it$, where $i = \sqrt{-1}$ and correspondingly make the time basis vector be imaginary, so that $\vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta}$. When this approach is adopted, the resulting formalism does not care whether indices are placed up or down; one can place them wherever one's stomach or liver dictate without asking one's brain. However, this $x^0 = it$ approach has severe disadvantages: (i) it hides the true physical geometry of Minkowski spacetime, (ii) it cannot be extended in any reasonable manner to non-orthonormal bases in flat spacetime, and (iii) it cannot be extended in any reasonable manner to the curvilinear coordinates that one must use in general relativity. For these reasons, most modern texts (including the third editions of Goldstein and Jackson) take an alternative approach, one always used in general relativity. This alternative, which we shall adopt, requires introducing two different types of components for vectors, and analogously for tensors: *contravariant components* denoted by superscripts, e.g. $T^{\alpha\beta\gamma}$, and *covariant components* denoted by subscripts, e.g. $T_{\alpha\beta\gamma}$. In Parts I–VI of this book we introduce these components only for orthonormal bases; in Part VII we develop a more sophisticated version of them, valid for nonorthonormal bases.

A vector or tensor's *contravariant components* are defined as its expansion coefficients in the chosen basis [analog of Eq. (1.9d) in Euclidean 3-space]:

$$\vec{A} \equiv A^\alpha \vec{e}_\alpha \ , \quad \boldsymbol{T} \equiv T^{\alpha\beta\gamma} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma \ . \tag{2.23a}$$

Here and throughout this book, *Greek (spacetime) indices are to be summed whenever they are repeated with one up and the other down.* The *covariant components* are defined as the numbers produced by evaluating the vector or tensor on its basis vectors [analog of Eq. (1.9e) in Euclidean 3-space]:

$$A_\alpha \equiv \vec{A}(\vec{e}_\alpha) = \vec{A} \cdot \vec{e}_\alpha \ , \quad T_{\alpha\beta\gamma} \equiv \boldsymbol{T}(\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_\gamma) \ . \tag{2.23b}$$

These definitions have a number of important consequences. We shall derive them one after another and then at the end shall summarize them succinctly with equation numbers:

(i) The covariant components of the metric tensor are $g_{\alpha\beta} = \boldsymbol{g}(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha\beta}$. Here the first equality is the definition (2.23b) of the covariant components and the second equality is the orthonormality relation (2.21) for the basis vectors.

(ii) The covariant components of any tensor can be computed from the contravariant components by $T_{\lambda\mu\nu} = \boldsymbol{T}(\vec{e}_\lambda, \vec{e}_\mu, \vec{e}_\nu) = T^{\alpha\beta\gamma} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \vec{e}_\gamma (\vec{e}_\lambda, \vec{e}_\mu, \vec{e}_\nu) = T^{\alpha\beta\gamma} (\vec{e}_\alpha \cdot \vec{e}_\lambda)(\vec{e}_\beta \cdot \vec{e}_\mu)(\vec{e}_\gamma \cdot \vec{e}_\nu) = T^{\alpha\beta\gamma} g_{\alpha\lambda} g_{\beta\mu} g_{\gamma\nu}$. The first equality is the definition (2.23b) of the covariant components, the second is the expansion (2.23a) of $\boldsymbol{T}$ on the chosen basis, the third is the definition (1.5a) of the tensor product, and the fourth is one version of our result (i) for the covariant components of the metric.

(iii) This result, $T_{\lambda\mu\nu} = T^{\alpha\beta\gamma} g_{\alpha\lambda} g_{\beta\mu} g_{\gamma\nu}$, together with the numerical values (i) of $g_{\alpha\beta}$, implies that when one lowers a spatial index there is no change in the numerical value of a component, and when one lowers a temporal index, the sign changes: $T_{ijk} = T^{ijk}$, $T_{0jk} = -T^{0jk}$, $T_{0j0} = +T^{0j0}$, $T_{000} = -T^{000}$. We shall call this the "sign-flip-if-temporal" rule. As a special case, $-1 = g_{00} = g^{00}$, $0 = g_{0j} = -g^{0j}$, $\delta_{jk} = g_{jk} = g^{jk}$ — i.e., the metric's covariant and contravariant components are numerically identical; they are both equal to the orthonormality values $\eta_{\alpha\beta}$.

(iv) It is easy to see that this sign-flip-if-temporal rule for lowering indices implies the same sign-flip-if-temporal rule for raising them, which in turn can be written in terms of metric components as $T^{\alpha\beta\gamma} = T_{\lambda\mu\nu} g^{\lambda\alpha} g^{\mu\beta} g^{\nu\gamma}$.

(v) It is convenient to define *mixed components* of a tensor, components with some indices up and others down, as having numerical values obtained by raising or lowering some but not all of its indices using the metric, e.g. $T^{\alpha}{}_{\mu\nu} = T^{\alpha\beta\gamma} g_{\beta\mu} g_{\gamma\nu} = T_{\lambda\mu\nu} g^{\lambda\alpha}$. Numerically, this continues to follow the sign-flip-if-temporal rule: $T^{0}{}_{0k} = -T^{00k}$, $T^{0}{}_{jk} = T^{0jk}$, and it implies, in particular, that the mixed components of the metric are $g^{\alpha}{}_{\beta} = \delta_{\alpha\beta}$ (the Kronecker-delta values; +1 if $\alpha = \beta$ and zero otherwise).

Summarizing these results: *The numerical values of the components of the metric in Minkowski spacetime are*

$$g_{\alpha\beta} = \eta_{\alpha\beta}\ , \quad g^{\alpha}{}_{\beta} = \delta_{\alpha\beta}\ , \quad g_{\alpha}{}^{\beta} = \delta_{\alpha\beta}\ , \quad g^{\alpha\beta} = \eta_{\alpha\beta}\ ; \tag{2.23c}$$

and *indices on all vectors and tensors can be raised and lowered using these components of the metric*

$$A_{\alpha} = g_{\alpha\beta} A^{\beta}\ , \quad A^{\alpha} = g^{\alpha\beta} A_{\beta}\ , \quad T^{\alpha}{}_{\mu\nu} \equiv g_{\mu\beta} g_{\nu\gamma} T^{\alpha\beta\gamma} \quad T^{\alpha\beta\gamma} \equiv g^{\beta\mu} g^{\gamma\nu} T^{\alpha}{}_{\mu\nu}\ , \tag{2.23d}$$

which is equivalent to the sign-flip-if-temporal rule.

This index notation gives rise to formulas for tensor products, inner products, values of tensors on vectors, and tensor contractions, that are the obvious analogs of those in Euclidean space:

$$[\text{Contravariant components of } \boldsymbol{T}(\_,\_,\_) \otimes \boldsymbol{S}(\_,\_)] = T^{\alpha\beta\gamma} S^{\delta\epsilon}\ , \tag{2.23e}$$

$$\vec{A} \cdot \vec{B} = A^{\alpha} B_{\alpha} = A_{\alpha} B^{\alpha}\ , \quad \boldsymbol{T}(\mathbf{A}, \mathbf{B}, \mathbf{C}) = T_{\alpha\beta\gamma} A^{\alpha} B^{\beta} C^{\gamma} = T^{\alpha\beta\gamma} A_{\alpha} B_{\beta} C_{\gamma}\ , \tag{2.23f}$$

$$\begin{aligned} \text{Covariant components of } [\text{1\&3contraction of } \boldsymbol{R}] &= R^{\mu}{}_{\alpha\mu\beta}\ , \\ \text{Contravariant components of } [\text{1\&3contraction of } \boldsymbol{R}] &= R^{\mu\alpha}{}_{\mu}{}^{\beta}\ . \end{aligned} \tag{2.23g}$$

Notice the very simple pattern in Eqs. (2.23), which universally permeates the rules of index gymnastics, a pattern that permits one to reconstruct the rules without any memorization: *Free indices (indices not summed over) must agree in position (up versus down) on the two sides of each equation.* In keeping with this pattern, one can regard the two indices in a pair that is summed as "destroying each other by contraction", and one speaks of "lining up the indices" on the two sides of an equation to get them to agree.

In Part VII, when we use non-orthonormal bases, all of these index-notation equations (2.23) will remain valid unchanged except for the numerical values (2.23c) of the metric components and the sign-flip-if-temporal rule.

In Minkowski spacetime, as in Euclidean space, we can (and often we shall) use slot-naming index notation to represent frame-independent geometric objects and equations and physical laws. (Readers who have not studied Sec. 1.5.1 on slot-naming index notation should do so now.)

For example, we shall often write the frame-independent Lorentz force law $d\vec{p}/d\tau = q\boldsymbol{F}(\_, \vec{u})$ as $dp_\mu/d\tau = qF_{\mu\nu}u^\nu$.

Notice that, because the components of the metric in any Lorentz basis are $g_{\alpha\beta} = \eta_{\alpha\beta}$, we can write the invariant interval between two events x^α and $x^\alpha + dx^\alpha$ as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{2.24}$$

This is called the special relativistic *line element.*

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EXERCISES

Exercise 2.5 *Derivation: Component Manipulation Rules*
Derive the relativistic component manipulation rules (2.23e)–(2.23g).

Exercise 2.6 *Numerics of Component Manipulations*
In some inertial reference frame, the vector $\vec{A}$ and second-rank tensor $\boldsymbol{T}$ have as their only nonzero components $A^0 = 1$, $A^1 = 2$, $A^2 = A^3 = 0$; $T^{00} = 3$, $T^{01} = T^{10} = 2$, $T^{11} = -1$. Evaluate $\boldsymbol{T}(\vec{A}, \vec{A})$ and the components of $\boldsymbol{T}(\vec{A}, \_)$ and $\vec{A} \otimes \boldsymbol{T}$.

Exercise 2.7 *Practice: Meaning of Slot-Naming Index Notation*

(a) Convert the following expressions and equations into geometric, index-free notation: $A^\alpha B_{\gamma\delta}$; $A_\alpha B_\gamma{}^\delta$; $S_\alpha{}^{\beta\gamma} = S^{\gamma\beta}{}_\alpha$; $A^\alpha B_\alpha = A_\alpha B^\beta g^\alpha{}_\beta$.

(d) Convert $\boldsymbol{T}(\_, \boldsymbol{S}(\boldsymbol{R}(\vec{C}, \_), \_), \_)$ into slot-naming index notation.

Exercise 2.8 *Practice: Index Gymnastics*

(a) Simplify the following expression so the metric does not appear in it: $A^{\alpha\beta\gamma} g_{\beta\rho} S_{\gamma\lambda} g^{\rho\delta} g^\lambda{}_\alpha$.

(b) The quantity $g_{\alpha\beta} g^{\alpha\beta}$ is a scalar since it has no free indices. What is its numerical value?

(c) What is wrong with the following expression and equation? $A_\alpha{}^{\beta\gamma} S_{\alpha\gamma}$; $A_\alpha{}^{\beta\gamma} S_\beta T_\gamma = R_{\alpha\beta\delta} S^\beta$.

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2.6 Particle Kinetics in Index Notation and in a Lorentz Frame

As an illustration of the component representation of tensor algebra, let us return to the relativistic, accelerated particle of Fig. 2.3 and, from the frame-independent equations for the particle's 4-velocity $\vec{u}$ and 4-momentum $\vec{p}$ (Sec. 2.4), derive the component description given in elementary textbooks.

We introduce a specific inertial reference frame and associated Lorentz coordinates x^α and basis vectors $\{\vec{e}_\alpha\}$. In this Lorentz frame, the particle's world line $\vec{x}(\tau)$ is represented by its coordinate location $x^\alpha(\tau)$ as a function of its proper time τ. The contravariant components of the separation vector $d\vec{x}$ between two neighboring events along the particle's world line are the events' coordinate separations dx^α [Eq. (2.1)]; and correspondingly, the components of the particle's 4-velocity $\vec{u} = d\vec{x}/d\tau$ are

$$u^\alpha = dx^\alpha/d\tau \tag{2.25a}$$

(the time derivatives of the particle's spacetime coordinates). Note that Eq. (2.25a) implies

$$v^j \equiv \frac{dx^j}{dt} = \frac{dx^j/d\tau}{dt/d\tau} = \frac{u^j}{u^0} . \tag{2.25b}$$

This relation, together with $-1 = \vec{u}^2 = g_{\alpha\beta}u^\alpha u^\beta = -(u^0)^2 + \delta_{ij}u^i u^j = -(u^0)^2(1 - \delta_{ij}v^i v^j)$, implies that the components of the 4-velocity have the forms familiar from elementary textbooks:

$$u^0 = \gamma \ , \quad u^j = \gamma v^j \ , \quad \text{where} \quad \gamma = \frac{1}{(1 - \delta_{ij}v^i v^j)^{\frac{1}{2}}} . \tag{2.25c}$$

It is useful to think of v^j as the components of a 3-dimensional vector $\mathbf{v}$, the *ordinary velocity*, that lives in the 3-dimensional Euclidean space $t =$ const of the chosen Lorentz frame (the stippled space in Fig. 2.5). This 3-space is sometimes called the frame's *slice of simultaneity* or *3-space of simultaneity* because all events lying in it are simultaneous, as measured by the frame's observers. This 3-space is not well defined until a Lorentz frame has been chosen, and correspondingly, $\mathbf{v}$ relies for its existence on a specific choice of frame. However, once the frame has been chosen, $\mathbf{v}$ can be regarded as a coordinate-independent, basis-independent 3-vector lying in the frame's slice of simultaneity. Similarly, the spatial part of the 4-velocity $\vec{u}$ (the part with components u^j in our chosen frame) can be regarded as a 3-vector $\mathbf{u}$ lying in the frame's 3-space; and Eqs. (2.25c) become the component versions of the coordinate-independent, basis-independent 3-space relations

$$\mathbf{u} = \gamma\mathbf{v} \ , \quad \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2}} . \tag{2.25d}$$

The components of the particle's 4-momentum $\vec{p}$ in our chosen Lorentz frame have special names and special physical significances: The time component of the 4-momentum is the particle's (relativistic) *energy* $\mathcal{E}$ as measured in that frame

$$\begin{aligned} \mathcal{E} \equiv p^0 = mu^0 = m\gamma = \frac{m}{\sqrt{1 - \mathbf{v}^2}} &= \text{ (the particle's energy)} \\ \simeq m + \frac{1}{2}m\mathbf{v}^2 \quad &\text{for } |\mathbf{v}| \ll 1 . \end{aligned} \tag{2.26a}$$

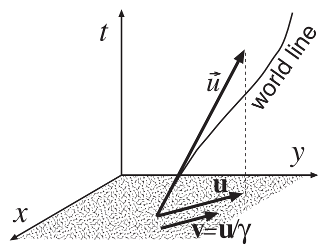

Fig. 2.5: Spacetime diagram in a specific Lorentz frame, showing the frame's 3-space $t = 0$ (stippled region), the world line of a particle, the 4-velocity $\vec{u}$ of the particle as it passes through the 3-space; and two 3-dimensional vectors that lie in the 3-space: the spatial part of the particle's 4-velocity, $\mathbf{u}$, and the particle's ordinary velocity $\mathbf{v}$.

Note that this energy is the sum of the particle's *rest mass-energy* $m = mc^2$ and its *kinetic energy*

$$\begin{aligned} E &\equiv \mathcal{E} - m = m\left(\frac{1}{1-\mathbf{v}^2} - 1\right) \\ &\simeq \frac{1}{2}m\mathbf{v}^2 \quad \text{for } |\mathbf{v}| \ll 1 \; . \end{aligned} \tag{2.26b}$$

The spatial components of the 4-momentum, when regarded from the viewpoint of 3-dimensional physics, are the same as the components of the *momentum*, a 3-vector residing in the chosen Lorentz frame's 3-space:

$$p^j = mu^j = m\gamma v^j = \frac{mv^j}{\sqrt{1-\mathbf{v}^2}} = \mathcal{E}v_j = (j\text{-component of particle's momentum}) \; ; \tag{2.26c}$$

or, in basis-independent, 3-dimensional vector notation,

$$\mathbf{p} = m\mathbf{u} = m\gamma\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1-\mathbf{v}^2}} = \mathcal{E}\mathbf{v} = \text{ (particle's momentum)} \; . \tag{2.26d}$$

For a zero-rest-mass particle, as for one with finite rest mass, we identify the time component of the 4-momentum, in a chosen Lorentz frame, as the particle's energy, and the spatial part as its momentum. Moreover, if—appealing to quantum theory—we regard a zero-rest-mass particle as a quantum associated with a monochromatic wave, then quantum theory tells us that the wave's angular frequency ω as measured in a chosen Lorentz frame is related to its energy by

$$\mathcal{E} \equiv p^0 = \hbar\omega = (\text{particle's energy}) \; ; \tag{2.27a}$$

and, since the particle has $\vec{p}^2 = -(p^0)^2 + \mathbf{p}^2 = -m^2 = 0$ (in accord with the lightlike nature of its world line), its momentum as measured in the chosen Lorentz frame is

$$\mathbf{p} = \mathcal{E}\mathbf{n} = \hbar\omega\mathbf{n} \; . \tag{2.27b}$$

Here $\mathbf{n}$ is the unit 3-vector that points in the direction of the particle's travel, as measured in the chosen frame; i.e. (since the particle moves at the speed of light $v = 1$), $\mathbf{n}$ is the particle's ordinary velocity. Eqs. (2.27a) and (2.27b) are the temporal and spatial components of the geometric, frame-independent relation $\vec{p} = \hbar\vec{k}$ [Eq. (2.13), which is valid for zero-rest-mass particles as well as finite-mass ones].

The introduction of a specific Lorentz frame into spacetime can be said to produce a "3+1" split of every 4-vector into a 3-dimensional vector plus a scalar (a real number). The 3+1 split of a particle's 4-momentum $\vec{p}$ produces its momentum $\mathbf{p}$ plus its energy $\mathcal{E} = p^0$; and correspondingly, the 3+1 split of the law of 4-momentum conservation (2.15) produces a law of conservation of momentum plus a law of conservation of energy:

$$\sum_{\bar{A}} \mathbf{p}_{\bar{A}} = \sum_{A} \mathbf{p}_A \ , \quad \sum_{\bar{A}} \mathcal{E}_{\bar{A}} = \sum_{A} \mathcal{E}_A \ . \tag{2.28}$$

Here the unbarred quantities are momenta and energies of the particles entering the interaction region, and the barred quantities are those of the particles leaving; cf. Fig. 2.4 above.

Because the concept of energy does not even exist until one has chosen a Lorentz frame, and neither does that of momentum, the laws of energy conservation and momentum conservation separately are frame-dependent laws. In this sense, they are far less fundamental than their combination, the frame-independent law of 4-momentum conservation.

By learning to think about the 3+1 split in a geometric, frame-independent way, one can gain conceptual and computational power. As an example, consider a particle with 4-momentum $\vec{p}$, being studied by an observer with 4-velocity $\vec{U}$. In the observer's own Lorentz reference frame, her 4-velocity has components $U^0 = 1$ and $U^j = 0$, and therefore, her 4-velocity is $\vec{U} = U^\alpha \vec{e}_\alpha = \vec{e}_0$, i.e. it is identically equal to the time basis vector of her Lorentz frame. This means that the particle energy that she measures is $\mathcal{E} = p^0 = -p_0 = -\vec{p} \cdot \vec{e}_0 = -\vec{p}\cdot\vec{U}$. This equation, derived in the observer's Lorentz frame, is actually a geometric, frame-independent relation: the inner product of two 4-vectors. It says that *when an observer with 4-velocity $\vec{U}$ measures the energy of a particle with 4-momentum $\vec{p}$, the result she gets (the time part of the 3+1 split of $\vec{p}$ as seen by her) is*

$$\boxed{\mathcal{E} = -\vec{p} \cdot \vec{U}} \ . \tag{2.29}$$

We shall use this equation in later chapters. In Exs. 2.9 and 2.10, the reader can get experience at deriving and interpreting other frame-independent equations for 3+1 splits. Exercise 2.11 exhibits the power of this geometric way of thinking by using it to derive the Doppler shift of a photon.

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EXERCISES

Exercise 2.9 ***Practice: Frame-Independent Expressions for Energy, Momentum, and Velocity*[4]

An observer with 4-velocity $\vec{U}$ measures the properties of a particle with 4-momentum $\vec{p}$. The energy she measures is $\mathcal{E} = -\vec{p} \cdot \vec{U}$, Eq. (2.29).

[4]Exercises marked with double stars are important expansions of the material presented in the text.

(a) Show that the particle's rest mass can be expressed in terms of $\vec{p}$ as

$$m^2 = -\vec{p}^{\,2} \ . \tag{2.30a}$$

(b) Show that the momentum the observer measures has the magnitude

$$|\mathbf{p}| = [(\vec{p}\cdot\vec{U})^2 + \vec{p}\cdot\vec{p}]^{\frac{1}{2}} \ . \tag{2.30b}$$

(c) Show that the ordinary velocity the observer measures has the magnitude

$$|\mathbf{v}| = \frac{|\mathbf{p}|}{\mathcal{E}}, \tag{2.30c}$$

where $|\mathbf{p}|$ and $\mathcal{E}$ are given by the above frame-independent expressions.

(d) Show that the ordinary velocity $\mathbf{v}$, thought of as a 4-vector that happens to lie in the observer's slice of simultaneity, is given by

$$\vec{v} = \frac{\vec{p} + (\vec{p}\cdot\vec{U})\vec{U}}{-\vec{p}\cdot\vec{U}} \ . \tag{2.30d}$$

Exercise 2.10 ***Example: 3-Metric as a Projection Tensor*
Consider, as in Exercise 2.9, an observer with 4-velocity $\vec{U}$ who measures the properties of a particle with 4-momentum $\vec{p}$.

(a) Show that the Euclidean metric of the observer's 3-space, when thought of as a tensor in 4-dimensional spacetime, has the form

$$\boxed{\boldsymbol{P} \equiv \boldsymbol{g} + \vec{U}\otimes\vec{U}\ .} \tag{2.31a}$$

Show, further, that if $\vec{A}$ is an arbitrary vector in spacetime, then $-\vec{A}\cdot\vec{U}$ is the component of $\vec{A}$ along the observer's 4-velocity $\vec{U}$, and

$$\boldsymbol{P}(\_\,,\vec{A}) = \vec{A} + (\vec{A}\cdot\vec{U})\vec{U} \tag{2.31b}$$

is the projection of $\vec{A}$ into the observer's 3-space; i.e., it is the spatial part of $\vec{A}$ as seen by the observer. For this reason, $\boldsymbol{P}$ is called a *projection tensor*. In quantum mechanics, one introduces the concept of a *projection operator* $\hat{P}$ as one that satisfies the equation $\hat{P}^2 = \hat{P}$. Show that the projection tensor $\boldsymbol{P}$ is a projection operator in the same sense:

$$P_{\alpha\mu}P^{\mu}{}_{\beta} = P_{\alpha\beta} \ . \tag{2.31c}$$

(b) Show that Eq. (2.30d) for the particle's ordinary velocity, thought of as a 4-vector, can be rewritten as

$$\vec{v} = \frac{\boldsymbol{P}(\_\,,\vec{p})}{-\vec{p}\cdot\vec{U}} \ . \tag{2.32}$$

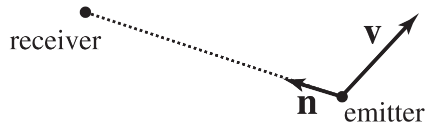

Fig. 2.6: Geometry for Doppler shift, drawn in a slice of simultaneity of the receiver's inertial frame.

Exercise 2.11 ***Example: Doppler Shift Derived without Lorentz Transformations*

(a) An observer at rest in some inertial frame receives a photon that was emitted in a direction $\mathbf{n}$ by an atom moving with ordinary velocity $\mathbf{v}$ (Fig. 2.6). The photon frequency and energy as measured by the emitting atom are $\nu_{\rm em}$ and $\mathcal{E}_{\rm em}$; those measured by the receiving observer are $\nu_{\rm rec}$ and $\mathcal{E}_{\rm rec}$. By a calculation carried out solely in the receiver's inertial frame (the frame of Fig. 2.6), and without the aid of any Lorentz transformation, derive the standard formula for the photon's Doppler shift,

$$\frac{\nu_{\rm rec}}{\nu_{\rm em}} = \frac{\sqrt{1-v^2}}{1-\mathbf{v}\cdot\mathbf{n}} . \tag{2.33}$$

Hint: Use Eq. (2.29) to evaluate $\mathcal{E}_{\rm em}$ using receiver-frame expressions for the emitting atom's 4-velocity $\vec{U}$ and the photon's 4-momentum $\vec{p}$.

(b) Suppose that instead of emitting a photon, the emitter produces a particle with finite rest mass m. Using the same method, derive an expression for the ratio of received energy to emitted energy, $\mathcal{E}_{\rm rec}/\mathcal{E}_{\rm em}$, expressed in terms of the emitter's ordinary velocity $\mathbf{v}$ and the particle's ordinary velocity $\mathbf{V}$ (both as measured in the receiver's frame).

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2.7 Lorentz Transformations

Consider two different inertial reference frames in Minkowski spacetime. Denote their Lorentz coordinates by $\{x^\alpha\}$ and $\{x^{\bar\mu}\}$ and their bases by $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}_{\bar\mu}\}$, and write the transformation from one basis to the other as

$$\vec{e}_\alpha = \vec{e}_{\bar\mu} L^{\bar\mu}{}_\alpha , \quad \vec{e}_{\bar\mu} = \vec{e}_\alpha L^\alpha{}_{\bar\mu} . \tag{2.34}$$

As in Euclidean 3-space, $L^{\bar\mu}{}_\alpha$ and $L^\alpha{}_{\bar\mu}$ are elements of two different transformation matrices, and since these matrices operate in opposite directions, they must be the inverse of each other:

$$L^{\bar\mu}{}_\alpha L^\alpha{}_{\bar\nu} = \delta^{\bar\mu}{}_{\bar\nu} , \quad L^\alpha{}_{\bar\mu} L^{\bar\mu}{}_\beta = \delta^\alpha{}_\beta . \tag{2.35a}$$

Notice the up/down placement of indices on the elements of the transformation matrices: the first index is always up, and the second is always down. This is just a convenient convention, which helps systematize the index shuffling rules in a way that can easily be remembered. Our rules about summing on the same index when up and down, and matching unsummed indices on the two sides of an equation automatically dictate the matrix to use in each of the transformations (2.34); and similarly for all other equations in this section.

In Euclidean 3-space the orthonormality of the two bases dictated that the transformations must be orthogonal, i.e. must be reflections or rotations. In Minkowski spacetime, orthonormality implies $g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta = (\vec{e}_{\bar\mu} L^{\bar\mu}{}_\alpha) \cdot (\vec{e}_{\bar\nu} L^{\bar\nu}{}_\beta) = L^{\bar\mu}{}_\alpha L^{\bar\nu}{}_\beta g_{\bar\mu\bar\nu}$; i.e.,

$$g_{\bar\mu\bar\nu} L^{\bar\mu}{}_\alpha L^{\bar\nu}{}_\beta = g_{\alpha\beta} \ , \quad \text{and similarly } g_{\alpha\beta} L^\alpha{}_{\bar\mu} L^\beta{}_{\bar\nu} = g_{\bar\mu\bar\nu} \ . \tag{2.35b}$$

Any matrices whose elements satisfy these equations is a *Lorentz transformation*.

From the fact that vectors and tensors are geometric, frame-independent objects, one can derive the Minkowski-space analogs of the Euclidean transformation laws for components (1.13a), (1.13b):

$$A^{\bar\mu} = L^{\bar\mu}{}_\alpha A^\alpha \ , \quad T^{\bar\mu\bar\nu\bar\rho} = L^{\bar\mu}{}_\alpha L^{\bar\nu}{}_\beta L^{\bar\rho}{}_\gamma T^{\alpha\beta\gamma} \ , \quad \text{and similarly in the opposite direction.} \tag{2.36a}$$

Notice that here, as elsewhere, these equations can be constructed by lining up indices in accord with our standard rules.

If (as is conventional) we choose the spacetime origins of the two Lorentz coordinate systems to coincide, then the vector $\vec{x}$ extending from the origin to some event $\mathcal{P}$, whose coordinates are x^α and $x^{\bar\alpha}$, has components equal to those coordinates. As a result, the transformation law for the coordinates takes the same form as that (2.36a) for components of a vector:

$$x^\alpha = L^\alpha{}_{\bar\mu} x^{\bar\mu} \ , \quad x^{\bar\mu} = L^{\bar\mu}{}_\alpha x^\alpha \ . \tag{2.36b}$$

The product $L^\alpha{}_{\bar\mu} L^{\bar\mu}{}_{\bar{\bar\rho}}$ of two Lorentz transformation matrices is a Lorentz transformation matrix; and under this product rule, the Lorentz transformations form a mathematical group, the *Lorentz group*, whose "representations" play an important role in quantum field theory.

An important specific example of a Lorentz transformation is the following

$$|| L^\alpha{}_{\bar\mu} || = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad || L^{\bar\mu}{}_\alpha || = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \tag{2.37a}$$

where β and γ are related by

$$|\beta| < 1 \ , \quad \gamma \equiv (1 - \beta^2)^{-\frac{1}{2}} \ . \tag{2.37b}$$

One can readily verify [Ex. 2.12] that these matrices are the inverses of each other and that they satisfy the Lorentz-transformation relation (2.35b). These transformation matrices produce the following change of coordinates [Eq. (2.36b)]

$$\begin{aligned} t &= \gamma(\bar t + \beta \bar x) \ , \quad x = \gamma(\bar x + \beta \bar t) \ , \quad y = \bar y \ , \quad z = \bar z \ , \\ \bar t &= \gamma(t - \beta x) \ , \quad \bar x = \gamma(x - \beta t) \ , \quad \bar y = y \ , \quad \bar z = z \ . \end{aligned} \tag{2.37c}$$

These expressions reveal that any particle at rest in the unbarred frame (a particle with fixed, time-independent x, y, z) is seen in the barred frame to move along the world line $\bar{x} = \text{const} - \beta\bar{t}$, $\bar{y} = \text{const}$, $\bar{z} = \text{const}$. In other words, the unbarred frame is seen by observers at rest in the barred frame to move with uniform velocity $\vec{v} = -\beta\vec{e}_{\bar{x}}$, and correspondingly the barred frame is seen by observers at rest in the unbarred frame to move with the opposite uniform velocity $\vec{v} = +\beta\vec{e}_x$. This special Lorentz transformation is called a *pure boost* along the x direction.

****

EXERCISES

Exercise 2.12 *Derivation: Lorentz Boosts*
Show that the matrices (2.37a), with β and γ satisfying (2.37b), are the inverses of each other, and that they obey the condition (2.35b) for a Lorentz transformation.

Exercise 2.13 ***Example: General Boosts and Rotations*

(a) Show that, if n^j is a 3-dimensional unit vector and β and γ are defined as in Eq. (2.37b), then the following is a Lorentz transformation; i.e., it satisfies Eq. (2.35b).

$$L^0{}_{\bar{0}} = \gamma \;, \quad L^0{}_{\bar{j}} = L^j{}_{\bar{0}} = \beta\gamma n^j \;, \quad L^j{}_{\bar{k}} = L^k{}_{\bar{j}} = (\gamma - 1) n^j n^k + \delta^{jk} \;. \tag{2.38}$$

Show, further, that this transformation is a *pure boost along the direction* **n** *with speed* β, and show that the inverse matrix $L^{\bar{\mu}}{}_\alpha$ for this boost is the same as $L^\alpha{}_{\bar{\mu}}$, but with β changed to $-\beta$.

(b) Show that the following is also a Lorentz transformation:

$$[\,L^\alpha{}_{\bar{\mu}}\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & [R_{i\bar{j}}] & \\ 0 & & & \end{bmatrix} , \tag{2.39}$$

where $[R_{i\bar{j}}]$ is a three-dimensional rotation matrix for Euclidean 3-space. Show, further, that this Lorentz transformation rotates the inertial frame's spatial axes (its latticework of measuring rods), while leaving the frame's velocity unchanged; i.e., the new frame is at rest with respect to the old.

One can show (not surprisingly) that the general Lorentz transformation [i.e., the general solution of Eqs. (2.35b)] can be expressed as a sequence of pure boosts, pure rotations, and pure inversions (in which one or more of the coordinate axes are reflected through the origin, so $x^\alpha = -x^{\bar{\alpha}}$).

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2.8 Spacetime Diagrams for Boosts

Figure 2.7 illustrates the pure boost (2.37c). Diagram (a) in that figure is a two-dimensional spacetime diagram, with the y- and z-coordinates suppressed, showing the $\bar{t}$ and $\bar{x}$ axes of the boosted Lorentz frame $\bar{\mathcal{F}}$ in the t, x Lorentz coordinate system of the unboosted frame $\mathcal{F}$. That the barred axes make angles $\tan^{-1}\beta$ with the unbarred axes, as shown, can be inferred from the Lorentz transformation equation (2.37c). Note that the orthogonality of the $\bar{t}$ and $\bar{x}$ axes to each other ($\vec{e}_{\bar{t}} \cdot \vec{e}_{\bar{x}} = 0$) shows up as the two axes making the same angle $\pi/2 - \beta$ with the null line $x = t$. The invariance of the interval guarantees that for $a = 1$ or 2, the event $\bar{x} = a$ on the $\bar{x}$-axis lies at the intersection of that axis with the dashed hyperbola $x^2 - t^2 = a^2$; and similarly, the event $\bar{t} = a$ on the $\bar{t}$-axis lies at the intersection of that axis with the dashed hyperbola $t^2 - x^2 = a^2$.

As is shown in diagram (b) of the figure, the barred coordinates $\bar{t}$, $\bar{x}$ of an event $\mathcal{P}$ can be inferred by projecting from $\mathcal{P}$ onto the $\bar{t}$- and $\bar{x}$-axes, with the projection going parallel to the $\bar{x}$- and $\bar{t}$- axes respectively. Diagram (c) shows the 4-velocity $\vec{u}$ of an observer at rest in frame $\mathcal{F}$ and that, $\bar{\vec{u}}$, of an observer at rest in frame $\bar{\mathcal{F}}$. The events which observer $\mathcal{F}$ regards as all simultaneous, with time $t = 0$, lie in a 3-space that is orthogonal to $\vec{u}$ and includes the x-axis. This is a slice of simultaneity of reference frame $\mathcal{F}$. Similarly, the events which observer $\bar{\mathcal{F}}$ regards as all simultaneous, with $\bar{t} = 0$, live in the 3-space that is orthogonal to $\bar{\vec{u}}$ and includes the $\bar{x}$-axis, i.e. in a slice of simultaneity of frame $\bar{\mathcal{F}}$.

Exercise 2.14 uses spacetime diagrams, similar to Fig. 2.7, to deduce a number of important relativistic phenomena, including the contraction of the length of a moving object ("length contraction"), the breakdown of simultaneity as a universally agreed upon concept, and the dilation of the ticking rate of a moving clock ("time dilation"). This exercise is extremely important; every reader who is not already familiar with it should study it.

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EXERCISES

Exercise 2.14 ***Example: Spacetime Diagrams*

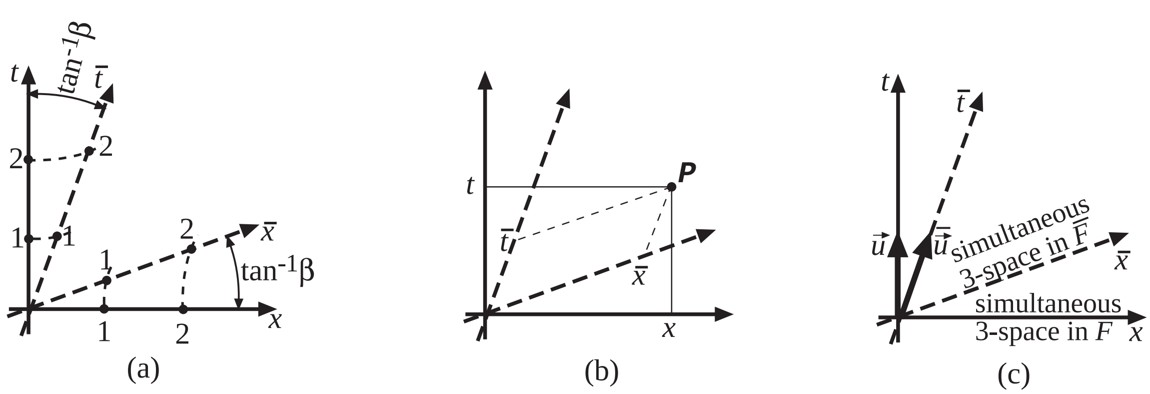

Fig. 2.7: Spacetime diagrams illustrating the pure boost (2.37c) from one Lorentz reference frame to another.

Use spacetime diagrams to prove the following:

(a) *Two events that are simultaneous in one inertial frame are not necessarily simultaneous in another.* More specifically, if frame $\bar{\mathcal{F}}$ moves with velocity $\vec{v} = \beta\vec{e}_x$ as seen in frame $\mathcal{F}$, where $\beta > 0$, then of two events that are simultaneous in $\bar{\mathcal{F}}$ the one farther "back" (with the more negative value of $\bar{x}$) will occur in $\mathcal{F}$ before the one farther "forward".

(b) Two events that occur at the same spatial location in one inertial frame do not necessarily occur at the same spatial location in another.

(c) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two events with a timelike separation, then there exists an inertial reference frame in which they occur at the same spatial location; and in that frame the time lapse between them is equal to the square root of the negative of their invariant interval, $\Delta t = \Delta\tau \equiv \sqrt{-(\Delta s)^2}$.

(d) If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two events with a spacelike separation, then there exists an inertial reference frame in which they are simultaneous; and in that frame the spatial distance between them is equal to the square root of their invariant interval, $\sqrt{g_{ij}\Delta x^i \Delta x^j} = \Delta s \equiv \sqrt{(\Delta s)^2}$.

(e) If the inertial frame $\bar{\mathcal{F}}$ moves with speed β relative to the frame $\mathcal{F}$, then a clock at rest in $\bar{\mathcal{F}}$ ticks more slowly as viewed from $\mathcal{F}$ than as viewed from $\bar{\mathcal{F}}$—more slowly by a factor $\gamma^{-1} = (1-\beta^2)^{\frac{1}{2}}$. This is called *relativistic time dilation.*

(f) If the inertial frame $\bar{\mathcal{F}}$ moves with velocity $\vec{v} = \beta\vec{e}_x$ relative to the frame $\mathcal{F}$, then an object at rest in $\bar{\mathcal{F}}$ as studied in $\mathcal{F}$ appears shortened by a factor $\gamma^{-1} = (1-\beta^2)^{\frac{1}{2}}$ along the x direction, but its length along the y and z directions is unchanged. This is called *Lorentz contraction.*

Exercise 2.15 *Problem: Allowed and Forbidden Electron-Photon Reactions*
Show, using spacetime diagrams and also using frame-independent calculations, that the law of conservation of 4-momentum forbids a photon to be absorbed by an electron, $e + \gamma \to e$ and also forbids an electron and a positron to annihilate and produce a single photon $e^+ + e^- \to \gamma$ (in the absence of any other particles to take up some of the 4-momentum); but the annihilation to form two photons, $e^+ + e^- \to 2\gamma$, is permitted.

****

2.9 Time Travel

Time dilation is one facet of a more general phenomenon: Time, as measured by ideal clocks, is a "personal thing," different for different observers who move through spacetime on different world lines. This is well illustrated by the infamous "twins paradox," in which one twin, Methuselah, remains forever at rest in an inertial frame and the other, Florence, makes a spacecraft journey at high speed and then returns to rest beside Methuselah.

The twins' world lines are depicted in Fig. 2.8a, a spacetime diagram whose axes are those of Methuselah's inertial frame. The time measured by an ideal clock that Methuselah carries is the coordinate time t of his inertial frame; and its total time lapse, from Florence's departure to her return, is $t_{\rm return} - t_{\rm departure} \equiv T_{\rm Methuselah}$. By contrast, the time measured by an ideal clock that Florence carries is her proper time τ, i.e. the square root of the invariant interval (2.4) along her world line; and thus her total time lapse from departure to return is

$$T_{\rm Florence} = \int d\tau = \int \sqrt{dt^2 - \delta_{ij} dx^i dx^j} = \int_0^{T_{\rm Methuselah}} \sqrt{1-v^2} dt \ . \tag{2.40}$$

Here (t, x^i) are the time and space coordinates of Methuselah's inertial frame, and v is Florence's ordinary speed, $v = \sqrt{\delta_{ij}(dx^i/dt)(dx^j/dt)}$, relative to Methuselah's frame. Obviously, Eq. (2.40) predicts that $T_{\rm Florence}$ is less than $T_{\rm Methuselah}$. In fact (Ex. 2.16), even if Florence's acceleration is kept no larger than one Earth gravity throughout her trip, and her trip lasts only $T_{\rm Florence}$ = (a few tens of years), $T_{\rm Methuselah}$ can be hundreds or thousands or millions or billions of years.

Does this mean that Methuselah actually "experiences" a far longer time lapse, and actually ages far more than Florence? Yes! The time experienced by humans and the aging of the human body are governed by chemical processes, which in turn are governed by the natural oscillation rates of molecules, rates that are constant to high accuracy when measured in terms of ideal time (or, equivalently, proper time τ). Therefore, a human's experiential time and aging time are the same as the human's proper time—so long as the human is not subjected to such high accelerations as to damage her body.

In effect, then, Florence's spacecraft has functioned as a time machine to carry her far into Methuselah's future, with only a modest lapse of her own proper time (ideal time; experiential time; aging time).

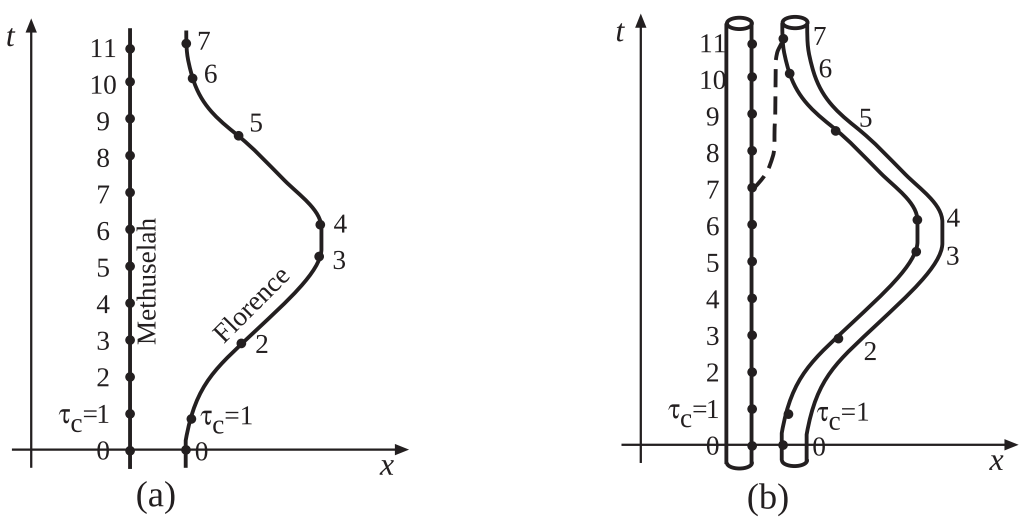

Fig. 2.8: (a) Spacetime diagram depicting the so-called "twins paradox". Marked along the two world lines are intervals of proper time as measured by the two twins. (b) Spacetime diagram depicting the motions of the two mouths of a wormhole. Marked along the mouths' world tubes are intervals of proper time τ_c as measured by the single clock that sits in the common mouths.

Is it also possible, at least in principle, for Florence to construct a time machine that carries her into Methuselah's past—and also her own past? At first sight, the answer would seem to be Yes. Figure 2.8b shows one possible method, using a *wormhole*. [Papers on other methods are cited in Everett and Roman (2011) and Friedman and Higuchi (2006).]

Wormholes are hypothetical "handles" in the topology of space. A simple model of a wormhole can be obtained by taking a flat 3-dimensional space, removing from it the interiors of two identical spheres, and identifying the spheres' surfaces so that if one enters the surface of one of the spheres, one immediately finds oneself exiting through the surface of the other. When this is done, there is a bit of strongly localized spatial curvature at the spheres' common surface, so to analyze such a wormhole properly, one must use general relativity rather than special relativity. In particular, it is the laws of general relativity, combined with the laws of quantum field theory, that tell one how to construct such a wormhole and what kinds of materials are required to hold it open, so things can pass through it. Unfortunately, despite considerable effort, theoretical physicists have not yet deduced definitively whether those laws permit such wormholes to exist and stay open, though indications are pessimistic (Friedman and Higuchi 2006, Everett and Roman 2011). On the other hand, assuming such wormholes *can* exist, the following special relativistic analysis (Morris, Thorne and Yurtsever 1988) shows how one might be used to construct a machine for backward time travel.

The two identified spherical surfaces are called the wormhole's mouths. Ask Methuselah to keep one mouth with himself, forever at rest in his inertial frame, and ask Florence to take the other mouth with herself on her high-speed journey. The two mouths' *world tubes* (analogs of world lines for a 3-dimensional object) then have the forms shown in Fig. 2.8b. Suppose that a single ideal clock sits in the wormhole's identified mouths, so that from the external Universe one sees it both in Methuselah's wormhole mouth and in Florence's. As seen in Methuselah's mouth, the clock measures his proper time, which is equal to the coordinate time t (see tick marks along the left world tube in Fig. 2.8b). As seen in Florence's mouth, the clock measures her proper time, Eq. (2.40) (see tick marks along the right world tube in Fig. 2.8b). The result should be obvious, if surprising: When Florence returns to rest beside Methuselah, the wormhole has become a time machine. If she travels through the wormhole when the clock reads $\tau_c = 7$, she goes backward in time as seen in Methuselah's (or anyone else's) inertial frame; and then, in fact, traveling along the everywhere timelike, dashed world line, she is able to meet her younger self before she entered the wormhole.

This scenario is profoundly disturbing to most physicists because of the dangers of science-fiction-type paradoxes (e.g., the older Florence might kill her younger self, thereby preventing herself from making the trip through the wormhole and killing herself). Fortunately perhaps, it seems likely (though far from certain) that vacuum fluctuations of quantum fields will destroy the wormhole at the moment when its mouths' motion first makes backward time travel possible; and it may be that this mechanism will *always* prevent the construction of backward-travel time machines, no matter what tools one uses for their construction.[5] Whether this is so we likely will not know until the laws of quantum gravity have been mastered.

[5]Kim and Thorne (1991), Kay, Radzikowski, Marek and Wald (1997). But see also contrary indications in research reviewed by Friedman and Higuchi (2006), and by Everett and Roman (2011).

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EXERCISES

Exercise 2.16 *Example: Twins Paradox*

(a) The 4-acceleration of a particle or other object is defined by $\vec{a} \equiv d\vec{u}/d\tau$, where $\vec{u}$ is its 4-velocity and τ is proper time along its world line. Show that, if an observer carries an accelerometer, the magnitude $|\mathbf{a}|$ of the 3-dimensional acceleration $\mathbf{a}$ measured by the accelerometer will always be equal to the magnitude of the observer's 4-acceleration, $|\mathbf{a}| = |\vec{a}| \equiv \sqrt{\vec{a} \cdot \vec{a}}$.

(b) In the twins paradox of Fig. 2.8a, suppose that Florence begins at rest beside Methuselah, then accelerates in Methuselah's x-direction with an acceleration a equal to one Earth gravity, g, for a time $T_{\rm Florence}/4$ as measured by her, then accelerates in the $-x$-direction at g for a time $T_{\rm Florence}/2$ thereby reversing her motion, and then accelerates in the $+x$-direction at g for a time $T_{\rm Florence}/4$ thereby returning to rest beside Methuselah. (This is the type of motion shown in the figure.) Show that the total time lapse as measured by Methuselah is

$$T_{\rm Methuselah} = \frac{4}{g} \sinh\left(\frac{gT_{\rm Florence}}{4}\right) \ . \tag{2.41}$$

(c) Show that in the geometrized units used here, Florence's acceleration (equal to acceletion of gravity at the surface of the Earth) is $g = 1.033/\text{yr}$. Plot $T_{\rm Methuselah}$ as a function of $T_{\rm Florence}$, and from your plot deduce that, if $T_{\rm Florence}$ is several tens of years, then $T_{\rm Methuselah}$ can be hundreds or thousands or millions or even billions of years.

Exercise 2.17 *Challenge: Around the World on TWA*

In a long-ago era when an airline named Trans World Airlines (TWA) flew around the world, J. C. Hafele and R. E. Keating carried out a real live twins paradox experiment: They synchronized two atomic clocks, and then flew one around the world eastward on TWA, and on a separate trip, around the world westward, while the other clock remained at home at the Naval Research Laboratory near Washington D.C. When the clocks were compared after each trip, they were found to have aged differently. Making reasonable estimates for the airplane routing and speeds, compute the difference in aging, and compare your result with the experimental data (Hafele and Keating, 1972). [Note: The rotation of the Earth is important, as is the general relativistic gravitational redshift associated with the clocks' altitudes; but the gravitational redshift drops out of the *difference* in aging, if the time spent at high altitude is the same eastward as westward.]

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2.10 Directional Derivatives, Gradients, Levi-Civita Tensor

Derivatives of vectors and tensors in Minkowski spacetime are defined precisely the same way as in Euclidean space; see Sec. 1.7. Any reader who has not studied that section should do so now. In particular (in extreme brevity, as the explanations and justifications are the same as in Euclidean space):

The directional derivative of a tensor $\boldsymbol{T}$ along a vector $\vec{A}$ is $\nabla_{\vec{A}}\boldsymbol{T} \equiv \lim_{\epsilon\to 0}(1/\epsilon)[\boldsymbol{T}(\vec{x}_{\mathcal{P}} + \epsilon\vec{A}) - \boldsymbol{T}(\vec{x}_{\mathcal{P}})]$; and the gradient $\vec{\nabla}\boldsymbol{T}$ is the tensor that produces the directional derivative when one inserts $\vec{A}$ into its last slot: $\nabla_{\vec{A}}\boldsymbol{T} = \vec{\nabla}\boldsymbol{T}(\_\,, \_\,, \_\,, \vec{A})$. In slot-naming index notation (or in components on a basis), the gradient is denoted $T_{\alpha\beta\gamma;\mu}$. In a Lorentz basis (the basis vectors associated with an inertial reference frame), the components of the gradient are simply the partial derivatives of the tensor, $T_{\alpha\beta\gamma;\mu} = \partial T_{\alpha\beta\gamma}/\partial x^\mu \equiv T_{\alpha\beta\gamma,\mu}$. (The comma means partial derivative in a Lorentz basis, as in a Cartesian bases.)

The gradient and the directional derivative obey all the familiar rules for differentiation of products, e.g. $\nabla_{\vec{A}}(\boldsymbol{S}\otimes\boldsymbol{T}) = (\nabla_{\vec{A}}\boldsymbol{S})\otimes\boldsymbol{T} + \boldsymbol{S}\otimes\nabla_{\vec{A}}\boldsymbol{T}$. The gradient of the metric vanishes, $g_{\alpha\beta;\mu} = 0$. The divergence of a vector is the contraction of its gradient, $\vec{\nabla}\cdot\vec{A} = A_{\alpha;\beta}g^{\alpha\beta} = A^\alpha{}_{;\alpha}$.

Recall that the divergence of the gradient of a tensor in Euclidean space is the Laplacian: $T_{abc;jk}g_{jk} = T_{abc,jk}\delta_{jk} = \partial^2 T_{abc}\partial x^j \partial x^j$. By contrast, in Minkowski spacetime, because $g^{00} = -1$ and $g^{jk} = \delta^{jk}$ in a Lorentz frame, the divergence of the gradient is the wave operator (also called the d'Alembertian):

$$T_{\alpha\beta\gamma;\mu\nu}g^{\mu\nu} = T_{\alpha\beta\gamma,\mu\nu}g^{\mu\nu} = -\frac{\partial^2 T_{\alpha\beta\gamma}}{\partial t^2} + \frac{\partial^2 T_{\alpha\beta\gamma}}{\partial x^j \partial x^k}\delta^{jk} = \Box T_{\alpha\beta\gamma}\ . \tag{2.42}$$

When one sets this to zero, one gets the wave equation.

As in Euclidean space, so also in Minkowski spacetime there are two tensors that embody the space's geometry: the metric tensor $\boldsymbol{g}$ and the Levi-Civita tensor $\boldsymbol{\epsilon}$. The Levi-Civita tensor in Minkowski spacetime is the tensor that is completely antisymmetric in all its slots and has value $\boldsymbol{\epsilon}(\vec{A}, \vec{B}, \vec{C}, \vec{D}) = +1$ when evaluated on any *right-handed set of orthonormal 4-vectors*—i.e., by definition, any orthonormal set for which $\vec{A}$ is timelike and future directed, and $\{\vec{B}, \vec{C}, \vec{D}\}$ are spatial and right-handed. This means that in any right-handed Lorentz basis, the only nonzero components of $\boldsymbol{\epsilon}$ are

$$\begin{aligned}\epsilon_{\alpha\beta\gamma\delta} &= +1 \ \text{ if } \alpha,\ \beta,\ \gamma,\ \delta \text{ is an even permutation of } 0,\ 1,\ 2,\ 3 \\ &= -1 \ \text{ if } \alpha,\ \beta,\ \gamma,\ \delta \text{ is an odd permutation of } 0,\ 1,\ 2,\ 3 \\ &= 0 \quad \text{ if } \alpha,\ \beta,\ \gamma,\ \delta \text{ are not all different.}\end{aligned} \tag{2.43}$$

By the sign-flip-if-temporal rule, $\epsilon_{0123} = +1$ implies that $\epsilon^{0123} = -1$.

2.11 Nature of Electric and Magnetic Fields; Maxwell's Equations

Now that we have introduced the gradient and the Levi-Civita tensor, we can study the relationship of the relativistic version of electrodynamics to the nonrelativistic ("Newtonian") version. In doing so, we shall use Gaussian units (with the speed of light set to one) for the same reason as Jackson (1999) switches from SI to Gaussian when moving from nonrelativistic electrodynamics to the relativistic theory: The equations of the Gaussian formalism are noticeably simpler than those with SI units. One does not have to worry about where to put the factors of $\epsilon_o = c^2/\mu_o = 1/\mu_o$.

Consider a particle with charge q, rest mass m and 4-velocity $\vec{u}$ interacting with an electromagnetic field $\boldsymbol{F}(\_,\_)$. In index notation, the electromagnetic 4-force acting on the particle [Eq. (2.20)] is

$$dp^\alpha/d\tau = qF^{\alpha\beta}u_\beta \ . \tag{2.44}$$

Let us examine this 4-force in some arbitrary inertial reference frame in which particle's ordinary-velocity components are $v^j = v_j$ and its 4-velocity components are $u^0 = \gamma$, $u^j = \gamma v^j$ [Eqs. (2.25c)]. Anticipating the connection with the nonrelativistic viewpoint, we introduce the following notation for the contravariant components of the antisymmetric electromagnetic field tensor:

$$F^{0j} = -F^{j0} = +F_{j0} = -F_{0j} = E_j \ , \quad F^{ij} = F_{ij} = \epsilon_{ijk}B_k \ . \tag{2.45}$$

Inserting these components of $\boldsymbol{F}$ and $\vec{u}$ into Eq. (2.44) and using the relationship $dt/d\tau = u^0 = \gamma$ between t and τ derivatives, we obtain for the components of the 4-force $dp_j/d\tau = \gamma dp_j/dt = q(F_{j0}u^0 + F_{jk}u^k) = qu^0(F_{j0} + F_{jk}v^k) = q\gamma(E_j + \epsilon_{ijk}v_jB_k)$ and $dp^0/d\tau = \gamma dp^0/dt = qF^{0j}u_j = q\gamma E_j v_j$. Dividing by γ, converting into 3-space index notation, and denoting the particle's energy by $\mathcal{E} = p^0$, we bring these into the familiar Lorentz-force form

$$d\mathbf{p}/dt = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \ , \quad d\mathcal{E}/dt = q\mathbf{v}\cdot\mathbf{E} \ . \tag{2.46}$$

Evidently $\mathbf{E}$ is the electric field and $\mathbf{B}$ the magnetic field as measured in our chosen Lorentz frame.

This may be familiar from standard electrodynamics textbooks, e.g. Jackson (1999). Not so familiar, but very important, is the following geometric interpretation of $\mathbf{E}$ and $\mathbf{B}$:

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are spatial vectors as measured in the chosen inertial frame. We can also regard them as 4-vectors that lie in a 3-surface of simultaneity $t =$ const of the chosen frame, i.e. that are orthogonal to the 4-velocity (denote it $\vec{w}$) of the frame's observers (cf. Figs. 2.7 and 2.9). We shall denote this 4-vector version of $\mathbf{E}$ and $\mathbf{B}$ by $\vec{E}_{\vec{w}}$ and $\vec{B}_{\vec{w}}$, where the subscript $\vec{w}$ identifies the 4-velocity of the observer who measures these fields. These fields are depicted in Fig. 2.9.

In the rest frame of the observer $\vec{w}$, the components of $\vec{E}_{\vec{w}}$ are $E^0_{\vec{w}} = 0$, $E^j_{\vec{w}} = E_j = F_{j0}$ [the E_j appearing in Eqs. (2.45)], and similarly for $\vec{B}_{\vec{w}}$; and the components of $\vec{w}$ are $w^0 = 1$, $w^j = 0$. Therefore, in this frame Eqs. (2.45) can be rewritten as

$$\boxed{E^\alpha_{\vec{w}} = F^{\alpha\beta}w_\beta \ , \quad B^\beta_{\vec{w}} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}w_\alpha \ .} \tag{2.47a}$$

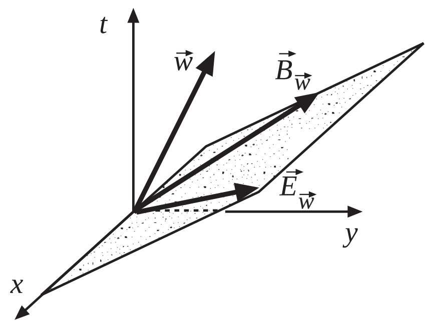

Fig. 2.9: The electric and magnetic fields measured by an observer with 4-velocity $\vec{w}$, shown as 4-vectors $\vec{E}_{\vec{w}}$ and $\vec{B}_{\vec{w}}$ that lie in the observer's 3-surface of simultaneity (stippled 3-surface orthogonal to $\vec{w}$).

(To verify this, insert the above components of $\boldsymbol{F}$ and $\vec{w}$ into these equations and, after some algebra, recover Eqs. (2.45) along with $E^0_{\vec{w}} = B^0_{\vec{w}} = 0$.) Equations (2.47a) say that in one special reference frame, that of the observer $\vec{w}$, the components of the 4-vectors on the left and on the right are equal. This implies that in every Lorentz frame the components of these 4-vectors will be equal; i.e., it implies that Eqs. (2.47a) are true when one regards them as geometric, frame-independent equations written in slot-naming index notation. *These equations enable one to compute the electric and magnetic fields measured by an observer (viewed as 4-vectors in the observer's 3-surface of simultaneity) from the observer's 4-velocity and the electromagnetic field tensor, without the aid of any basis or reference frame.*

Equations (2.47a) embody explicitly the following important fact: Although the electromagnetic field tensor $\boldsymbol{F}$ is a geometric, frame-independent quantity, the electric and magnetic fields $\vec{E}_{\vec{w}}$ and $\vec{B}_{\vec{w}}$ individually depend for their existence on a specific choice of observer (with 4-velocity $\vec{w}$), i.e., a specific choice of inertial reference frame, i.e., a specific choice of the split of spacetime into a 3-space (the 3-surface of simultaneity orthogonal to the observer's 4-velocity $\vec{w}$) and corresponding time (the Lorentz time of the observer's reference frame). *Only after making such an observer-dependent "3+1 split" of spacetime into space plus time do the electric field and the magnetic field come into existence as separate entities.* Different observers with different 4-velocities $\vec{w}$ make this spacetime split in different ways, thereby resolving the frame-independent $\boldsymbol{F}$ into different electric and magnetic fields $\vec{E}_{\vec{w}}$ and $\vec{B}_{\vec{w}}$.

By the same procedure as we used to derive Eqs. (2.47a), one can derive the inverse relationship, the following expression for the electromagnetic field tensor in terms of the (4-vector) electric and magnetic fields measured by some observer:

$$F^{\alpha\beta} = w^\alpha E^\beta_{\vec{w}} - E^\alpha_{\vec{w}} w^\beta + \epsilon^{\alpha\beta}{}_{\gamma\delta} w^\gamma B^\delta_{\vec{w}} \ . \tag{2.47b}$$

Maxwell's equations in geometric, frame-independent form are (in Gaussian units)

$$\begin{aligned} &F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha \ , \\ &\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta;\beta} = 0 \ ; \quad \text{i.e.} \quad F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0 \ . \end{aligned} \tag{2.48}$$

Here $\vec{J}$ is the charge-current 4-vector, which in any inertial frame has components

$$J^0 = \rho_e = (\text{charge density}) \ , \quad J^i = j_i = (\text{current density}). \tag{2.49}$$

Exercise 2.19 describes how to think about this charge density and current density as geometric objects determined by the observer's 4-velocity or 3+1 split of spacetime into space plus time. Exercise 2.20 shows how the frame-independent Maxwell equations (2.48) reduce to the more familiar ones in terms of **E** and **B**. Exercise 2.21 explores potentials for the electromagnetic field in geometric, frame-independent language and the 3+1 split.

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EXERCISES

Exercise 2.18 *Derivation and Practice: Reconstruction of* $\boldsymbol{F}$
Derive Eq. (2.47b) by the same method as was used to derive (2.47a). Then show, by a geometric, frame-independent calculation, that Eq. (2.47b) implies Eq. (2.47a).

Exercise 2.19 *Problem: 3+1 Split of Charge-Current 4-Vector*
Just as the electric and magnetic fields measured by some observer can be regarded as 4-vectors $\vec{E}_{\vec{w}}$ and $\vec{B}_{\vec{w}}$ that live in the observer's 3-space of simultaneity, so also the charge density and current density that the observer measures can be regarded as a scalar $\rho_{\vec{w}}$ and 4-vector $\vec{j}_{\vec{w}}$ that live in the 3-space of simultaneity. Derive geometric, frame-independent equations for $\rho_{\vec{w}}$ and $\vec{j}_{\vec{w}}$ in terms of the charge-current 4-vector $\vec{J}$ and the observer's 4-velocity $\vec{w}$, and derive a geometric expression for $\vec{J}$ in terms of $\rho_{\vec{w}}$, $\vec{j}_{\vec{w}}$, and $\vec{w}$.

Exercise 2.20 *Problem: Frame-Dependent Version of Maxwell's Equations*
From the geometric version of Maxwell's equations (2.48), derive the elementary, frame-dependent version

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= 4\pi\rho_e \ , \quad & \boldsymbol{\nabla}\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} &= 4\pi\mathbf{j} \ , \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \ , & \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 \ . \end{aligned} \tag{2.50}$$

Exercise 2.21 *Problem: Potentials for the Electromagnetic Field*

(a) Express the electromagnetic field tensor as an antisymmetrized gradient of a 4-vector potential: in slot-naming index notation

$$F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta} \ . \tag{2.51a}$$

Show that, whatever may be the 4-vector potential $\vec{A}$, the second of the Maxwell equations (2.48) is automatically satisfied. Show further that the electromagnetic field tensor is unaffected by a gauge change of the form

$$\vec{A}_{\rm new} = \vec{A}_{\rm old} + \vec{\nabla}\psi \ , \tag{2.51b}$$

where ψ is a scalar field (the generator of the gauge change). Show, finally, that it is possible to find a gauge-change generator that enforces "Lorenz gauge"

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{2.51c}$$

on the new 4-vector potential, and show that in this gauge, the first of the Maxwell equations (2.48) becomes

$$\Box \vec{A} = 4\pi \vec{J}\,; \qquad \text{i.e. } A^{\alpha;\mu}{}_{\mu} = 4\pi J^{\alpha}\ . \tag{2.51d}$$

(b) Introduce an inertial reference frame, and in that frame split $\boldsymbol{F}$ into the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, split $\vec{J}$ into the charge and current densities ρ_e and $\mathbf{j}$, and split the vector potential into a scalar potential and a 3-vector potential

$$\phi \equiv A_0\ , \quad \mathbf{A} = \text{spatial part of } \vec{A}\ . \tag{2.51e}$$

Deduce the 3+1 splits of Eqs. (2.51a)–(2.51d) and show that they take the form given in standard textbooks on electrodynamics.

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2.12 Volumes, Integration, and Conservation Laws

2.12.1 Spacetime Volumes and Integration

In Minkowski spacetime as in Euclidean 3-space (Sec. 1.8), the Levi-Civita tensor is the tool by which one constructs volumes: The 4-dimensional parallelopiped whose legs are the four vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ has a 4-dimensional volume given by the analog of Eqs. (1.25) and (1.26):

$$\text{4-Volume} = \epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta = \boldsymbol{\epsilon}(\vec{A}, \vec{B}, \vec{C}, \vec{D}) = \det \begin{bmatrix} A^0 & B^0 & C^0 & D^0 \\ A^1 & B^1 & C^1 & D^1 \\ A^2 & B^2 & C^2 & D^2 \\ A^3 & B^3 & C^3 & D^3 \end{bmatrix}\ . \tag{2.52}$$

Note that this 4-volume is positive if the set of vectors $\{\vec{A}, \vec{B}, \vec{C}, \vec{D}\}$ is right-handed and negative if left-handed.

Equation (2.52) provides us a way to perform volume integrals over 4-dimensional Minkowski spacetime: To integrate a tensor field $\boldsymbol{T}$ over some 4-dimensional region $\mathcal{V}$ of spacetime, we need only divide $\mathcal{V}$ up into tiny parallelopipeds, multiply the 4-volume $d\Sigma$ of each parallelopiped by the value of $\boldsymbol{T}$ at its center, and add. In any right-handed Lorentz coordinate system, the 4-volume of a tiny parallelopiped whose edges are dx^α along the four orthogonal coordinate axes is $d\Sigma = \boldsymbol{\epsilon}(dt\,\vec{e}_0, dx\,\vec{e}_x, dy\,\vec{e}_y, dz\,\vec{e}_z) = \epsilon_{0123}\,dt\,dx\,dy\,dz = dt\,dx\,dy\,dz$ (the analog of $dV = dx\,dy\,dz$). Correspondingly the integral of $\boldsymbol{T}$ over $\mathcal{V}$ can be expressed as

$$\int_{\mathcal{V}} T^{\alpha\beta\gamma} d\Sigma = \int_{\mathcal{V}} T^{\alpha\beta\gamma} dt\,dx\,dy\,dz\ . \tag{2.53}$$

By analogy with the vectorial area (1.27) of a parallelogram in 3-space, any 3-dimensional parallelopiped in spacetime with legs $\vec{A}, \vec{B}, \vec{C}$ has a vectorial 3-volume $\vec{\Sigma}$ (not to be confused with the scalar 4-volume Σ) defined by

$$\vec{\Sigma}(\_) = \boldsymbol{\epsilon}(\_, \vec{A}, \vec{B}, \vec{C}) \; ; \quad \Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma} A^\alpha B^\beta C^\gamma \; . \tag{2.54}$$

Here we have written the 3-volume vector both in abstract notation and in slot-naming index notation. This 3-volume vector has one empty slot, ready and waiting for a fourth vector ("leg") to be inserted, so as to compute the 4-volume Σ of a 4-dimensional parallelopiped.

Notice that the 3-volume vector $\vec{\Sigma}$ is orthogonal to each of its three legs (because of the antisymmetry of $\boldsymbol{\epsilon}$), and thus (unless it is null) it can be written as $\vec{\Sigma} = V\vec{n}$ where V is the magnitude of the 3-volume and $\vec{n}$ is the unit normal to the three legs.

Interchanging any two legs of the parallelopiped reverses the 3-volume's sign. Consequently, the 3-volume is characterized not only by its legs but also by the order of its legs, or equally well, in two other ways: (i) by the direction of the vector $\vec{\Sigma}$ (reverse the order of the legs, and the direction of $\vec{\Sigma}$ will reverse); and (ii) by the *sense* of the 3-volume, defined as follows. Just as a 2-volume (i.e., a segment of a plane) in 3-dimensional space has two sides, so a 3-volume in 4-dimensional spacetime has two sides; cf. Fig. 2.10. Every vector $\vec{D}$ for which $\vec{\Sigma} \cdot \vec{D} > 0$ points out the *positive side* of the 3-volume $\vec{\Sigma}$. Vectors $\vec{D}$ with $\vec{\Sigma} \cdot \vec{D} < 0$ point out its *negative side*. When something moves through or reaches through or points through the 3-volume from its negative side to its positive side, we say that this thing is moving or reaching or pointing in the "positive sense"; and similarly for "negative sense". The examples shown in Fig. 2.10 should make this more clear.

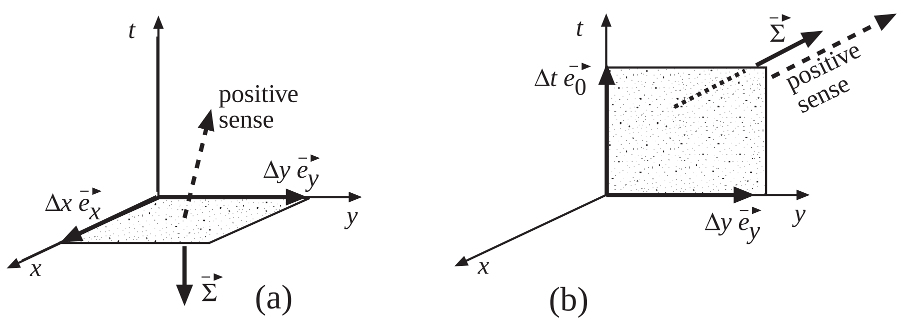

Fig. 2.10: Spacetime diagrams depicting 3-volumes in 4-dimensional spacetime, with one spatial dimension (that along the z-direction) suppressed.

Figure 2.10a shows two of the three legs of the volume vector $\vec{\Sigma} = \boldsymbol{\epsilon}(\_, \Delta x \vec{e}_x, \Delta y \vec{e}_y, \Delta z \vec{e}_z)$, where $\{t, x, y, z\}$ are the coordinates and $\{\vec{e}_\alpha\}$ is the corresponding right-handed basis of a specific Lorentz frame. It is easy to show that this $\vec{\Sigma}$ can also be written as $\vec{\Sigma} = -\Delta V \vec{e}_0$, where ΔV is the ordinary volume of the parallelopiped as measured by an observer in the chosen Lorentz frame, $\Delta V = \Delta x \Delta y \Delta z$. Thus, the direction of the vector $\vec{\Sigma}$ is toward the past (direction of decreasing Lorentz time t). From this, and the fact that timelike vectors have negative squared length, it is easy to infer that $\vec{\Sigma} \cdot \vec{D} > 0$ if and only if the vector $\vec{D}$ points out of the "future" side of the 3-volume (the side of increasing Lorentz time t); therefore, the positive side of $\vec{\Sigma}$ is the future side. This means that the vector $\vec{\Sigma}$ points in the negative sense of its own 3-volume.

Figure 2.10b shows two of the three legs of the volume vector $\vec{\Sigma} = \boldsymbol{\epsilon}(\_, \Delta t\vec{e}_t, \Delta y\vec{e}_y, \Delta z\vec{e}_z) = -\Delta t\Delta A\vec{e}_x$ (with $\Delta A = \Delta y\Delta z$). In this case, $\vec{\Sigma}$ points in its own positive sense.

This peculiar behavior is completely general: When the normal to a 3-volume is timelike, its volume vector $\vec{\Sigma}$ points in the negative sense; when the normal is spacelike, $\vec{\Sigma}$ points in the positive sense; and—it turns out—when the normal is null, $\vec{\Sigma}$ lies in the 3-volume (parallel to its one null leg) and thus points neither in the positive sense nor the negative.[6]

Note the physical interpretations of the 3-volumes of Fig. 2.10: That in Fig. 2.10a is an instantaneous snapshot of an ordinary, spatial, parallelopiped, while that in Fig. 2.10b is the 3-dimensional region in spacetime swept out during time Δt by the parallelogram with legs $\Delta y\vec{e}_y$, $\Delta z\vec{e}_z$ and with area $\Delta A = \Delta y\Delta z$.

Vectorial 3-volume elements can be used to construct integrals over 3-dimensional volumes (also called 3-dimensional surfaces) in spacetime, e.g. $\int_{\mathcal{V}_3} \vec{A}\cdot d\vec{\Sigma}$. More specifically: Let (a, b, c) be (possibly curvilinear) coordinates in the 3-surface (3-volume) $\mathcal{V}_3$, and denote by $\vec{x}(a, b, c)$ the spacetime point $\mathcal{P}$ on $\mathcal{V}_3$ whose coordinate values are (a, b, c). Then $(\partial\vec{x}/\partial a)da$, $(\partial\vec{x}/\partial b)db$, $(\partial\vec{x}/\partial c)dc$ are the vectorial legs of the elementary parallelopiped whose corners are at (a, b, c), $(a+da, b, c)$, $(a, b+db, c)$, etc; and the spacetime components of these vectorial legs are $(\partial x^\alpha/\partial a)da$, $(\partial x^\alpha/\partial b)db$, $(\partial x^\alpha/\partial c)dc$. The 3-volume of this elementary parallelopiped is $d\vec{\Sigma} = \boldsymbol{\epsilon}\big(\_, (\partial\vec{x}/\partial a)da, (\partial\vec{x}/\partial b)db, (\partial\vec{x}/\partial c)dc\big)$, which has spacetime components

$$d\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial a}\frac{\partial x^\beta}{\partial b}\frac{\partial x^\gamma}{\partial c}dadbdc\ . \tag{2.55}$$

This is the integration element to be used when evaluating

$$\int_{\mathcal{V}_3} \vec{A}\cdot d\vec{\Sigma} = \int_{\mathcal{V}_3} A^\mu d\Sigma_\mu\ . \tag{2.56}$$

See Ex. 2.22 for an example.

Just as there are Gauss and Stokes theorems (1.28a) and (1.28b) for integrals in Euclidean 3-space, so also there are Gauss and Stokes theorems in spacetime. The Gauss theorem has the obvious form

$$\boxed{\int_{\mathcal{V}_4}(\vec{\nabla}\cdot\vec{A})d\Sigma = \int_{\partial\mathcal{V}_4}\vec{A}\cdot d\vec{\Sigma}\ ,} \tag{2.57}$$

where the first integral is over a 4-dimensional region $\mathcal{V}_4$ in spacetime, and the second is over the 3-dimensional boundary $\partial\mathcal{V}_4$ of $\mathcal{V}_4$, with the boundary's positive sense pointing outward, away from $\mathcal{V}_4$ (just as in the 3-dimensional case). We shall not write down the 4-dimensional Stokes theorem because it is complicated to formulate with the tools we have developed thus far; easy formulation requires *differential forms,* which we shall not introduce in this book.

[6] This peculiar behavior gets replaced by a simpler description if one uses one-forms rather than vectors to describe 3-volumes; see, e.g., Box 5.2 of Misner, Thorne, and Wheeler (1973).

2.12.2 Conservation of Charge in Spacetime

We shall use integration over a 3-dimensional region in 4-dimensional spacetime to construct an elegant, frame-independent formulation of the law of conservation of electric charge:

We begin by examining the geometric meaning of the charge-current 4-vector $\vec{J}$. We defined $\vec{J}$ in Eq. (2.49) in terms of its components. The spatial component $J^x = J_x = J(\vec{e}_x)$ is equal to the x component of current density j_x; i.e. it is the amount Q of charge that flows across a unit surface area lying in the y-z plane, in a unit time; i.e., the charge that flows across the unit 3-surface $\vec{\Sigma} = \vec{e}_x$. In other words, *$\vec{J}(\vec{\Sigma}) = \vec{J}(\vec{e}_x)$ is the total charge Q that flows across $\vec{\Sigma} = \vec{e}_x$ in $\vec{\Sigma}$'s positive sense*; and similarly for the other spatial directions. The temporal component $J^0 = -J_0 = \vec{J}(-\vec{e}_0)$ is the charge density ρ_e; i.e., it is the total charge Q in a unit spatial volume. This charge is carried by particles that are traveling through spacetime from past to future, and pass through the unit 3-surface (3-volume) $\vec{\Sigma} = -\vec{e}_0$. Therefore, *$\vec{J}(\vec{\Sigma}) = \vec{J}(-\vec{e}_0)$ is the total charge Q that flows through $\vec{\Sigma} = -\vec{e}_0$ in its positive sense.* This is the same interpretation as we deduced for the spatial components of $\vec{J}$.

This makes it plausible, and indeed one can show, that *for any small 3-surface $\vec{\Sigma}$, $\vec{J}(\vec{\Sigma}) \equiv J^\alpha \Sigma_\alpha$ is the total charge Q that flows across $\vec{\Sigma}$ in its positive sense.*

This property of the charge-current 4-vector is the foundation for our frame-independent formulation of the law of charge conservation. Let $\mathcal{V}$ be a compact, 4-dimensional region of spacetime and denote by $\partial\mathcal{V}$ its boundary, a closed 3-surface in 4-dimensional spacetime (Fig. 2.11). The charged media (fluids, solids, particles, ...) present in spacetime carry electric charge through $\mathcal{V}$, from the past toward the future. The law of charge conservation says that all the charge that enters $\mathcal{V}$ through the past part of its boundary $\partial\mathcal{V}$ must exit through the future part of its boundary. If we choose the positive sense of the boundary's 3-volume element $d\vec{\Sigma}$ to point out of $\mathcal{V}$ (toward the past on the bottom boundary and toward the future on the top), then this *global law of charge conservation* can be expressed as

$$\boxed{\int_{\partial\mathcal{V}} J^\alpha d\Sigma_\alpha = 0\ .} \tag{2.58}$$

When each tiny charge q enters $\mathcal{V}$ through its past boundary, it contributes negatively to the integral, since it travels through $\partial\mathcal{V}$ in the negative sense (from positive side of $\partial\mathcal{V}$ toward negative side); and when that same charge exits $\mathcal{V}$ through its future boundary, it contributes positively. Therefore its net contribution is zero, and similarly for all other charges.

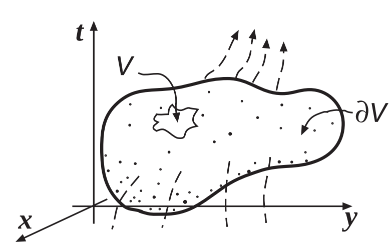

Fig. 2.11: The 4-dimensional region $\mathcal{V}$ in spacetime, and its closed 3-boundary $\partial\mathcal{V}$, used in formulating the law of charge conservation. The dashed lines symbolize, heuristically, the flow of charge from past toward future.

In Ex. 2.23 we show that, when this *global law of charge conservation* (2.58) is subjected to a 3+1 split of spacetime into space plus time, it becomes the nonrelativistic integral law of charge conservation (1.29).

This global conservation law can be converted into a *local conservation law* with the help of the 4-dimensional Gauss theorem (2.57), $\int_{\partial\mathcal{V}} J^\alpha d\Sigma_\alpha = \int_{\mathcal{V}} J^\alpha{}_{;\alpha} d\Sigma$. Since the left-hand side vanishes, so must the right-hand side; and in order for this 4-volume integral to vanish for every choice of $\mathcal{V}$, it is necessary that the integrand vanish everywhere in spacetime:

$$\boxed{J^\alpha{}_{;\alpha} = 0 \; ; \quad \text{i.e. } \vec{\nabla} \cdot \vec{J} = 0 \; .} \tag{2.59}$$

In a specific but arbitrary Lorentz frame (i.e., in a 3+1 split of spacetime into space plus time), this becomes the standard differential law of charge conservation (1.30).

2.12.3 Conservation of Particles, Baryons and Rest Mass

Any conserved scalar quantity obeys conservation laws of the same form as those for electric charge. For example, if the number of particles of some species (e.g. electrons or protons or photons) is conserved, then we can introduce for that species a *number-flux 4-vector* $\vec{S}$ (analog of charge-current 4-vector $\vec{J}$): In any Lorentz frame, S^0 is the number density of particles n and S^j is the particle flux. If $\vec{\Sigma}$ is a small 3-volume (3-surface) in spacetime, then $\vec{S}(\vec{\Sigma}) = S^\alpha \Sigma_\alpha$ is the number of particles that pass through Σ from its negative side to its positive side. The frame-invariant global and local conservation laws for these particles take the same form as those for electric charge:

$$\int_{\partial\mathcal{V}} S^\alpha d\Sigma_\alpha = 0, \quad \text{where } \partial\mathcal{V} \text{ is any closed 3-surface in spacetime,} \tag{2.60a}$$

$$S^\alpha{}_{;\alpha} = 0 \; ; \quad \text{i.e. } \vec{\nabla} \cdot \vec{S} = 0 \; . \tag{2.60b}$$

When fundamental particles (e.g. protons and antiprotons) are created and destroyed by quantum processes, the total baryon number (number of baryons minus number of antibaryons) is still conserved—or, at least this is so to the accuracy of all experiments performed thus far. We shall assume it so in this book. This law of baryon-number conservation takes the forms (2.60), with $\vec{S}$ the number-flux 4-vector for baryons (with antibaryons counted negatively).

It is useful to express this baryon-number conservation law in Newtonian-like language by introducing a universally agreed upon mean rest mass per baryon $\bar{m}_B$ This $\bar{m}_B$ is often taken to be 1/56 the mass of an ^{56}Fe (iron-56) atomic nucleus, since ^{56}Fe is the nucleus with the tightest nuclear binding, i.e. the endpoint of thermonuclear evolution in stars. We multiply the baryon number-flux 4-vector $\vec{S}$ by this mean rest mass per baryon to obtain a rest-mass-flux 4-vector

$$\vec{S}_{\rm rm} = \bar{m}_B \vec{S} \; , \tag{2.61}$$

which (since $\bar{m}_B$ is, by definition, a constant) satisfies the same conservation laws (2.60) as baryon number.

For media such as fluids and solids, in which the particles travel only short distances between collisions or strong interactions, it is often useful to resolve the particle number-flux 4-vector and the rest-mass-flux 4-vector into a 4-velocity of the medium $\vec{u}$ (i.e., the 4-velocity of the frame in which there is a vanishing net spatial flux of particles), and the particle number density n_o or rest mass density ρ_o as measured in the medium's rest frame:

$$\vec{S} = n_o \vec{u} \;, \quad \vec{S}_{\rm rm} = \rho_o \vec{u} \;. \tag{2.62}$$

See Ex. 2.24.

We shall make use of the conservation laws $\vec{\nabla} \cdot \vec{S} = 0$ and $\vec{\nabla} \cdot \vec{S}_{\rm rm} = 0$ for particles and rest mass later in this book, e.g. when studying relativistic fluids; and we shall find the expressions (2.62) for the number-flux 4-vector and rest-mass-flux 4-vector quite useful. See, e.g., the discussion of relativistic shock waves in Ex. 17.11.

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EXERCISES

Exercise 2.22 *Practice and Example: Evaluation of 3-Surface Integral in Spacetime*
In Minkowski spacetime the set of all events separated from the origin by a timelike interval a^2 is a 3-surface, the hyperboloid $t^2 - x^2 - y^2 - z^2 = a^2$, where $\{t, x, y, z\}$ are Lorentz coordinates of some inertial reference frame. On this hyperboloid, introduce coordinates $\{\chi, \theta, \phi\}$ such that

$$t = a\cosh\chi \;, \quad x = a\sinh\chi\sin\theta\cos\phi \;, \quad y = a\sinh\chi\sin\theta\sin\phi; \;, \quad z = a\sinh\chi\cos\theta \;. \tag{2.63}$$

Note that χ is a radial coordinate and (θ, ϕ) are spherical polar coordinates. Denote by $\mathcal{V}_3$ the portion of the hyperboloid with radius $\chi \le b$.

(a) Verify that for all values of (χ, θ, ϕ), the points (2.63) do lie on the hyperboloid.

(b) On a spacetime diagram, draw a picture of $\mathcal{V}_3$, the $\{\chi, \theta, \phi\}$ coordinates, and the elementary volume element (vector field) $d\vec{\Sigma}$ [Eq. (2.55)].

(c) Set $\vec{A} \equiv \vec{e}_0$ (the temporal basis vector), and express $\int_{\mathcal{V}_3} \vec{A} \cdot d\vec{\Sigma}$ as an integral over $\{\chi, \theta, \phi\}$. Evaluate the integral.

(d) Consider a closed 3-surface consisting of the segment $\mathcal{V}_3$ of the hyperboloid as its top, the hypercylinder $\{x^2 + y^2 + z^2 = a^2\sinh^2 b, \; 0 < t < a\cosh b\}$ as its sides, and the sphere $\{x^2 + y^2 + z^2 \le a^2\sinh^2 b \;, \; t = 0\}$ as its bottom. Draw a picture of this closed 3-surface on a spacetime diagram. Use Gauss's theorem, applied to this 3-surface, to show that $\int_{\mathcal{V}_3} \vec{A} \cdot d\vec{\Sigma}$ is equal to the 3-volume of its spherical base.

Exercise 2.23 *Derivation and Example: Global Law of Charge Conservation in an Inertial Frame*

Consider the global law of charge conservation $\int_{\partial V} J^\alpha d\Sigma_\alpha = 0$ for a special choice of the closed 3-surface $\partial\mathcal{V}$: The bottom of $\partial\mathcal{V}$ is the ball $\{t = 0, x^2 + y^2 + z^2 \leq a^2\}$, where $\{t, x, y, z\}$ are the Lorentz coordinates of some inertial frame. The sides are the spherical world tube $\{0 \leq t \leq T, x^2 + y^2 + z^2 = a^2\}$. The top is the ball $\{t = T, x^2 + y^2 + z^2 \leq a^2\}$.

(a) Draw this 3-surface in a spacetime diagram.

(b) Show that for this $\partial\mathcal{V}$, $\int_{\partial V} J^\alpha d\Sigma_\alpha = 0$ is a time integral of the nonrelativistic integral conservation law (1.29) for charge.

Exercise 2.24 *Example: Rest-mass-flux 4-vector, Lorentz contraction of rest-mass density, and rest-mass conservation for a fluid*
Consider a fluid with 4-velocity $\vec{u}$, and rest-mass density ρ_o as measured in the fluid's rest frame.

(a) From the physical meanings of $\vec{u}$, ρ_o, and the rest-mass-flux 4-vector $\vec{S}_{\rm rm}$, deduce Eq. (2.62).

(b) Examine the components of $\vec{S}_{\rm rm}$ in a reference frame where the fluid moves with ordinary velocity $\mathbf{v}$. Show that $S^0 = \rho_o\gamma$, $S^j = \rho_o\gamma v^j$, where $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$. Explain the physical interpretation of these formulas in terms of Lorentz contraction.

(c) Show that the law of conservation of rest-mass $\vec{\nabla} \cdot \vec{S}_{\rm rm} = 0$, takes the form

$$\frac{d\rho_o}{d\tau} = -\rho_o \vec{\nabla} \cdot \vec{u} \,, \tag{2.64}$$

where $d/d\tau$ is derivative with respect to proper time moving with the fluid.

(d) Consider a small 3-dimensional volume V of the fluid, whose walls move with the fluid (so if the fluid expands, V goes up). Explain why the law of rest-mass conservation must take the form $d(\rho_o V)/d\tau = 0$. Thereby deduce that

$$\vec{\nabla} \cdot \vec{u} = (1/V)(dV/d\tau) \,. \tag{2.65}$$

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2.13 The Stress-energy Tensor and Conservation of 4-Momentum

2.13.1 Stress-Energy Tensor

We conclude this chapter by formulating the law of 4-momentum conservation in ways analogous to our laws of conservation of charge, particles, baryons and rest mass. This task is not trivial, since 4-momentum is a vector in spacetime, while charge, particle number, baryon number, and rest mass are scalar quantities. Correspondingly, the density-flux of 4-momentum must have one more slot than the density-fluxes of charge, baryon number and rest mass, $\vec{J}$, $\vec{S}$ and $\vec{S}_{\rm rm}$; it must be a second-rank tensor. We call it the *stress-energy tensor* and denote it $\boldsymbol{T}(\_\,,\_)$.

Consider a medium or field flowing through 4-dimensional spacetime. As it crosses a tiny 3-surface $\vec{\Sigma}$, it transports a net electric charge $\vec{J}(\vec{\Sigma})$ from the negative side of $\vec{\Sigma}$ to the positive side, and net baryon number $\vec{S}(\vec{\Sigma})$ and net rest mass $\vec{S}_{\rm rm}(\vec{\Sigma})$; and similarly, it transports a net 4-momentum $\boldsymbol{T}(\_\,,\vec{\Sigma})$ from the negative side to the positive side:

$$\boldsymbol{T}(\_\,,\vec{\Sigma}) \equiv (\text{total 4-momentum } \vec{P} \text{ that flows through } \vec{\Sigma}); \quad \text{i.e., } T^{\alpha\beta}\Sigma_\beta = P^\alpha\,. \tag{2.66}$$

From this definition of the stress-energy tensor we can read off the physical meanings of its components on a specific, but arbitrary, Lorentz-coordinate basis: Making use of method (2.23b) for computing the components of a vector or tensor, we see that in a specific, but arbitrary, Lorentz frame (where $\vec{\Sigma} = -\vec{e}_0$ is a volume vector representing a parallelopiped with unit volume $\Delta V = 1$, at rest in that frame, with its positive sense toward the future):

$$\begin{aligned} -T_{\alpha 0} = \boldsymbol{T}(\vec{e}_\alpha, -\vec{e}_0) = \vec{P}(\vec{e}_\alpha) &= \begin{pmatrix} \alpha\text{-component of 4-momentum that} \\ \text{flows from past to future across a unit} \\ \text{volume } \Delta V = 1 \text{ in the 3-space } t = \text{const} \end{pmatrix} \\ &= (\alpha\text{-component of density of 4-momentum })\,. \end{aligned} \tag{2.67a}$$

Specializing α to be a time or space component and raising indices, we obtain the specialized versions of (2.67a)

$$\begin{aligned} T^{00} &= (\text{energy density as measured in the chosen Lorentz frame}), \\ T^{j0} &= (\text{density of } j\text{-component of momentum in that frame}). \end{aligned} \tag{2.67b}$$

Similarly, the αx component of the stress-energy tensor (also called the $\alpha 1$ component since $x = x^1$ and $\vec{e}_x = \vec{e}_1$) has the meaning

$$\begin{aligned} T_{\alpha 1} \equiv T_{\alpha x} \equiv \boldsymbol{T}(\vec{e}_\alpha, \vec{e}_x) &= \begin{pmatrix} \alpha\text{-component of 4-momentum that crosses} \\ \text{a unit area } \Delta y \Delta z = 1 \text{ lying in a surface of} \\ \text{constant } x, \text{ during unit time } \Delta t, \text{ crossing} \\ \text{from the } -x \text{ side toward the } +x \text{ side} \end{pmatrix} \\ &= \begin{pmatrix} \alpha \text{ component of flux of 4-momentum} \\ \text{across a surface lying perpendicular to } \vec{e}_x \end{pmatrix}\,. \end{aligned} \tag{2.67c}$$

The specific forms of this for temporal and spatial α are (after raising indices)

$$T^{0x} = \begin{pmatrix} \text{energy flux across a surface perpendicular to } \vec{e}_x, \\ \text{from the } -x \text{ side to the } +x \text{ side} \end{pmatrix}, \tag{2.67d}$$

$$T^{jx} = \begin{pmatrix} \text{flux of } j\text{-component of momentum across a surface} \\ \text{perpendicular to } \vec{e}_x, \text{ from the } -x \text{ side to the } +x \text{ side} \end{pmatrix} = \begin{pmatrix} jx \text{ component} \\ \text{of stress} \end{pmatrix}\,. \tag{2.67e}$$

The αy and αz components have the obvious, analogous interpretations.

These interpretations, restated much more briefly, are:

$$\boxed{T^{00} = (\text{energy density}),\ T^{j0} = (\text{momentum density}),\ T^{0j} = (\text{energy flux}),\ T^{jk} = (\text{stress}).} \tag{2.67f}$$

Although it might not be obvious at first sight, *the 4-dimensional stress-energy tensor is always symmetric*: in index notation (where indices can be thought of as representing the names of slots, or equally well components on an arbitrary basis)

$$T^{\alpha\beta} = T^{\beta\alpha} . \tag{2.68}$$

This symmetry can be deduced by physical arguments in a specific, but arbitrary, Lorentz frame: Consider, first, the $x0$ and $0x$ components, i.e., the x-components of momentum density and energy flux. A little thought, symbolized by the following heuristic equation, reveals that they must be equal

$$T^{x0} = \begin{pmatrix} \text{momentum} \\ \text{density} \end{pmatrix} = \frac{(\Delta\mathcal{E})dx/dt}{\Delta x \Delta y \Delta z} = \frac{\Delta\mathcal{E}}{\Delta y \Delta z \Delta t} = \begin{pmatrix} \text{energy} \\ \text{flux} \end{pmatrix} , \tag{2.69}$$

and similarly for the other space-time and time-space components: $T^{j0} = T^{0j}$. [In Eq. (2.69), in the first expression $\Delta\mathcal{E}$ is the total energy (or equivalently mass) in the volume $\Delta x \Delta y \Delta z$, $(\Delta\mathcal{E})dx/dt$ is the total momentum, and when divided by the volume we get the momentum density. The third equality is just elementary algebra, and the resulting expression is obviously the energy flux.] The space-space components, being equal to the stress tensor, are also symmetric, $T^{jk} = T^{kj}$, by the argument embodied in Fig. 1.6 above. Since $T^{0j} = T^{j0}$ and $T^{jk} = T^{kj}$, all components in our chosen Lorentz frame are symmetric, $T^{\alpha\beta} = T^{\beta\alpha}$. This means that, if we insert arbitrary vectors into the slots of $\boldsymbol{T}$ and evaluate the resulting number in our chosen Lorentz frame, we will find

$$\boldsymbol{T}(\vec{A}, \vec{B}) = T^{\alpha\beta} A_\alpha B_\beta = T^{\beta\alpha} A_\alpha B_\beta = \boldsymbol{T}(\vec{B}, \vec{A}) ; \tag{2.70}$$

i.e., $\boldsymbol{T}$ is symmetric under interchange of its slots.

Let us return to the physical meanings (2.67f) of the components of the stress-energy tensor. With the aid of $\boldsymbol{T}$'s symmetry, we can restate those meanings in the language of a 3+1 split of spacetime into space plus time: *When one chooses a specific reference frame, that choice splits the stress-energy tensor up into three parts. Its time-time part is the energy density T^{00}, Its time-space part $T^{0j} = T^{j0}$ is the energy flux or equivalently the momentum density, and its space-space part T^{jk} is the symmetric stress tensor.*

2.13.2 4-Momentum Conservation

Our interpretation of $\vec{J}(\vec{\Sigma}) \equiv J^\alpha \Sigma_\alpha$ as the net charge that flows through a small 3-surface $\vec{\Sigma}$ from its negative side to its positive side gave rise to the global conservation law for charge, $\int_{\partial\mathcal{V}} J^\alpha d\Sigma_\alpha = 0$ [Eqs. (2.58) and Fig. 2.11]. Similarly the role of $\boldsymbol{T}(\_\,, \vec{\Sigma})$ [$T^{\alpha\beta}\Sigma_\beta$ in slot naming index notation] as the net 4-momentum that flows through $\vec{\Sigma}$ from its negative side to positive gives rise to the following equation for conservation of 4-momentum:

$$\boxed{\int_{\partial\mathcal{V}} T^{\alpha\beta} d\Sigma_\beta = 0 \ .} \tag{2.71}$$

This equation says that all the 4-momentum that flows into the 4-volume $\mathcal{V}$ of Fig. 2.11 through its 3-surface $\partial\mathcal{V}$ must also leave $\mathcal{V}$ through $\partial\mathcal{V}$; it gets counted negatively when it enters (since it is traveling from the positive side of $\partial\mathcal{V}$ to the negative), and it gets counted positively when it leaves, so its net contribution to the integral (2.71) is zero.

This *global law of 4-momentum conservation* can be converted into a *local law* (analogous to $\vec{\boldsymbol{\nabla}} \cdot \vec{J} = 0$ for charge) with the help of the 4-dimensional Gauss's theorem (2.57). Gauss's theorem, generalized in the obvious way from a vectorial integrand to a tensorial one, says:

$$\int_{\mathcal{V}} T^{\alpha\beta}{}_{;\beta}\, d\Sigma \ = \int_{\partial\mathcal{V}} T^{\alpha\beta} d\Sigma_\beta \ . \tag{2.72}$$

Since the right-hand side vanishes, so must the left-hand side; and in order for this 4-volume integral to vanish for every choice of $\mathcal{V}$, the integrand must vanish everywhere in spacetime:

$$\boxed{T^{\alpha\beta}{}_{;\beta} = 0 \ ; \quad \text{i.e.} \ \ \vec{\nabla} \cdot \boldsymbol{T} = 0 \ .} \tag{2.73a}$$

In the second, index-free version of this local conservation law, the ambiguity about which slot the divergence is taken on is unimportant, since $\boldsymbol{T}$ is symmetric in its two slots: $T^{\alpha\beta}{}_{;\beta} = T^{\beta\alpha}{}_{;\beta}$.

In a specific but arbitrary Lorentz frame, the local conservation law (2.73a) for 4-momentum has as its temporal and spatial parts

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0k}}{\partial x^k} = 0 \ , \tag{2.73b}$$

i.e., the time derivative of the energy density plus the 3-divergence of the energy flux vanishes; and

$$\frac{\partial T^{j0}}{\partial t} + \frac{\partial T^{jk}}{\partial x^k} = 0 \ , \tag{2.73c}$$

i.e., the time derivative of the momentum density plus the 3-divergence of the stress (i.e., of momentum flux) vanishes. Thus, as one should expect, the geometric, frame-independent law of 4-momentum conservation includes as special cases both the conservation of energy and the conservation of momentum; and their differential conservation laws have the standard form that one expects both in Newtonian physics and in special relativity: time derivative of density plus divergence of flux vanishes; cf. Eq. (1.36) and associated discussion.

2.13.3 Stress-Energy Tensors for Perfect Fluid and Electromagnetic Field

As an important example that illustrates the stress-energy tensor, consider a *perfect fluid* — i.e., a medium whose stress-energy tensor, evaluated in its *local rest frame* (a Lorentz frame where $T^{j0} = T^{0j} = 0$), has the form

$$T^{00} = \rho \ , \quad T^{jk} = P\delta^{jk} \ . \tag{2.74a}$$

[Eq. (1.33) and associated discussion]. Here ρ is a short-hand notation for the energy density T^{00} (density of total mass-energy, including rest mass), as measured in the local rest frame; and the stress tensor T^{jk} in that frame is an isotropic pressure P. From this special form of $T^{\alpha\beta}$ in the local rest frame, one can derive the following geometric, frame-independent expression for the stress-energy tensor in terms of the 4-velocity $\vec{u}$ of the local rest frame, i.e., of the fluid itself, the metric tensor of spacetime $\boldsymbol{g}$, and the rest-frame energy density ρ and pressure P:

$$\boxed{T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta + P g^{\alpha\beta} \; ; \quad \text{i.e., } \boldsymbol{T} = (\rho + P)\vec{u} \otimes \vec{u} + P\boldsymbol{g} \; .} \tag{2.74b}$$

See Ex. 2.26, below.

In Sec. 13.8, we shall develop and explore the laws of relativistic fluid dynamics that follow from energy-momentum conservation $\vec{\nabla} \cdot \boldsymbol{T} = 0$ for this stress-energy tensor and from rest-mass conservation $\vec{\nabla} \cdot \vec{S}_{\rm rm} = 0$. By constructing the Newtonian limit of the relativistic laws, we shall deduce the nonrelativistic laws of fluid mechanics, which are the central theme of Part V. Notice, in particular, that the Newtonian limit ($P \ll \rho$, $u^0 \simeq 1$, $u^j \simeq v^j$) of the stress part of the stress-energy tensor (2.74b) is $T^{jk} = \rho v^j v^k + P\delta^{jk}$, which we met in Ex. 1.13.

Another example of a stress-energy tensor is that for the electromagnetic field, which takes the following form:

$$T^{\alpha\beta} = \frac{1}{4\pi}\left(F^{\alpha\mu}F^{\beta}{}_{\mu} - \frac{1}{4}g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right) \; . \tag{2.75}$$

We shall explore this stress-energy tensor in Ex. 2.28.

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EXERCISES

Exercise 2.25 *Example: Global Conservation of 4-Momentum in an Inertial Frame*
Consider the 4-dimensional parallelopiped $\mathcal{V}$ whose legs are $\Delta t\vec{e}_t$, $\Delta x\vec{e}_x$, $\Delta y\vec{e}_y$ $\Delta z\vec{e}_z$, where $(t, x, y, z) = (x^0, x^1, x^2, x^3)$ are the coordinates of some inertial frame. The boundary $\partial\mathcal{V}$ of this $\mathcal{V}$ has eight 3-dimensional "faces". Identify these faces, and write the integral $\int_{\partial\mathcal{V}} T^{0\beta}d\Sigma_\beta$ as the sum of contributions from each of them. According to the law of energy conservation, this sum must vanish. Explain the physical interpretation of each of the eight contributions to this energy conservation law. (See Ex. 2.23 for an analogous interpretation of charge conservation.)

Exercise 2.26 ***Derivation and Example: Stress-Energy Tensor and Energy-Momentum Conservation for a Perfect Fluid*

(a) Derive the frame-independent expression (2.74b) for the perfect fluid stress-energy tensor from its rest-frame components (2.74a).

(b) Explain why the projection of $\vec{\nabla} \cdot \boldsymbol{T} = 0$ along the fluid 4-velocity, $\vec{u} \cdot (\vec{\nabla} \cdot \boldsymbol{T}) = 0$, should represent energy conservation as viewed by the fluid itself. Show that this equation reduces to

$$\frac{d\rho}{d\tau} = -(\rho + P)\vec{\nabla} \cdot \vec{u} \,. \tag{2.76a}$$

With the aid of Eq. (2.65), bring this into the form

$$\frac{d(\rho V)}{d\tau} = -P\frac{dV}{d\tau} \,, \tag{2.76b}$$

where V is the 3-volume of some small fluid element as measured in the fluid's local rest frame. What are the physical interpretations of the left and right sides of this equation, and how is it related to the first law of thermodynamics?

(c) Read the discussion, in Ex. 2.10, of the tensor $\boldsymbol{P} = \boldsymbol{g} + \vec{u} \otimes \vec{u}$ that projects into the 3-space of the fluid's rest frame. Explain why $P_{\mu\alpha} T^{\alpha\beta}{}_{;\beta} = 0$ should represent the law of force balance (momentum conservation) as seen by the fluid. Show that this equation reduces to

$$(\rho + P)\vec{a} = -\boldsymbol{P} \cdot \vec{\nabla} P \,, \tag{2.76c}$$

where $\vec{a} = d\vec{u}/d\tau$ is the fluid's 4-acceleration. This equation is a relativistic version of Newton's "$\mathbf{F} = m\mathbf{a}$". Explain the physical meanings of the left and right hand sides. Infer that $\rho + P$ must be the fluid's inertial mass per unit volume.

Exercise 2.27 ***Example: Inertial Mass Per Unit Volume*

Suppose that some medium has a rest frame (unprimed frame) in which its energy flux and momentum density vanish, $T^{0j} = T^{j0} = 0$. Suppose that the medium moves in the x direction with speed very small compared to light, $v \ll 1$, as seen in a (primed) laboratory frame, and ignore factors of order v^2. The "ratio" of the medium's momentum density $\mathcal{G}_{j'} = T^{j'0'}$ as measured in the laboratory frame to its velocity $v_i = v\delta_{ix}$ is called its total *inertial mass per unit volume*, and is denoted $\rho^{\rm inert}_{ji}$:

$$T^{j'0'} = \rho^{\rm inert}_{ji} v_i \,. \tag{2.77}$$

In other words, $\rho^{\rm inert}_{ji}$ is the 3-dimensional tensor that gives the momentum density $\mathcal{G}_{j'}$ when the medium's small velocity is put into its second slot.

(a) Show, using a Lorentz transformation from the medium's (unprimed) rest frame to the (primed) laboratory frame, that

$$\rho^{\rm inert}_{ji} = T^{00}\delta_{ji} \; + \; T_{ji} \,. \tag{2.78}$$

(b) Give a physical explanation of the contribution $T_{ji} v_i$ to the momentum density.

(c) Show that for a perfect fluid [Eq. (2.74b)] the inertial mass per unit volume is isotropic and has magnitude $\rho + P$, where ρ is the mass-energy density and P is the pressure measured in the fluid's rest frame:

$$\boxed{\rho_{ji}^{\rm inert} = (\rho + P)\delta_{ji}\ .} \tag{2.79}$$

See Ex. 2.26 above for this inertial-mass role of $\rho + P$ in the law of force balance.

Exercise 2.28 ***Example: Stress-Energy Tensor, and Energy-Momentum Conservation for the Electromagnetic Field*

(a) Compute from Eqs. (2.75) and (2.45) the components of the electromagnetic stress-energy tensor in an inertial reference frame (in Gaussian units). Your answer should be the expressions given in electrodynamics textbooks:

$$\begin{aligned} T^{00} &= \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}\ , \quad \boldsymbol{\mathcal{G}} = T^{0j}\mathbf{e}_j = T^{j0}\mathbf{e}_j = \frac{\mathbf{E}\times\mathbf{B}}{4\pi}\ , \\ T^{jk} &= \frac{1}{8\pi}\left[(\mathbf{E}^2 + \mathbf{B}^2)\delta_{jk} - 2(E_jE_k + B_jB_k)\right]\ . \end{aligned} \tag{2.80}$$

See also Ex. 1.14 above for an alternative derivation of the stress tensor T_{jk}.

(b) Show that the divergence of the stress-energy tensor (2.75) is given by

$$T^{\mu\nu}{}_{;\nu} = \frac{1}{4\pi}(F^{\mu\alpha}{}_{;\nu}F^{\nu}{}_{\alpha} + F^{\mu\alpha}F^{\nu}{}_{\alpha;\nu} - \frac{1}{2}F_{\alpha\beta}{}^{;\mu}F^{\alpha\beta})\ . \tag{2.81a}$$

(c) Combine this with the Maxwell equations (2.48) to show that

$$\boldsymbol{\nabla}\cdot\boldsymbol{T} = -\boldsymbol{F}(\_\,,\vec{J})\ ; \quad \text{i.e., } T^{\alpha\beta}{}_{;\beta} = -F^{\alpha\beta}J_\beta\ . \tag{2.81b}$$

(c) The matter that carries the electric charge and current can exchange energy and momentum with the electromagnetic field. Explain why Eq. (2.81b) is the rate per unit volume at which that matter feeds 4-momentum into the electromagnetic field, and conversely, $+F^{\alpha\mu}J_\mu$ is the rate per unit volume at which the electromagnetic field feeds 4-momentum into the matter. Show, further, that (as viewed in any reference frame) the time and space components of this quantity are

$$\frac{d\mathcal{E}_{\rm matter}}{dtdV} = -F^{0j}J_j = \mathbf{E}\cdot\mathbf{j}\ , \quad \frac{d\mathbf{p}_{\rm matter}}{dtdV} = \rho_e\mathbf{E} + \mathbf{j}\times\mathbf{B}\ , \tag{2.81c}$$

where ρ_e is charge density and $\mathbf{j}$ is current density [Eq. (2.49)]. The first of these equations is ohmic heating of the matter by the electric field; the second is the Lorentz force per unit volume on the matter; cf. Ex. 1.14b.

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Box 2.5
Important Concepts in Chapter 2

- **Foundational Concepts**
  - Inertial reference frame, Sec. 2.2.1.
  - Events, and 4-vectors as arrows between events, Sec. 2.2.1
  - Invariant interval and how it defines the geometry of spacetime, Sec. 2.2.2.
- **Principle of Relativity:** Laws of physics are frame-independent geometric relations between geometric objects (same as Geometric Principal for physical laws in Newtonian physics), Sec. 2.2.2. Important examples:
  - Relativistic particle kinetics, Sec. 2.4.1.
  - Lorentz force law (2.20) in terms of the electromagnetic field tensor $\boldsymbol{F}$, and its connection to the 3-dimensional version in terms of $\mathbf{E}$ and $\mathbf{B}$, Sec. 2.11.
  - Conservation of 4-momentum in particle interactions, Eq. (2.15).
  - Global and local conservation laws for charge, baryon number, and 4-momentum, Secs. 2.12.2, 2.12.3, 2.13.2.
- **Differential geometry**
  - Tensor as a linear function of vectors, Sec. 2.3. Important examples: metric tensor (2.6), Levi-Civita tensor (2.43), Electromagnetic field tensor (2.18) and stress-energy tensor (2.66).
  - Slot-naming index notation, end of Sec. 2.5; all of Sec. 1.5.1.
  - Differentiation and integration of tensors, Secs. 2.10 and 2.12.1.
  - Gauss's theorem in Minkowski spacetime (2.57).
  - Geometric computations without coordinates or Lorentz transformations (e.g. derive Lorentz force law, Ex. 2.4.2; derive Dopper shift, Ex. 2.11).
  - Lorentz transformations, Sec. 2.7.
- **3+1 Split of spacetime into space plus time** induced by choice of inertial frame, Sec. 2.6; resulting 3+1 split of physical quantities and laws:
  - 4-momentum $\rightarrow$ energy and momentum, Eqs. (2.26), (2.27), (2.29); Ex. 2.9.
  - Electromagnetic tensor $\rightarrow$ electric field and magnetic field, Sec. 2.11.
  - Charge-current 4-vector $\rightarrow$ charge density and current density, Ex. 2.19.
  - 3-vectors as 4-vectors living in observer's 3-surface of simultaneity, Sec. 2.11 and Fig. 2.9.
- **Spacetime diagrams**, Secs. 2.2.1 and 2.8; used to understand Lorentz contraction, time dilation, simultaneity breakdown (Ex. 2.14) and conservation laws (Fig. 2.11).

Bibliographic Note

For an inspiring taste of the history of special relativity, see the original papers by Einstein, Lorentz, and Minkowski, translated into English and archived in Einstein et. al. (1923).

Early relativity textbooks [see the bibliography on pp. 566–567 of Jackson (1999)] emphasized the transformation properties of physical quantities, in going from one inertial frame to another, rather than their roles as frame-invariant geometric objects. Minkowski (1908) introduced geometric thinking, but only in recent decades — in large measure due to the influence of John Wheeler — has the geometric viewpoint gained ascendancy.

In our opinion, the best elementary introduction to special relativity is the first edition of Taylor and Wheeler (1966); the more ponderous second edition (1992) is also good. At an intermediate level we strongly recommend the special relativity portions of Hartle (2003).

At a more advanced level, comparable to this chapter, we recommend Goldstein, Poole and Safko (2002) and the special relativity sections of Misner, Thorne and Wheeler (1973) and of Carroll (2004) and Schutz (2009). These all adopt the geometric viewpoint that we espouse. In this chapter, so far as possible, we have minimized the proliferation of mathematical concepts (avoiding, e.g., differential forms and dual bases). By contrast, the other advanced treatments, cited above, embrace the richer mathematics.

Much less geometric than the above references but still good, in our view, are the special relativity sections of popular electrodynamics texts: Griffiths (1999) at an intermediate level, and Jackson (1999) at a more advanced level. We recommend avoiding special relativity treatments that use imaginary time and thereby obfuscate — e.g. earlier editions of Goldstein and of Jackson.

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