# Contents

# Chapter 12

# Elastodynamics

Version 1212.1.K, 7 Sept 2012

*Please send comments, suggestions, and errata via email to kip@caltech.edu, or on paper to Kip Thorne, 350-17 Caltech, Pasadena CA 91125*

**Box 12.1**
**Reader's Guide**

- This chapter is a companion to Chap. 10 (Elastostatics) and relies heavily on it.
- This chapter also relies rather heavily on geometric-optics concepts and formalism, as developed in Secs. 7.2 and 7.3, especially: phase velocity, group velocity, dispersion relation, rays and the propagation of waves, information and energy along them, the role of the dispersion relation as a Hamiltonian for the rays, and ray tracing.
- The discussion of continuum-mechanics wave equations in Box 12.2 underlies this book's treatment of waves in fluids (Part IV), especially in Plasmas (Part V), and in general relativity (Part VI).
- The experience that the reader gains in this chapter with waves in solids will be useful when we encounter much more complicated waves in plasmas in Part V.
- No other portions of this chapter are of great importance for subsequent Parts of this book.

## 12.1 Overview

In the previous chapter we considered elastostatic equilibria in which the forces acting on elements of an elastic solid were balanced so that the solid remained at rest. When this equilibrium is disturbed, the solid will undergo accelerations. This is the subject of this chapter — Elastodynamics.

In Sec. 12.2, we derive the equations of motion for elastic media, paying particular attention to the underlying conservation laws and focusing especially on elastodynamic waves. We show that there are two distinct wave modes that propagate in a uniform, isotropic solid, *longitudinal* waves and *shear* waves, and both are nondispersive (their phase speeds are independent of frequency).

A major use of elastodynamics is in structural engineering, where one encounters vibrations (usually standing waves) on the beams that support buildings and bridges. In Sec. 12.3 we discuss the types of waves that propagate on bars, rods and beams and find that the boundary conditions at the free transverse surfaces make the waves dispersive. We also return briefly to the problem of bifurcation of equilibria (treated in Sec. 11.8) and show how, by changing the parameters controlling an equilibrium, a linear wave can be made to grow exponentially in time, thereby rendering the equilibrium unstable.

A second application of elastodynamics is to seismology (Sec. 12.4). The earth is mostly a solid body through which waves can propagate. The waves can be excited naturally by earthquakes or artificially using man-made explosions. Understanding how waves propagate through the earth is important for locating the sources of earthquakes, for diagnosing the nature of an explosion (was it an illicit nuclear bomb test?) and for analyzing the structure of the earth. We briefly describe some of the wave modes that propagate through the earth and some of the inferences about the earth's structure that have been drawn from studying their propagation. In the process, we gain some experience in applying the tools of geometric optics to new types of waves, and we learn how rich can be the Green's function for elastodynamic waves, even when the medium is as simple as a homogeneous half space.

Finally (Sec. 12.5), we return to physics to consider the quantum theory of elastodynamic waves. We compare the classical theory with the quantum theory, specializing to quantised vibrations in an elastic solid: phonons.

## 12.2 Basic Equations of Elastodynamics; Waves in a Homogeneous Medium

In subsection 12.2.1 of this section, we shall derive a vectorial equation that governs the dynamical displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ of a dynamically disturbed elastic medium. We shall then specialize to monochromatic plane waves in a homogeneous medium (Subsec. 12.2.2) and shall show how the monochromatic plane-wave equation can be converted into two wave equations, one for "longitudinal" waves (Subsec. 12.2.3) and the other for "transverse" waves (Subsec. 12.2.4). From those two wave equations we shall deduce the waves' dispersion relations, which act as Hamiltonians for geometric-optics wave propagation through inhomogeneous media. Our method of analysis is a special case of a very general approach to deriving wave equations in continuum mechanics. That general approach is sketched in Box 12.2. We shall follow that approach not only here, for elastic waves, but also in Part IV for waves in fluids, Part V for waves in plasmas and Part VI for general relativistic gravitational waves. We shall conclude this section in Subsec. 12.2.5 with a discussion of the energy density and energy flux of these waves, and in Ex. 12.4 we shall explore the relationship of this energy density and flux to a Lagrangian for elastodynamic waves.

### 12.2.1 Equation of Motion for a Strained Elastic Medium

In Chap. 10, we learned that, when an elastic medium undergoes a displacement $\boldsymbol{\xi}(\mathbf{x})$, it builds up a strain $\mathbf{S} = \nabla\boldsymbol{\xi}$, which in turn produces an internal stress $\mathbf{T} = -K\Theta\mathbf{g} - 2\mu\boldsymbol{\Sigma}$, where $\Theta \equiv \nabla\cdot\boldsymbol{\xi}$ is the expansion and $\boldsymbol{\Sigma} \equiv$(the symmetric trace-free part of $\mathbf{S}$) is the shear; see Eqs. (11.5) and (11.18). The stress $\mathbf{T}$ produces an elastic force per unit volume

$$\mathbf{f} = -\nabla\cdot\mathbf{T} = \left(K + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi} \tag{12.1}$$

[Eq. (11.19)], where $K$ and $\mu$ are the bulk and shear moduli.

In Chap. 10, we restricted ourselves to elastic media that are in elastostatic equilibrium, so they are static. This equilibrium required that the net force per unit volume acting on the medium vanish. If the only force is elastic, then $\mathbf{f}$ must vanish. If the pull of gravity is also significant, then $\mathbf{f} + \rho\boldsymbol{g}$ vanishes, where $\rho$ is the medium's mass density and $\boldsymbol{g}$ the acceleration of gravity.

In this chapter we shall focus on dynamical situations, in which an unbalanced force per unit volume causes the medium to move — with the motion, in this chapter, taking the form of an elastodynamic wave. For simplicity, we shall assume that the only significant force is elastic; i.e., that the gravitational force is negligible by comparison. In Ex. 12.2 we shall show that this is the case for elastodynamic waves in most media on Earth whenever the wave frequency $\omega/2\pi$ is higher than about 0.001 Hz (which is usually the case in practice). Stated more precisely, in a homogeneous medium we can ignore the gravitational force whenever the elastodynamic wave's angular frequency $\omega$ is much larger than $g/c$, where $g$ is the acceleration of gravity and $c$ is the wave's propagation speed.

Consider, then, a dynamical, strained medium with elastic force per unit volume (12.1) and no other significant force (negligible gravity), and with velocity

$$\boxed{\mathbf{v} = \frac{\partial\boldsymbol{\xi}}{\partial t}\ .} \tag{12.2a}$$

The law of momentum conservation states that the force per unit volume $\mathbf{f}$, if nonzero, must produce a rate of change of momentum per unit volume $\rho\mathbf{v}$ according to the equation[1]

$$\frac{\partial(\rho\mathbf{v})}{\partial t} = \mathbf{f} = -\nabla\cdot\mathbf{T} = \left(K + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi}\ . \tag{12.2b}$$

[1] In Sec. 13.5 of the next chapter we shall learn that the motion of the medium produces a stress $\rho\mathbf{v}\otimes\mathbf{v}$ that must be included in this equation if the velocities are large. However, this subtle dynamical stress is always negligible in elastodynamic waves because the displacements and hence velocities $\mathbf{v}$ are tiny and $\rho\mathbf{v}\otimes\mathbf{v}$ is second order in the displacement. For this reason we shall delay studying this subtle nonlinear effect until Chap. 12.

Notice that when rewritten in the form

$$\boxed{\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} = 0\ ,}$$

this is the version of the law of momentum conservation discussed in Chap. 1 [Eq. (2.40)], and it has the standard form for a conservation law (time derivative of density of something, plus divergence of flux of that something, vanishes; Sec. ??); $\rho\mathbf{v}$ is the density of momentum, and the stress tensor $\mathbf{T}$ is by definition the flux of momentum. Equations (12.2a) and (12.2b), together with the law of mass conservation [the obvious analog of Eqs. (1.30) for conservation of charge and particle number],

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0} \tag{12.2c}$$

are a complete set of equations for the evolution of the displacement $\boldsymbol{\xi}(\mathbf{x},t)$, the velocity $\mathbf{v}(\mathbf{x},t)$ and the density $\rho(\mathbf{x},t)$.

The elastodynamic equations (12.2) are nonlinear because of the $\rho\mathbf{v}$ terms (see below). From them we shall derive a linear wave equation for the displacement vector $\boldsymbol{\xi}(\mathbf{x},t)$. Our derivation provides us with a simple (almost trivial) example of the general procedure discussed in Box 12.2.

To derive a linear wave equation, we must find some small parameter in which to expand. The obvious choice in elastodynamics is the strain $\mathbf{S} = \boldsymbol{\nabla}\boldsymbol{\xi}$ and its components, which are all dimensionless and must be less than about $10^{-3}$ to remain within the non-yielding, non-breaking, linear elastic regime (Sec. 11.2.1). Equally well, we can regard the displacement $\boldsymbol{\xi}$ itself as our small parameter.

If the medium's equilibrium state were homogeneous, the linearization would be trivial. However, we wish to be able to treat perturbations of inhomogeneous equilibria such as seismic waves in the Earth, or perturbations of slowly changing equilibria such as vibrations of a pipe or mirror that is gradually changing temperature. In almost all situations the lengthscale $\mathcal{L}$ and timescale $\mathcal{T}$ on which the medium's equilibrium properties ($\rho$, $K$, $\mu$) vary are extremely large compared to the lengthscale and timescale of the dynamical perturbations (their reduced wavelength $\lambda\!\!\!^{-}$ =wavelength/$2\pi$ and $1/\omega$ =period/$2\pi$). This permits us to perform a two-lengthscale expansion (like the one that underlies geometric optics, Sec. 7.3) alongside our small-strain expansion.

In analyzing a dynamical perturbation of an equilibrium state, we use $\boldsymbol{\xi}(\mathbf{x},t)$ to denote the dynamical displacement (i.e., we omit from it the equilibrium's static displacement, and similarly we omit from $\boldsymbol{\nabla}\boldsymbol{\xi}$ the equilibrium strain). We write the density as $\rho+\delta\rho$, where $\rho(\mathbf{x})$ is the equilibrium density distribution and $\delta\rho(\mathbf{x},t)$ is the dynamical density perturbation, which is first-order in the dynamical displacement $\boldsymbol{\xi}$. Inserting these into the equation of mass conservation (12.2c), we obtain $\partial\delta\rho/\partial t + \boldsymbol{\nabla}\cdot[(\rho+\delta\rho)\mathbf{v}] = 0$. Because $\mathbf{v} = \partial\boldsymbol{\xi}/\partial t$ is first order, the term $(\delta\rho)\mathbf{v}$ is second order and can be dropped, resulting in the linearized equation $\partial\delta\rho/\partial t + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$. Because $\rho$ varies on a much longer lengthscale than $\mathbf{v}$ ($\mathcal{L}$ vs. $\lambda\!\!\!^{-}$), we can pull $\rho$ out of the derivative; setting $\mathbf{v} = \partial\boldsymbol{\xi}/\partial t$ and interchanging the time derivative and divergence, we then obtain $\partial\delta\rho/\partial t + \rho\partial(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})/\partial t = 0$. Noting that $\rho$ varies

**Box 12.2**
**Wave Equations in Continuum Mechanics**

In this box, we make an investment for future chapters by considering wave equations in some generality.

Most wave equations arise as approximations to the full set of equations that govern a dynamical physical system. It is usually possible to arrange those full equations as a set of first order partial differential equations that describe the dynamical evolution of a set of $n$ physical quantities, $V_A$, with $A = 1, 2, ..., n$: i.e.

$$\frac{\partial V_A}{\partial t} + F_A(V_B) = 0 \ . \tag{1}$$

[For elastodynamics there are $n = 7$ quantities $V_A$: $\{\rho, \rho v_x, \rho v_y, \rho v_z, \xi_x, \xi_y, \xi_z\}$ (in Cartesian coordinates); and the seven equations (1) are mass conservation, momentum conservation, and $\partial \xi_j / \partial t = v_j$; Eqs. (12.2).]

Now, most dynamical systems are intrinsically nonlinear (Maxwell's equations *in vacuo* being a conspicuous exception) and it is usually quite hard to find nonlinear solutions. However, it is generally possible to make a perturbation expansion in some small physical quantity about a time-independent equilibrium and just retain terms that are linear in this quantity. We then have a set of $n$ linear partial differential equations that are much easier to solve than the nonlinear ones—and that usually turn out to have the character of wave equations (i.e., to be "hyperbolic"). Of course the solutions will only be a good approximation for small amplitude waves. [In elastodynamics, we justify linearization by requiring that the strains be below the elastic limit, we linearize in the strain or displacement of the dynamical perturbation, and the resulting linear wave equation is $\rho \partial^2 \boldsymbol{\xi} / \partial t^2 = (K + \frac{1}{3}\mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) + \mu \nabla^2 \boldsymbol{\xi}$; Eq. (12.4b).]

*Boundary Conditions*

In some problems, e.g. determining the normal modes of vibration of a building during an earthquake, or analyzing the sound from a violin or the vibrations of a finite-length rod, the boundary conditions are intricate and have to be incorporated as well as possible, to have any hope of modeling the problem. The situation is rather similar to that familiar from elementary quantum mechanics. The waves are often localised within some region of space, like bound states, in such a way that the eigenfrequencies are discrete, for example, standing wave modes of a plucked string. In other problems the volume in which the wave propagates is essentially infinite, as happens with unbound states (e.g. waves on the surface of the ocean or seismic waves propagating through the earth). Then the only boundary condition is essentially that the wave amplitude remain finite at large distances. In this case, the wave spectrum is usually continuous.

**Box 12.2, Continued**

*Geometric Optics Limit and Dispersion Relations*

The solutions to the wave equation will reflect the properties of the medium through which the wave is propagating, as well as its boundaries. If the medium and boundaries have a finite number of discontinuities but are otherwise smoothly varying, there is a simple limiting case: waves of short enough wavelength and high enough frequency that they can be analyzed in the *geometric optics approximation* (Chap. 6).

The key to geometric optics is the *dispersion relation*, which (as we learned in Sec. 7.3) acts as a Hamiltonian for the propagation. Recall from Chap. 6 that, although the medium may actually be inhomogeneous and might even be changing with time, when deriving the dispersion relation we can approximate it as precisely homogeneous and time-independent, and can resolve the waves into plane-wave modes, i.e. modes in which the perturbations vary $\propto \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Here $\mathbf{k}$ is the *wave vector* and $\omega$ is the *angular frequency*. This allows us to remove all the temporal and spatial derivatives and converts our set of partial differential equations into a set of homogeneous, linear algebraic equations. When we do this, we say that our normal modes are *local*. If, instead, we were to go to the trouble of solving the partial differential wave equation with its attendant boundary conditions, the modes would be referred to as *global*.

The linear algebraic equations for a local problem can be written in the form $M_{AB}V_B = 0$, where $V_A$ is the vector of $n$ dependent variables and the elements $M_{AB}$ of the $n \times n$ matrix $||M_{AB}||$ depend on $\mathbf{k}$ and $\omega$ as well as on parameters $p_\alpha$ that describe the local conditions of the medium. This set of equations can be solved in the usual manner by requiring that the determinant of $\|M_{AB}\|$ vanish. Carrying through this procedure yields a polynomial, usually of $n$'th order, for $\omega(\mathbf{k}, p_\alpha)$. This polynomial is the dispersion relation. It can be solved (analytically in simple cases and numerically in general) to yield a number of complex solutions for $\omega$, with $\mathbf{k}$ regarded as real. (Of course, we might just as well treat the wave vector as a complex number, but for the moment we will regard it as real.) Armed with these solutions, we can solve for the associated eigenvectors. The eigenfrequencies fully characterize the solution of the local problem, and can be used to solve for the waves' temporal evolution from some given initial conditions in the usual manner. (As we shall see several times, especially when we discuss Landau damping in Chap. 21, there are some subtleties that can arise.)

What does a complex value of the angular frequency $\omega$ mean? We have posited that all small quantities vary $\propto \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. If $\omega$ has a positive imaginary part, then the small perturbation quantities will grow exponentially with time. Conversely, if it has a negative imaginary part, they will decay. Now, polynomial equations with real coefficients have complex conjugate solutions. Therefore if there is a decaying mode there must also be a growing mode. Growing modes correspond to *instability*, a topic that we shall encounter often.

on a timescale $\mathcal{T}$ long compared to that $1/\omega$ of $\boldsymbol{\xi}$ and $\delta\rho$, we can integrate this to obtain the linear relation

$$\boxed{\frac{\delta\rho}{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\ .} \tag{12.3}$$

This linearized equation for the fractional perturbation of density could equally well have been derived by considering a small volume $V$ of the medium that contains mass $M = \rho V$, and by noting that the dynamical perturbations lead to a volume change $\delta V/V = \Theta = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ [Eq. (11.8)], so conservation of mass requires $0 = \delta M = \delta(\rho V) = V\delta\rho + \rho\delta V = V\delta\rho + \rho V\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, which implies $\delta\rho/\rho = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$. This is the same as Eq. (12.3).

The equation of momentum conservation (12.2b) can be handled similarly. By linearizing and pulling the slowly varying density out from under the time derivative, we convert $\partial(\rho\mathbf{v})/\partial t$ into $\rho\partial\mathbf{v}/\partial t = \rho\partial^2\boldsymbol{\xi}/\partial t^2$. Inserting this into Eq. (12.2b), we obtain the linear wave equation

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\boldsymbol{\nabla}\cdot\mathbf{T}_{el} \tag{12.4a}$$

i.e.,

$$\boxed{\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = (K + \frac{1}{3}\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi}\ .} \tag{12.4b}$$

In this equation, terms involving a derivative of $K$ or $\mu$ have been omitted because the two-lengthscale assumption $\mathcal{L} \gg \lambda\!\!\!^{-}$ makes them negligible compared to the terms we have kept.

Equation (12.4b) is the first of many wave equations we shall encounter in elastodynamics, fluid mechanics, and plasma physics.

### 12.2.2 Elastodynamic Waves

Continuing to follow our general procedure for deriving and analyzing wave equations as outlined in Box 12.2, we next derive dispersion relations for two types of waves (longitudinal and transverse) that are jointly incorporated into the general elastodynamic wave equation (12.4b).

Recall from Chap. 6 that, although a dispersion relation can be used as a Hamiltonian for computing wave propagation through an *inhomogeneous medium*, one can derive the dispersion relation most easily by specializing to monochromatic plane waves propagating through a medium that is precisely homogeneous. Therefore, we seek a plane-wave solution, i.e. a solution of the form

$$\boldsymbol{\xi}(\mathbf{x}, t) \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\ , \tag{12.5}$$

to the wave equation (12.4b) with $\rho$, $K$ and $\mu$ regarded as homogeneous (constant). (To deal with more complicated perturbations of a homogeneous medium, we can think of this wave as being an individual Fourier component and linearly superpose many such waves as a Fourier integral.) Since our wave is planar and monochromatic, we can remove the derivatives in Eq. (12.4b) by making the substitutions $\boldsymbol{\nabla} \to i\mathbf{k}$ and $\partial/\partial t \to -i\omega$ (the first of which implies $\nabla^2 \to -k^2$, $\boldsymbol{\nabla}\cdot \to i\mathbf{k}\cdot$, $\boldsymbol{\nabla}\times \to i\mathbf{k}\times$.) We thereby reduce the partial differential equation (12.4b) to a vectorial algebraic equation:

$$\rho\omega^2\boldsymbol{\xi} = (K + \frac{1}{3}\mu)\mathbf{k}(\mathbf{k}\cdot\boldsymbol{\xi}) + \mu k^2\boldsymbol{\xi}\ . \tag{12.6}$$

(This reduction is only possible because the medium is uniform, or in the geometric optics limit of near uniformity; otherwise, we must solve the second order partial differential equation (12.4b) using standard techniques.)

How do we solve this equation? The sure way is to write it as a $3 \times 3$ matrix equation $M_{ij}\xi_j = 0$ for the vector $\boldsymbol{\xi}$ and set the determinant of $M_{ij}$ to zero (Box 12.2 and Ex. 12.3). This is not hard for small or sparse matrices. However, some wave equations are more complicated and it often pays to think about the waves in a geometric, coordinate-independent way before resorting to brute force.

The quantity that oscillates in the elastodynamic waves (12.6) is the vector field $\boldsymbol{\xi}$. The nature of its oscillations is influenced by the scalar constants $\rho$, $\mu$, $K$, $\omega$ and by just one quantity that has directionality: the constant vector $\mathbf{k}$. It seems reasonable to expect the description (12.6) of the oscillations to simplify, then, if we resolve the oscillations into a "longitudinal" component (or "mode") along $\mathbf{k}$ and a "transverse" component (or "mode") perpendicular to $\mathbf{k}$, as shown in Fig. 12.1:

$$\boxed{\boldsymbol{\xi} = \boldsymbol{\xi}_L + \boldsymbol{\xi}_T\ , \quad \boldsymbol{\xi}_L = \xi_L\hat{\mathbf{k}}\ , \quad \boldsymbol{\xi}_T\cdot\hat{\mathbf{k}} = 0\ .} \tag{12.7a}$$

Here $\hat{\mathbf{k}} \equiv \mathbf{k}/k$ is the unit vector along the propagation direction. It is easy to see that the longitudinal mode $\boldsymbol{\xi}_L$ has nonzero expansion $\Theta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_L \neq 0$ but vanishing rotation $\boldsymbol{\phi} = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\xi}_L = 0$, and can therefore be written as the gradient of a scalar potential,

$$\boxed{\boldsymbol{\xi}_L = \boldsymbol{\nabla}\psi\ .} \tag{12.7b}$$

By contrast, the transverse mode has zero expansion but nonzero rotation and can thus be written as the curl of a vector potential,

$$\boxed{\boldsymbol{\xi}_T = \boldsymbol{\nabla}\times\mathbf{A}\ ;} \tag{12.7c}$$

cf. Ex. 12.1.

### 12.2.3 Longitudinal Sound Waves

For the longitudinal mode the algebraic wave equation (12.6) reduces to the following simple relation [as one can easily see by inserting $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_L = \xi_L\hat{\mathbf{k}}$ into Eq. (12.6) , or, alternatively, by taking the divergence of (12.6), which is equivalent to taking the scalar product with $\mathbf{k}$]:

$$\omega^2 = \frac{K + \frac{4}{3}\mu}{\rho}k^2\ ; \quad \text{i.e.} \quad \boxed{\omega = \Omega(\mathbf{k}) = \left(\frac{K + \frac{4}{3}\mu}{\rho}\mathbf{k}^2\right)^{1/2}}\ . \tag{12.8}$$

This relation between $\omega$ and $\mathbf{k}$ is the longitudinal mode's dispersion relation.

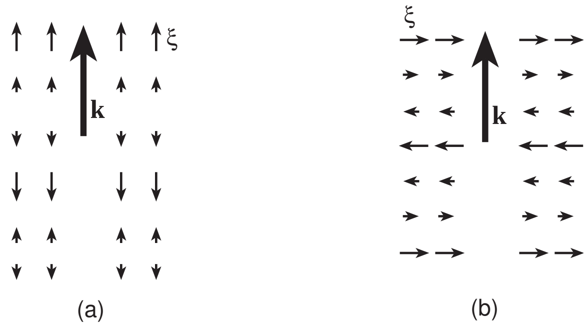



**Fig. 12.1:** Displacements in an isotropic, elastic solid, perturbed by a) a longitudinal mode, b) a transverse mode.

From the geometric-optics analysis in Sec. 7.3 we infer that, if $K$, $\mu$ and $\rho$ vary spatially on an inhomogeneity lengthscale $\mathcal{L}$ large compared to $1/k = \lambdabar$, and vary temporally on a timescale $\mathcal{T}$ large compared to $1/\omega$, then the dispersion relation (12.8), with $\Omega$ now depending on $\mathbf{x}$ and $t$ through $K$, $\mu$, and $\rho$, serves as a Hamiltonian for the wave propagation. In Sec. 12.4 and Fig. 12.6 below we shall use this to deduce details of the propagation of seismic waves through the Earth's inhomogeneous interior.

As we discussed in great detail in Sec. 7.2, associated with any wave mode is its phase velocity, $\mathbf{V}_{\rm ph} = (\omega/k)\hat{\mathbf{k}}$ and its phase speed $V_{\rm ph} = \omega/k$. The dispersion relation (12.8) implies that for longitudinal elastodynamic modes, the phase speed is

$$\boxed{C_L = \frac{\omega}{k} = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{1/2}} . \tag{12.9a}$$

As this does not depend on the wave number $k \equiv |\mathbf{k}|$, the mode is *non-dispersive*, and as it does not depend on the direction $\hat{\mathbf{k}}$ of propagation through the medium, the phase speed is also isotropic, naturally enough, and the group velocity $V_{g\,j} = \partial\Omega/\partial k_j$ is equal to the phase velocity:

$$\boxed{\mathbf{V}_g = \mathbf{V}_{\rm ph} = C_L \hat{\mathbf{k}}\ .} \tag{12.9b}$$

Elastodynamic longitudinal modes are similar to sound waves in a fluid. However, in a fluid, as we shall see in Eq. (16.48), the sound waves travel with phase speed $V_{\rm ph} = (K/\rho)^{1/2}$ [the limit of Eq. (12.9a) when the shear modulus vanishes].[2] This fluid sound speed is lower than the $C_L$ of a solid with the same bulk modulus because the longitudinal displacement necessarily entails shear (note that in Fig. 12.1a the motions are not an isotropic expansion), and in a solid there is a restoring shear stress (proportional to $\mu$) that is absent in a fluid.

Because the longitudinal phase velocity is independent of frequency, we can write down general planar longitudinal-wave solutions to the elastodynamic wave equation (12.4b) in

[2]Eq. (16.48) says the fluid sound speed is $C = \sqrt{(\partial P/\partial \rho)_s}$, i.e. the square root of the derivative of the fluid pressure with respect to density at fixed entropy. In the language of elasticity theory, the fractional change of density is related to the expansion $\Theta$ by $\delta\rho/\rho = -\Theta$ [Eq. (12.3)], and the accompanying change of pressure is $\delta P = -K\Theta$ [paragraph preceding Eq. (11.18)], i.e. $\delta P = K(\delta\rho/\rho)$. Therefore the fluid mechanical sound speed is $C = \sqrt{\delta P/\delta\rho} = \sqrt{K/\rho}$.

the following form:

$$\boldsymbol{\xi} = \xi_L \hat{\mathbf{k}} = F(\hat{\mathbf{k}} \cdot \mathbf{x} - C_L t)\hat{\mathbf{k}} \,, \tag{12.10}$$

where $F(x)$ is an arbitrary function. This describes a wave propagating in the (arbitrary) direction $\hat{\mathbf{k}}$ with an arbitrary profile determined by the function $F$.

### 12.2.4 Transverse Shear Waves

To derive the dispersion relation for a transverse wave we can simply make use of the transversality condition $\mathbf{k} \cdot \boldsymbol{\xi}_T = 0$ in Eq. (12.6); or, equally well, we can take the curl of Eq. (12.6) (multiply it by $i\mathbf{k}\times$), thereby projecting out the transverse piece, since the longitudinal part of $\boldsymbol{\xi}$ has vanishing curl. The result is

$$\omega^2 = \frac{\mu}{\rho} k^2 \; ; \quad \text{i.e.} \quad \boxed{\omega = \Omega(\mathbf{k}) \equiv \left(\frac{\mu}{\rho}\mathbf{k}^2\right)^{1/2}} \,. \tag{12.11}$$

This dispersion relation $\omega = \Omega(\mathbf{k})$ serves as a geometric-optics Hamiltonian for wave propagation when $\mu$ and $\rho$ vary slowly with $\mathbf{x}$ and/or $t$, and it also implies that the transverse waves propagate with a phase speed $C_T$ and phase and group velocities given by

$$\boxed{C_T = \left(\frac{\mu}{\rho}\right)^{1/2}} \; ; \tag{12.12a}$$

$$\boxed{\mathbf{V}_{\rm ph} = \mathbf{V}_g = C_T \hat{\mathbf{k}} \,.} \tag{12.12b}$$

As $K > 0$, the shear wave speed $C_T$ is always less than the speed $C_L$ of longitudinal waves [Eq. (12.9a)].

These transverse modes are known as *shear waves* because they are driven by the shear stress; cf. Fig. 12.1b. There is no expansion and therefore no change in volume associated with shear waves. They do not exist in fluids, but they are close analogs of the transverse vibrations of a string.

Longitudinal waves can be thought of as scalar waves, since they are fully describable by a single component $\xi_L$ of the displacement $\boldsymbol{\xi}$: that along $\hat{\mathbf{k}}$. Shear waves, by contrast, are inherently vectorial. Their displacement $\boldsymbol{\xi}_T$ can point in any direction orthogonal to $\mathbf{k}$. Since the directions orthogonal to $\mathbf{k}$ form a two-dimensional space, once $\mathbf{k}$ has been chosen, there are two independent states of polarization for the shear wave. These two polarization states, together with the single one for the scalar, longitudinal wave, make up the three independent degrees of freedom in the displacement $\boldsymbol{\xi}$.

In Ex. 12.3 we deduce these properties of $\boldsymbol{\xi}$ using matrix techniques.

### 12.2.5 Energy of Elastodynamic Waves

Elastodynamic waves transport energy, just like waves on a string. The waves' kinetic energy density is obviously $\frac{1}{2}\rho\mathbf{v}^2 = \frac{1}{2}\rho\dot{\boldsymbol{\xi}}^2$, where the dot means $\partial/\partial t$. The elastic energy density is given by Eq. (11.27), so the total energy density is

$$U = \frac{1}{2}\rho\dot{\boldsymbol{\xi}}^2 + \frac{1}{2}K\Theta^2 + \mu\Sigma_{ij}\Sigma_{ij} \ . \tag{12.13a}$$

In Ex. 12.4 we show that (as one might expect) the elastodynamic wave equation (12.4b) can be derived from an action whose Lagrangian density is the kinetic energy density minus the elastic energy density. We also show that associated with the waves is an energy flux $\mathbf{F}$ (not to be confused with a force for which we use the same notation) given by

$$F_i = -K\Theta\dot{\xi}_i - 2\mu\Sigma_{ij}\dot{\xi}_j \ . \tag{12.13b}$$

As the waves propagate, energy sloshes back and forth between the kinetic part and the elastic part, with the time averaged kinetic energy being equal to the time averaged elastic energy (equipartion of energy). For the planar, monochromatic, longitudinal mode, the time averaged energy density and flux are

$$\boxed{U_L = \rho\langle\dot{\xi}_L^2\rangle} \ , \quad \boxed{\mathbf{F}_L = U_L C_L \hat{\mathbf{k}}} \ , \tag{12.14}$$

where $\langle ... \rangle$ denotes an average over one period or wavelength of the wave. Similarly, for the planar, monochromatic, transverse mode, the time averaged density and flux of energy are

$$\boxed{U_T = \rho\langle\dot{\boldsymbol{\xi}}_T^2\rangle \ ,} \ , \quad \boxed{\mathbf{F}_T = U_T C_T \hat{\mathbf{k}}} \tag{12.15}$$

[Ex. 12.4]. Thus, elastodynamic waves transport energy at the same speed $c_{L,T}$ as the waves propagate, and in the same direction $\hat{\mathbf{k}}$. This is the same behavior as electromagnetic waves in vacuum, whose Poynting flux and energy density are related by $\mathbf{F}_{EM} = U_{EM} c\hat{\mathbf{k}}$ with $c$ the speed of light, and the same as all forms of dispersion-free scalar waves (e.g. sound waves in a medium), cf. Eq. (7.31). Actually, this is the dispersion-free limit of the more general result that the energy of any wave, in the geometric-optics limit, is transported with the wave's group velocity, $\mathbf{V}_g$; see Sec. 7.2.2.

****************************

## EXERCISES

**Exercise 12.1** *Example: Scalar and Vector Potentials for Elastic Waves in a Homogeneous Solid*
Just as in electromagnetic theory, it is sometimes useful to write the displacement $\boldsymbol{\xi}$ in terms of scalar and vector potentials,

$$\boldsymbol{\xi} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \mathbf{A} \ . \tag{12.16}$$

(The vector potential $\mathbf{A}$ is, as usual, only defined up to a gauge transformation, $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\varphi$, where $\varphi$ is an arbitrary scalar field.) By inserting Eq. (12.16) into the general elastodynamic wave equation (12.4b), show that the scalar and vector potentials satisfy the following wave equations in a homogeneous solid:

$$\frac{\partial^2 \psi}{\partial t^2} = c_L^2 \nabla^2 \psi \; , \quad \frac{\partial^2 \mathbf{A}}{\partial t^2} = c_T^2 \nabla^2 \mathbf{A} \; . \tag{12.17}$$

Thus, the scalar potential $\psi$ generates longitudinal waves, while the vector potential $\mathbf{A}$ generates transverse waves.

**Exercise 12.2** **** Problem: Influence of gravity on wave speed*
Modify the wave equation (12.4b) to include the effect of gravity. Assume that the medium is homogeneous and the gravitational field is constant. By comparing the orders of magnitude of the terms in the wave equation verify that the gravitational terms can be ignored for high-enough frequency elastodynamic modes: $\omega \gg g/c_{L,T}$. For wave speeds $\sim 3$ km/s, this says $\omega/2\pi \gg 0.0005$ Hz. Seismic waves are generally in this regime.

**Exercise 12.3** *Example: Solving the Algebraic Wave Equation by Matrix Techniques*
By using the matrix techniques discussed in the next-to-the-last paragraph of Box 12.2, deduce that the general solution to the algebraic wave equation (12.6) is the sum of a longitudinal mode with the properties deduced in Sec. 12.2.3, and two transverse modes with the properties deduced in Sec. 12.2.4. [Note: This matrix technique is necessary and powerful when the algebraic dispersion relation is complicated, e.g. for plasma waves; Secs. 21.4.1 and 21.5.1. Elastodynamic waves are simple enough that we did not need this matrix technique in the text.] Guidelines for solution:

(a) Rewrite the algebraic wave equation in the matrix form $M_{ij}\xi_j = 0$, obtaining thereby an explicit form for the matrix $||M_{ij}||$ in terms of $\rho$, $K$, $\mu$, $\omega$ and the components of $\mathbf{k}$.

(b) This matrix equation has a solution if and only if the determinant of the matrix $||M_{ij}||$ vanishes. (Why?) Show that $\det||M_{ij}|| = 0$ is a cubic equation for $\omega^2$ in terms of $k^2$, and that one root of this cubic equation is $\omega = C_L k$, while the other two roots are $\omega = C_T k$ with $C_L$ and $C_T$ given by Eqs. (12.9a) and (12.12a).

(c) Orient Cartesian axes so that $\mathbf{k}$ points in the $z$ direction. Then show that when $\omega = C_L k$, the solution to $M_{ij}\xi_j = 0$ is a longitudinal wave, i.e., a wave with $\boldsymbol{\xi}$ pointing in the $z$ direction, the same direction as $\mathbf{k}$.

(d) Show that when $\omega = C_T k$, there are two linearly independent solutions to $M_{ij}\xi_j = 0$, one with $\boldsymbol{\xi}$ pointing in the $x$ direction (transverse to $\mathbf{k}$) and the other in the $y$ direction (also transverse to $\mathbf{k}$).

**Exercise 12.4** *Example: Lagrangian and Energy for Elastodynamic Waves*
Derive the energy-density, energy-flux, and Lagrangian properties of elastodynamic waves that are stated in Sec. 12.2.5. Guidelines:

(a) For ease of calculation (and for greater generality), consider an elastodynamic wave in a possibly anisotropic medium, for which

$$T_{ij} = -Y_{ijkl}\xi_{k;l} \tag{12.18}$$

with $Y_{ijkl}$ the tensorial modulus of elasticity, which is symmetric under interchange of the first two indices $ij$, and under interchange of the last two indices $kl$, and under interchange of the first pair $ij$ with the last pair $kl$ [Eq. (11.17) and associated discussion]. Show that for an isotropic medium

$$Y_{ijkl} = \left(K - \frac{2}{3}\mu\right) g_{ij}g_{kl} + \mu(g_{ik}g_{jl} + g_{il}g_{jk}) \ . \tag{12.19}$$

(Recall that in the orthonormal bases to which we confine ourselves, the components of the metric are $g_{ij} = \delta_{ij}$, i.e. the Kronecker delta.)

(b) For these waves the elastic energy density is $\frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l}$ [Eq. (11.28)]. Show that the kinetic energy density minus the elastic energy density

$$\mathcal{L} = \frac{1}{2}\rho\,\dot{\xi}_i\dot{\xi}_i - \frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l} \tag{12.20}$$

is a Lagrangian density for the waves; i.e., show that the vanishing of its variational derivative $\delta\mathcal{L}/\delta\xi_j = 0$ is equivalent to the elastodynamic equations $\rho\ddot{\boldsymbol{\xi}} = -\boldsymbol{\nabla}\cdot\mathbf{T}$.

(c) The waves' energy density and flux can be constructed by the vector-wave analog of the canonical procedure of Eq. (7.35c):

$$\begin{aligned} U &= \frac{\partial\mathcal{L}}{\partial\dot{\xi}_i}\dot{\xi}_i - \mathcal{L} = \frac{1}{2}\rho\,\dot{\xi}_i\dot{\xi}_i + \frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l} \ , \\ F_j &= \frac{\partial\mathcal{L}}{\partial\xi_{i;j}}\dot{\xi}_i = -Y_{ijkl}\dot{\xi}_i\xi_{k;l} \ . \end{aligned} \tag{12.21}$$

Verify that these density and flux satisfy the energy conservation law, $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$. It is straightforward algebra to verify, using Eq. (12.19), that for an isotropic medium expressions (12.21) for the energy density and flux become the expressions (12.13) given in the text.

(d) Show that, in general (for an arbitrary mixture of wave modes), the time average of the total kinetic energy in some huge volume is equal to that of the total elastic energy. Show further that, for an individual longitudinal or transverse, planar, monochromatic, mode, the time averaged kinetic energy density and time averaged elastic energy density are both independent of spatial location. Combining these results, infer that for a single mode, the time averaged kinetic and elastic energy densities are equal, and therefore the time averaged total energy density is equal to twice the time averaged kinetic energy density. Show that this total time averaged energy density is given by the first of Eqs. (12.14) and (12.15).

(e) Show that the time average of the energy flux (12.13b) for the longitudinal and transverse modes is given by the second of Eqs. (12.14) and (12.15), so the energy propagates with the same speed and direction as the waves themselves.

****************************

## 12.3 Waves in Rods, Strings and Beams

Let us now illustrate some of these ideas using the types of waves that can arise in some practical applications. In particular we discuss how the waves get modified when the medium through which they propagate is not uniform but instead is bounded. Despite this situation being formally "global" in the sense of Box 12.2, elementary considerations enable us to derive the relevant dispersion relations without much effort.

### 12.3.1 Compression waves

First consider a longitudinal wave propagating along a light (negligible gravity), thin, unstressed rod. Introduce a Cartesian coordinate system with the $x$-axis parallel to the rod. When there is a small displacement $\xi_x$ independent of $y$ and $z$, the restoring stress is given by $T_{xx} = -E\partial\xi_x/\partial x$, where $E$ is Young's modulus (cf. end of Sec. 11.3). Hence the restoring force density $\mathbf{f} = -\boldsymbol{\nabla}\cdot\mathbf{T}$ is $f_x = E\partial^2\xi_x/\partial x^2$. The wave equation then becomes

$$\frac{\partial^2\xi_x}{\partial t^2} = \left(\frac{E}{\rho}\right)\frac{\partial^2\xi_x}{\partial x^2} \ , \tag{12.22}$$

and so the sound speed for compression waves in a long straight rod is

$$\boxed{C_C = \left(\frac{E}{\rho}\right)^{\frac{1}{2}}} \ . \tag{12.23}$$

Referring to Table 11.1 (in Chap. 10), we see that a typical value of Young's modulus in a solid is $\sim 100$ GPa. If we adopt a typical density $\sim 3\times10^3$ kg m$^{-3}$, then we estimate the compressional sound speed to be $\sim 5$ km s$^{-1}$. This is roughly 15 times the sound speed in air.

### 12.3.2 Torsion waves

Next consider a wire with circular cross section of radius $a$ subjected to a twisting force (Fig. 12.2). Let us introduce an angular displacement $\Delta\phi \equiv \varphi$ that depends on $x$. The only nonzero component of the displacement vector is then $\xi_\phi = \varpi\varphi$. We can calculate the total torque by integating over a circular cross section. For small twists, there will be no expansion and the only components of the shear tensor are

$$\Sigma_{\phi x} = \Sigma_{x\phi} = \frac{1}{2}\xi_{\phi,x} = \frac{\varpi}{2}\frac{\partial\phi}{\partial x} \ . \tag{12.24}$$

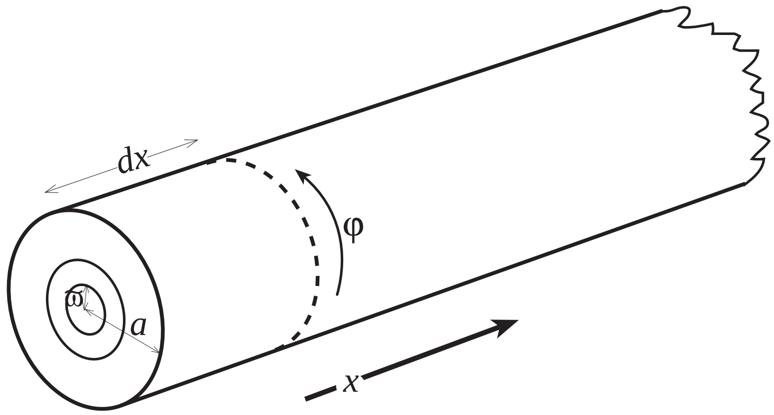



**Fig. 12.2:** When a wire of circular cross section is twisted, there will be a restoring torque.

The torque contributed by an annular ring of radius $\varpi$ and thickness $d\varpi$ is $\varpi \cdot T_{\phi x} \cdot 2\pi\varpi d\varpi$ and we substitute $T_{\phi x} = -2\mu\Sigma_{\phi x}$ to obtain the total torque

$$N = \int_0^a 2\pi\mu\varpi^3 d\varpi \frac{\partial\varphi}{\partial x} \; . \tag{12.25}$$

Now the moment of inertia per unit length is

$$I = \frac{\pi}{2}\rho a^4 \; , \tag{12.26}$$

so equating the net torque per unit length to the rate of change of angular momentum, also per unit length, we obtain

$$\frac{\partial N}{\partial x} = I \frac{\partial^2\varphi}{\partial t^2} \; , \tag{12.27}$$

or

$$\frac{\partial^2\varphi}{\partial x^2} = \left(\frac{\rho}{\mu}\right) \frac{\partial^2\varphi}{\partial t^2} \; . \tag{12.28}$$

The speed of torsional waves is thus

$$C_T = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \; . \tag{12.29}$$

Note that this is the same speed as that of shear waves in a uniform medium. This might have been anticipated as there is no change in volume in a torsional oscillation and so only the shear stress acts to produce a restoring force.

### 12.3.3 Waves on Strings

This example is surely all too familiar. When a string under a tension force $T$ (*not* force per unit area) is plucked, there will be a restoring force proportional to the curvature of the string. If $\xi_x \equiv \eta$ is the transverse displacement (in the same notation as we used for rods in Secs. 11.7 and 11.8), then the wave equation will be

$$T\frac{\partial^2\eta}{\partial x^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} \; , \tag{12.30}$$

where $\Lambda$ is the mass per unit length. The wave speed is thus

$$\boxed{C_S = \left(\frac{T}{\Lambda}\right)^{1/2}} \; . \tag{12.31}$$

### 12.3.4 Flexural Waves on a Beam

Now consider the small amplitude displacement of a rod or beam that can be flexed. In Sec. 11.7 we showed that such a flexural displacement produces a net elastic restoring force per unit length given by $D\partial^4\eta/\partial x^4$, and we considered a situation where that force was balanced by the beam's weight per unit length, $W = \Lambda g$ [Eq. (11.87)]. Here

$$D = \frac{1}{12}Ewh^3 \tag{12.32}$$

is the flexural rigidity [Eq. (11.82)], $h$ is the beam's thickness in the direction of bend, $w$ is its width, $\eta = \xi_z$ is the transverse displacement of the neutral surface from the horizontal, $\Lambda$ is the mass per unit length, and $g$ is the earth's acceleration of gravity. The solution of the resulting force-balance equation, $-D\partial^4\eta/\partial x^4 = W = \Lambda g$, was the quartic (11.88a), which described the equilibrium beam shape.

When gravity is absent and the beam is allowed to move, the acceleration of gravity $g$ gets replaced by a dynamical acceleration of the beam, $\partial^2\eta/\partial t^2$; the result is a wave equation for flexural waves on the beam:

$$-D\frac{\partial^4\eta}{\partial x^4} = \Lambda\frac{\partial^2\eta}{\partial t^2} \; . \tag{12.33}$$

[This derivation of the wave equation is an elementary illustration of the *Principle of Equivalence*—the equivalence of gravitational and inertial forces, or gravitational and inertial accelerations—which underlies Einstein's general relativity theory (Chap. 24).]

The wave equations we have encountered so far in this chapter have all described non-dispersive waves, for which the wave speed is independent of the frequency. Flexural waves, by contrast, are dispersive. We can see this by assuming that $\eta \propto \exp[i(kx - \omega t)]$ and thereby deducing from Eq. (12.33) the dispersion relation

$$\boxed{\omega = \sqrt{D/\Lambda}\; k^2 \; .} \tag{12.34}$$

Before considering the implications of this dispersion, we shall complicate the equilibrium a little. Let us suppose that, in addition to the net shearing force per unit length $-D\partial^4\eta/\partial x^4$, the beam is also held under a tension force $T$ as well. We can then combine the two wave equations (12.30), (12.33) to obtain

$$-D\frac{\partial^4\eta}{\partial x^4} + T\frac{\partial^2\eta}{\partial x^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} \; , \tag{12.35}$$

for which the dispersion relation is

$$\omega^2 = C_S^2 k^2 \left(1 + \frac{k^2}{k_c^2}\right) , \tag{12.36}$$

where $C_S = \sqrt{T/\Lambda}$ is the wave speed when the flexural rigidity $D$ is negligible so the beam is string-like, and

$$k_c = \sqrt{T/D} \tag{12.37}$$

is a critical wave number. If the average strain induced by the tension is $\epsilon = \xi_{x,x} = T/Ewh$, then $k_c = (12\epsilon)^{1/2}h^{-1}$, where $h$ is the thickness of the beam and $w$ is its width. [Notice that $k_c$ is also of order $1/\lambda$, where $\lambda$ is the lengthscale on which a pendulum's support wire ("beam") bends as discussed in Ex. 11.18.] For short wavelengths $k \gg k_c$, the shearing force dominates and the beam behaves like a tension-free beam; for long wavelengths $k \ll k_c$, it behaves like a string.

A consequence of dispersion is that waves with different wave numbers $k$ propagate with different speeds, and correspondingly the group velocity $V_g = d\omega/dk$ with which wave packets propagate differs from the phase velocity $V_{\rm ph} = \omega/k$ with which a wave's crests and troughs move (see Sec. 7.2.2). For the dispersion relation (12.36), the phase and group velocities are

$$\begin{aligned} V_{\rm ph} &\equiv \omega/k = C_S(1 + k^2/k_c^2)^{1/2} , \\ V_g &\equiv d\omega/dk = C_S(1 + 2k^2/k_c^2)(1 + k^2/k_c^2)^{-1/2} . \end{aligned} \tag{12.38}$$

As we discussed in detail in Sec. 7.2.2 and Ex. 7.2, for dispersive waves such as this one, the fact that different Fourier components in the wave packet propagate with different speeds causes the packet to gradually spread; we explore this quantitatively for longitudinal waves on a beam in Ex. 12.5.

### 12.3.5 Bifurcation of Equilibria and Buckling (once more)

We conclude this discussion by returning to the problem of buckling, which we introduced in Sec. 11.8. The example we discussed there was a playing card compressed until it wants to buckle. We can analyze small dynamical perturbations of the card, $\eta(x,t)$, by treating the tension $T$ of the previous section as negative, $T = -F$ where $F$ is the compressional force applied to the card's two ends in Fig. 11.13. Then the equation of motion (12.35) becomes

$$-D\frac{\partial^4\eta}{\partial x^4} - F\frac{\partial^2\eta}{\partial x^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} . \tag{12.39}$$

We seek solutions for which the ends of the playing card are held fixed (as shown in Fig. 11.13), $\eta = 0$ at $x = 0$ and $x = \ell$. Solving Eq. (12.39) by separation of variables, we see that

$$\eta = \eta_n \sin\left(\frac{n\pi}{\ell}x\right) e^{-i\omega_n t} . \tag{12.40}$$

Here $n = 1, 2, 3, ...$ labels the card's modes of oscillation, $n - 1$ is the number of nodes in the card's sinusoidal shape for mode $n$, $\eta_n$ is the amplitude of deformation for mode $n$, and the mode's eigenfrequency $\omega_n$ (of course) satisfies the same dispersion relation (12.36) as for waves on a long, stretched beam, with $T \to -F$ and $k \to n\pi/\ell$:

$$\omega_n^2 = \frac{1}{\Lambda}\left(\frac{n\pi}{\ell}\right)^2 \left[\left(\frac{n\pi}{\ell}\right)^2 D - F\right] . \tag{12.41}$$

Consider the lowest normal mode, $n = 1$, for which the playing card is bent in the single-arch manner of Fig. 11.13 as it oscillates. When the compressional force $F$ is small, $\omega_1^2$ is positive, so $\omega_1$ is real and the normal mode oscillates sinusoidally, stably. But for $F > F_{\rm crit} = \pi^2 D/\ell^2$, $\omega_1^2$ is negative, so $\omega_1$ is imaginary and there are two normal-mode solutions, one decaying exponentially with time, $\eta \propto \exp(-|\omega_1|t)$, and the other increasing exponentially with time, $\eta \propto \exp(+|\omega_1|t)$, signifying an *instability against buckling*.

Notice that the onset of instability occurs at identically the same compressional force, $F = F_{\rm crit} \equiv \pi^2 D/\ell^2$, as the *bifurcation of equilibria* [Eq. (11.96)], at which a new, bent, equilibrium state for the playing card comes into existence. Notice, moreover, that the card's $n = 1$ normal mode has zero frequency, $\omega_1 = 0$, at this onset of instability and bifurcation of equilibria; the card can bend by an amount that grows linearly in time, $\eta = A \sin(\pi x/\ell)\, t$, with no restoring force or exponential growth. This zero-frequency motion leads the card from its original, straight equilibrium shape, to its new, bent equilibrium shape. [For a *free-energy*-based analysis of the onset of this instability, see Ex. 12.8.]

This is an example of a very general phenomenon, which we shall meet again in fluid mechanics (Sec. 15.6): For mechanical systems without dissipation (no energy losses to friction or viscosity or radiation or ...), as one gradually changes some "control parameter" (in this case the compressional force $F$), there can occur bifurcation of equilibria. At each bifurcation point, a normal mode of the original equilibrium becomes unstable, and at its onset of instability the mode has zero frequency and represents a motion from the original equilibrium (which is becoming unstable) to the new, stable equilibrium.

In our simple playing-card example, we see this phenomenon repeated again and again as the control parameter $F$ is increased: One after another the modes $n = 1$, $n = 2$, $n = 3$, ... become unstable. At each onset of instability, $\omega_n$ vanishes, and the zero-frequency mode (with $n - 1$ nodes in its eigenfunction) leads from the original, straight-card equilibrium to the new, stable, $(n - 1)$-noded, bent equilibrium.

Buckling is a serious issue in engineering. Whenever one has a vertical beam supporting a heavy weight (e.g. in the construction of a tall building), one must make sure that the beam has a large enough flexural rigidity $D$ to be stable against buckling. The reason is that, although there is a new, stable equilibrium if $F$ is only slightly larger than $F_{\rm crit}$, the bend in that equilibrium increases rapidly with increasing $F$ [Eq. (11.97)] and becomes so large, when $F$ is only moderately larger than $F_{\rm crit}$, that the beam breaks. A large enough flexural rigidity $D$ to protect against this is generally achieved not by making the beam uniformly thick, but rather by fashioning its cross section into an H shape or I shape (with cross bars).

Whenever one has a long pipe exposed to night-to-day cooling-to-heating transitions (e.g. an oil or natural gas pipe, or the long vacuum tubes of a laser interferometer gravitational wave detector), one must make sure the pipe has enough flexural rigidity to avoid buckling in the heat of the day, when it wants to expand in length.[3] It can be overly expensive to make

[3]Much of the thermal expansion is dealt with by bellows.

the pipe walls thick enough to achieve the required flexural rigidity, so instead of thickening the walls everywhere, engineers weld "stiffening rings" onto the outside of the pipe to increase its rigidity. Notice, in Eq. (12.41), that the longer is the length $\ell$ of the beam or pipe, the larger must be the flexural rigidity $D$ to avoid buckling; the required rigidity scales as the square of the length.

****************************

**EXERCISES**

**Exercise 12.5** *Derivation: Dispersion of Flexural Waves*
Verify Eqs. (12.36) and (12.38). Sketch the dispersion-induced evolution of a Gaussian wave packet as it propagates along a stretched beam.

**Exercise 12.6** *Problem: Speeds of Elastic Waves*
Show that the sound speeds for the following types of elastic waves in an isotropic material are in the ratio $1 : (1-\nu^2)^{-1/2} : \left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)^{1/2} : [2(1+\nu)]^{-1/2} : [2(1+\nu)]^{-1/2}$. Longitudinal waves along a rod, longitudinal waves along a sheet, longitudinal waves along a rod embedded in an incompressible fluid, shear waves in an extended solid, torsional waves along a rod. [*Note:* Here and elsewhere in this book, if you encounter grungy algebra (e.g. frequent conversions from $\{K, \mu\}$ to $\{E, \nu\}$), do not hesitate to use Mathematica or Maple or other symbolic manipulation software to do the algebra!]

**Exercise 12.7** *Problem: Xylophones*
Consider a beam of length $\ell$, whose weight is neglible in the elasticity equations, supported freely at both ends (so the slope of the beam is unconstrained at the ends). Show that the frequencies of standing flexural waves satisfy

$$\omega = \left(\frac{n\pi}{\ell}\right)^2 \left(\frac{D}{\rho A}\right)^{1/2} ,$$

where $A$ is the cross-sectional area and $n$ is an integer. Now repeat the exercise when the ends are clamped. Hence explain why xylophones don't have clamped ends.

**Exercise 12.8** ****Example: Free-Energy Analysis of Buckling Instability*
In this exercise you will explore the relationship of the onset of the buckling instability to the concept of *free energy*, which we introduced in Chap. 4 in our study of statistical mechanics and phase transitions.

(a) Consider a rod with flexural rigidity $D$ and with a compressional force $F$ applied at each end, as in Sec. 12.3.5. Show that, if the rod gets bent slightly, with a transverse displacement $\eta(x)$, its elastic energy increases by an amount

$$\mathcal{E} = \int \frac{1}{2} E (\xi_{x,x})^2 dx dy dz = \frac{1}{2} D \int_0^\ell \left(\frac{\partial^2 \eta}{\partial x^2}\right)^2 dx \; . \tag{12.42}$$

Here $\xi_x$ is the longitudinal displacement inside the rod, $\xi_{x,x}$ is the longitudinal strain, and the first integral is over the entire interior of the rod. [Hint: recall the first few steps in the dimensional-reduction analysis for such a rod in Sec. 11.7.]

(b) If the compressional force $F$ were absent, then the most stable equilibrium shape would be the one that minimizes this elastic energy subject to the boundary conditions that $\eta(x) = 0$ at $x = 0$ and $x = \ell$: i.e. the unbent shape $\eta = 0$. However, when the force $F$ is present and is held fixed as the rod deforms, the deformation causes the rod's right end to move inward relative to the left end by an amount $\delta\ell = -\int_0^\ell \frac{1}{2}(\partial\eta/\partial x)^2 dx$. [Prove this using the Pythagorean theorem.] As the rod moves inward, the force on its right end does an amount of work $-F\delta\ell$ on the rod. Correspondingly, the amount of *free energy* that the rod has (the amount of energy adjusted for the energy $-F\delta\ell$ that gets exchanged with the force $F$ when the rod bends) is

$$H = \mathcal{E} + F\delta\ell = \frac{1}{2}\int_0^\ell \left[ D\left(\frac{\partial^2\eta}{\partial x^2}\right)^2 - F\left(\frac{\partial\eta}{\partial x}\right)^2 \right] dx \ . \tag{12.43}$$

(c) Think of the constant force $F$ as arising from a "volume bath" with pressure $P$, that the rod's ends (with cross sectional area $hw$) are in contact with. Show that the free energy (12.43) can be reexpressed as $H = \mathcal{E} + P\delta V$, where $-\delta V = -hw\delta\ell$ is the change in the bath's volume as a result of the rod's ends moving when it bends. This is the *enthalpy* of the rod, associated with the bending (Ex. 5.5), and the laws of statistical mechanics for any system in contact with a volume bath tell us that the system's most stable state is the one with *minimum enthalpy*, i.e. *minimum free energy* (Ex. **??**). This minimum-free-energy state $\eta(x)$ must be stationary under small changes $\delta\eta(x)$ of the rod's shape, subject to $\delta\eta(0) = \delta\eta(\ell) = 0$. Show that this stationarity implies $\eta$ satisfies the equation of elastostatic equilibrium, Eq. (12.39) with $\partial^2\eta/\partial t^2 = 0$, and therefore must have the shape $\eta = \eta_o \sin(n\pi x/\ell)$, i.e. the shape of one of the normal-mode eigenfunctions. [KIP: THERE IS SOME DELICACY OF BOUNDARY CONDITIONS THAT NEEDS TO BE SORTED OUT. PH136 STUDENTS: CAN YOU HELP?]

(d) Compare the free energy $H_1$ for the $n = 1$ bent shape (no nodes) with that $H_0$ of the straight rod by performing the integral (12.43). Your result should be

$$H_1 - H_0 = \left(\frac{\pi\eta_1}{2\ell}\right)^2 \ell(F_{\rm crit} - F) \ , \tag{12.44}$$

where $F_{\rm crit} = \pi^2 D/\ell^2$ is the critical force at which the bifurcation of equilibria occurs and the straight rod becomes unstable, and $\eta_1$ is the amplitude of the bend.

(e) Comment: When one includes higher order corrections in $\eta_o$, the difference in free energies between the straight rod and the $n = 1$ shape turns out to be

$$H_1 - H_0 = H_1 - H_0 = \left(\frac{\pi\eta_1}{2\ell}\right)^2 \ell \left\{ F_{\rm crit}\left[1 + \left(\frac{\pi\eta_1}{2\ell}\right)^2\right] - F \right\} \ . \tag{12.45}$$

This free-energy difference is plotted in Fig. 12.3, as a function of $\eta_1$ (the amplitude of the rod's deformation) for various applied forces $F$. For $F < F_{\rm crit}$ there is only one extremum: a minimum at $\eta_1 = 0$, so the only equilibrium state is that of the straight

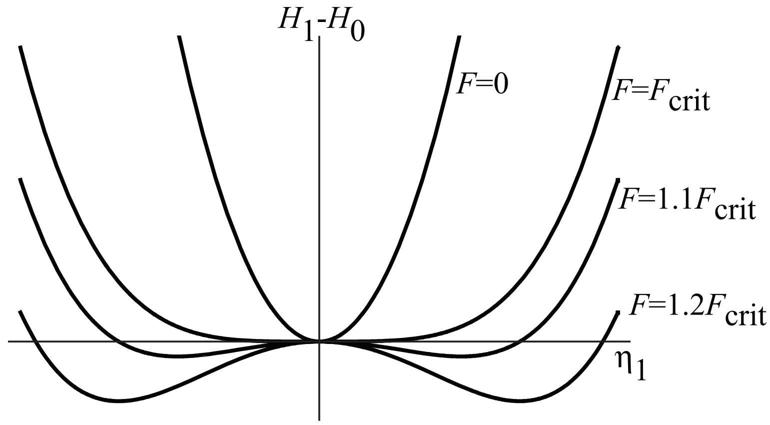



**Fig. 12.3:** The difference in free energy $H_1 - H_0$ between a rod bent in the $n = 1$ shape with bend amplitude $\eta_1$ and the unbent rod.

rod, and that equilibrium is stable. As $F$ increases past $F_{\rm crit}$ the minimum at $\eta = 0$ becomes a maximum, and two new minima are created, at $\eta = \pm\sqrt{(2\ell^2/\pi^2)(F - F_{\rm crit})}$. This is the bifurcation of equilibria, with the straight rod becoming unstable and the upward or downward bent rod, in the $n = 1$ shape, being stable.

****************************

## 12.4 Body Waves and Surface Waves — Seismology

In Sec. 12.2 we derived the dispersion relations $\omega = C_L k$ and $\omega = C_T k$ for elastodynamic waves in uniform media. We now consider how the waves are modified in an inhomogeneous, finite body, the earth. The earth is well approximated as a sphere of radius $R \sim 6000$ km and mean density $\bar\rho \sim 6000$ kg m$^{-3}$. The outer crust comprising rocks of high tensile strength rests on a denser but more malleable mantle, the two regions being separated by the famous Moho discontinuity. Underlying the mantle is an outer core mainly comprised of liquid iron, which itself surrounds a denser, solid inner core; see Table 12.1 and Fig.12.6 below.

The pressure in the Earth's interior is much larger than atmospheric and the rocks are therefore quite compressed. Their atomic structure cannot be regarded as a small perturbation from their structure *in vacuo*. Nevertheless, we can still use linear elasticity theory to discuss small perturbations about this equilibrium. This is because the crystal lattice has had plenty of time to re-establish a new equilibrium with a much smaller lattice spacing (Figure 12.4). The density of lattice defects and dislocations will probably not differ appreciably from the density on the earth's surface. The linear stress-strain relation should still apply below the elastic limit, though the elastic moduli are much greater than those measured at atmospheric pressure.

We can estimate the magnitude of the pressure $P$ in the Earth's interior by idealizing the earth as an isotropic medium with negligible shear stress so its stress tensor is like that

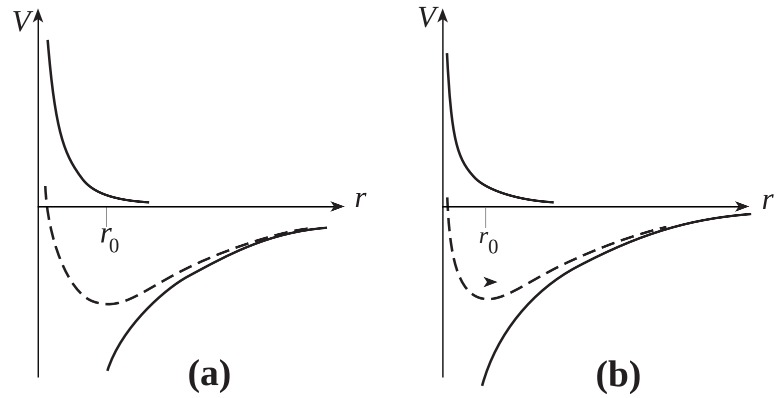



**Fig. 12.4:** Potential energy curves (dashed) for nearest neighbors in a crystal lattice. (a) At atmospheric (effectively zero) pressure, the equilibrium spacing is set by the minimum in the potential energy which is a combination of hard electrostatic repulsion by the nearest neighbors (upper solid curve) and a softer overall attraction associated with all the nearby ions (lower solid curve). (b) At much higher pressure, the softer, attractive component is moved inward and the equilibrium spacing is greatly reduced. The bulk modulus is proportional to the curvature of the potential energy curve at its minimum, and is considerably increased.

of a fluid, $\mathbf{T} = P\mathbf{g}$ (where $\mathbf{g}$ is the metric tensor). Then the equation of static equilibrium takes the form

$$\frac{dP}{dr} = -g\rho \; , \tag{12.46}$$

where $\rho$ is density and $g(r)$ is the acceleration of gravity at radius $r$. This equation can be approximated by

$$P \sim \bar{\rho} g R \sim 300\text{GPa} \sim 3 \times 10^6\text{atmospheres} \; , \tag{12.47}$$

where $g$ is now the acceleration of gravity at the earth's surface $r = R$, and $\bar{\rho}$ is the earth's mean density. This agrees well numerically with the accurate value of 360GPa at the earth's center. The bulk modulus produces the isotropic pressure $P = -K\Theta$ [Eq. (12.29)]; and since $\Theta = -\delta\rho/\rho$ [Eq. (11.18)], the bulk modulus can be expressed as

$$\boxed{K = \frac{dP}{d\ln\rho} \; .} \tag{12.48}$$

[Strictly speaking, we should distinguish between adiabatic and isothermal variations in Eq. (12.48), but the distinction is small for solids; see the passage following Eq. (11.71). It is significant for gases.] Typically, the bulk modulus inside the earth is 4-5 times the pressure and the shear modulus in the crust and mantle is about half the bulk modulus.

### 12.4.1 Body Waves

Virtually all our direct information about the internal structure of the earth comes from measurements of the propagation times of elastic waves generated by earthquakes. There are two fundamental kinds of body waves: the longitudinal and shear modes of Sec. 12.2.

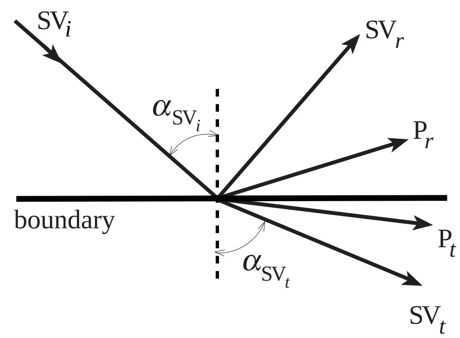



**Fig. 12.5:** An incident shear wave polarized in the vertical direction ($\mathrm{SV}_i$), incident from above on a boundary, produces both a longitudinal (P) wave and a SV wave in reflection and in transmission. If the wave speeds increase across the boundary (the case shown), then the transmitted waves, $\mathrm{SV}_t$, $\mathrm{P}_t$, will be refracted away from the vertical. A shear mode, $\mathrm{SV}_r$, will be reflected at the same angle as the incident wave. However, the reflected P mode, $\mathrm{P}_r$, will be reflected at a greater angle to the vertical as it is has a greater speed.

These are known in seismology as P-modes and S-modes respectively. The two polarizations of the shear waves are designated SH and SV, where H and V stand for "horizontal" and "vertical" displacements, i.e., displacements orthogonal to $\mathbf{k}$ that are fully horizontal, or that are obtained by projecting the vertical direction $\mathbf{e}_z$ orthogonal to $\hat{\mathbf{k}}$.

We shall first be concerned with what seismologists call high-frequency (of order 1Hz) modes. This leads to three related simplifications. As typical wave speeds lie in the range 3–14 km s$^{-1}$, the wavelengths lie in the range $\sim 1-10$ km which is generally small compared with the distance over which gravity causes the pressure to change significantly – the *pressure scale height*. It turns out that we then can ignore the effects of gravity on the propagation of small perturbations. In addition, we can regard the medium as effectively homogeneous and infinite and use the local dispersion relations $\omega = c_{L,T}k$, Finally, as the wavelengths are short we can trace rays through the earth using geometrical optics (Sec. 7.3).

| Zone | $R$ | $\rho$ | $K$ | $\mu$ | $C_P$ | $C_S$ |
|---|---|---|---|---|---|---|
| | $10^3$km | $10^3$kg m$^{-3}$ | GPa | GPa | km s$^{-1}$ | km s$^{-1}$ |
| Inner Core | 1.2 | 13 | 1400 | 160 | 11 | 2 |
| Outer Core | 3.5 | 10-12 | 600-1300 | - | 8-10 | - |
| Mantle | 6.35 | 3-5 | 100-600 | 70-250 | 8-14 | 5-7 |
| Crust | 6.37 | 3 | 50 | 30 | 6-7 | 3-4 |
| Ocean | 6.37 | 1 | 2 | - | 1.5 | - |

**Table 12.1:** Typical outer radii ($R$), densities ($\rho$), bulk moduli ($K$), shear moduli($\mu$), P-wave speeds and S-wave speeds within different zones of the earth. Note the absence of shear waves in the fluid regions. (Adapted from Stacey 1977.)

Despite these simplifications, the earth is quite inhomogeneous and the sound speeds vary significantly with radius; see Table 12.1. Two types of variation can be distinguished, the abrupt and the gradual. To a fair approximation, the earth is horizontally stratified below the outer crust. However, there are several abrupt changes in composition in the crust and mantle (including the Moho discontinuity) where the density, pressure and elastic constants apparently change over distances short compared with a wavelength. Seismic waves incident on these discontinuities behave like light incident on the surface of a glass plate; they can be reflected and refracted. In addition, as there are now two different waves with different phase speeds, it is possible to generate SV waves from pure P waves and *vice versa* at a discontinuity (Fig. 12.5). However, this wave-wave mixing is confined to SV and P; the SH waves do not mix with SV or P.

The junction conditions that control this wave mixing and all other details of the waves' behavior at a discontinuity are: (i) the displacement $\boldsymbol{\xi}$ must be continuous across the boundary (otherwise there would be infinite strain and infinite stress there); and (ii) the net force acting on an element of surface must be zero (otherwise the surface, having no mass, would have infinite acceleration), so the force per unit area acting from the front face of the boundary to the back must be balanced by that acting from the back to the front. If we take the unit normal to the horizontal boundary to be $\mathbf{e}_z$, then these boundary conditions become

$$\boxed{[\boldsymbol{\xi}_j] = [\mathbf{T}_{jz}] = 0\ ,} \tag{12.49}$$

where the notation $[X]$ signifies the difference in $X$ across the boundary and the $j$ is a vector index. (For an alternative, more formal derivation of $[T_{jz}] = 0$, see Ex. 12.9.)

One consequence of these boundary conditions is Snell's law for the directions of propagation of the waves: Since these continuity conditions must be satisfied all along the boundary and at all times, the phase $\phi = \mathbf{k}\cdot\mathbf{x} - \omega t$ of the wave must be continuous across the boundary at all locations $\mathbf{x}$ on it and all times, which means that the phase $\phi$ must be the same on the boundary for all transmitted waves and all reflected waves as for the incident waves. This is possible only if the frequency $\omega$, the horizontal wave number $k_H = k \sin\alpha$, and the horizontal phase speed $c_H = \omega/k_H = \omega/(k\sin\alpha)$, are the same for all the waves. (Here $k_H = k\sin\alpha$ is the magnitude of the horizontal component of a wave's propagation vector and $\alpha$ is the angle between its propagation direction and the vertical; cf. Fig. 12.5.) Thus, we arrive at Snell's law: For every reflected or transmitted wave $J$, the horizontal phase speed must be the same as for the incident wave:

$$\boxed{\frac{c_J}{\sin\alpha_J} = c_H \text{ is the same for all } J.} \tag{12.50}$$

It is straightforward though tedious to compute the reflection and transmission coefficients (e.g. the strength of transmitted P-wave produced by an incident SV wave) for the general case using the boundary conditions (12.49); see, e.g., Eringen and Suhubi (1975). The analysis is straightforward but algebraically complex. For the very simplest of examples, see Ex. 12.10.

Gradual variation in the wave speeds, due to gradual variations of the elastic moduli and density inside the earth, can be handled using geometrical optics:

In the regions between the discontinuities, the pressures and consequently the elastic moduli increase steadily, over many wavelengths, with depth. The elastic moduli generally increase more rapidly than the density so the wave speeds generally also increase with depth, i.e. $dc/dr < 0$. This radial variation in $c$ causes the rays along which the waves propagate to bend. The details of this bending are governed by Hamilton's equations, with the Hamiltonian $\Omega(\mathbf{x}, \mathbf{k})$ determined by the simple nondispersive dispersion relation $\Omega = c(\mathbf{x})k$ (Sec. 7.3.1). Hamilton's equations in this case reduce to the simple ray equation (7.49), which (since the index of refraction is $\propto 1/c$) can be rewritten as

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\mathbf{x}}{ds}\right) = \boldsymbol{\nabla}\left(\frac{1}{c}\right) . \tag{12.51}$$

Here $s$ is distance along the ray, so $d\mathbf{x}/ds = \mathbf{n}$ is the unit vector tangent to the ray. This ray equation can be reexpressed in the following form:

$$d\mathbf{n}/ds = -(\boldsymbol{\nabla} \ln c)_\perp , \tag{12.52}$$

where the subscript $\perp$ means "projected perpendicular to the ray;" and this in turn means that the ray bends *away* from the direction in which $c$ increases (i.e., it bends upward inside the earth since $c$ increases downward) with the radius of curvature of the bend given by

$$\mathcal{R} = \frac{1}{|(\boldsymbol{\nabla} \ln c)_\perp|} = \frac{1}{|(d \ln c/dr) \sin \alpha|} . \tag{12.53}$$

Here $\alpha$ is the angle between the ray and the radial direction; see the bending rays in Fig. 12.6.

Figure 12.6 shows schematically the propagation of seismic waves through the earth. At each discontinuity in the earth's material, Snell's law governs the directions of the reflected and transmitted waves. As an example, note from Eq. (12.50) that an SV mode incident on a boundary cannot generate any $P$ mode when its angle of incidence exceeds $\sin^{-1}(c_{Ti}/c_{Lt})$. (Here we use the standard notation $C_T$ for the phase speed of an $S$ wave and $C_L$ for that of a $P$ wave.) This is what happens at points $b$ and $c$ in Fig. 12.6.

### 12.4.2 Edge Waves

One phenomenon that is important in seismology but is absent for many other types of wave motion is the existence of "edge waves", i.e., waves that propagate along a discontinuity in the elastic medium. An important example is surface waves, which propagate along the surface of a medium (e.g., the earth) and that decay exponentially with depth. Waves with such exponential decay are sometimes called *evanescent*.

The simplest type of surface wave is called a *Rayleigh* wave. We shall now analyze Rayleigh waves for the idealisation of a plane semi-infinite solid. This discussion must be modified to allow for both the density stratification and the surface curvature when it is applied to the earth. However, the qualitative character of the mode is unchanged.

*Rayleigh* waves are an intertwined mixture of P and SV waves; and, in analyzing them, it is useful to resolve their displacement vector $\boldsymbol{\xi}$ into a sum of a (longitudinal) P-wave component, $\boldsymbol{\xi}^L$, and a (transverse) S-wave component, $\boldsymbol{\xi}^T$.

Consider a semi-infinite elastic medium and introduce a local Cartesian coordinate system with $\mathbf{e}_z$ normal to the surface, with $\mathbf{e}_x$ lying in the surface, and with the propagation vector $\mathbf{k}$ in the $\mathbf{e}_z$-$\mathbf{e}_x$ plane. The propagation vector will have a real component along the horizontal, $\mathbf{e}_x$ direction, corresponding to true propagation, and an imaginary component along the $\mathbf{e}_z$

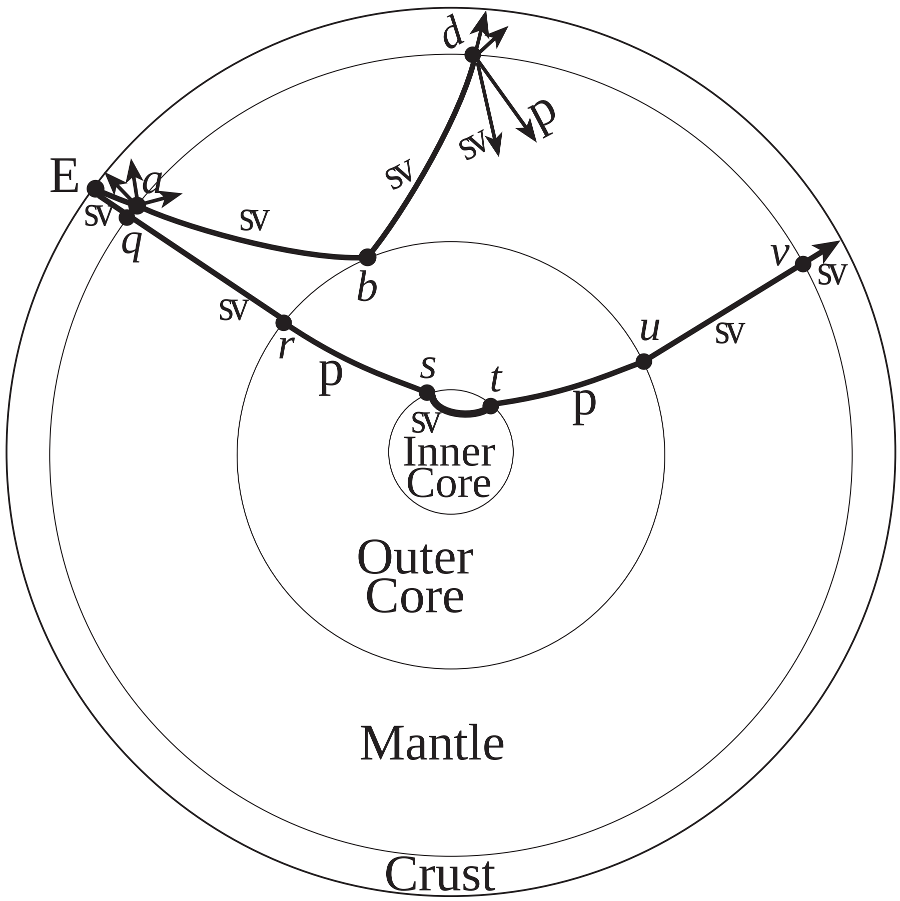



**Fig. 12.6:** Seismic wave propagation in a schematic earth model. A SV wave made by an earthquake, E, propagates to the crust-mantle boundary at $a$ where it generates two transmitted waves (SV and P) and two reflected waves (SV and P). The transmitted SV wave propagates along rays that bend upward a bit (geometric optics bending) and hits the mantle-outer-core boundary at $b$. There can be no transmitted SV wave at $b$ because the outer core is fluid; there can be no transmitted or reflected P wave because the angle of incidence of the SV wave is too great; so the SV wave is perfectly reflected. It then travels along an upward curving ray, to the crust-mantle interface at $d$, where it generates four waves, two of which hit the earth's surface. The earthquake E also generates an SV wave traveling almost radially inward, through the crust-mantle interface at $q$, to the mantle-outer-core interface at $r$. Because the outer core is liquid, it cannot support an SV wave, so only a P wave is transmitted into the outer core at $r$. That P wave propagates to the interface with the inner core at $s$, where it regenerates an SV wave (shown) along with the transmitted and reflected P waves. The SV wave refracts back upward in the inner core, and generates a P wave at the interface with the outer core $t$; that P wave propagates through the liquid outer core to $u$ where it generates an SV wave along with its transmitted and reflected P waves; that SV wave travels nearly radially outward, through $v$ to the earth's surface.

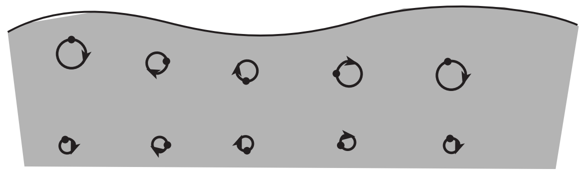


**Fig. 12.7:** Rayleigh waves in a semi-infinite elastic medium.

direction, corresponding to an exponential decay of the amplitude as one goes downward into the medium. In order for the longitudinal (P-wave) and transverse (SV-wave) parts of the wave to remain in phase with each other as they propagate along the boundary, they must have the same values of the frequency $\omega$ and horizontal wave number $k_x$. However, there is no reason why their vertical e-folding lengths should be the same, i.e. why their imaginary $k_z$'s should be the same. We therefore shall denote their imaginary $k_z$'s by $-iq_L$ for the longitudinal (P-wave) component and $-iq_T$ for the transverse (S-wave) component, and we shall denote $k_x$ by $k$.

Focus attention, first, on the longitudinal part of the wave. Its displacement must have the form

$$\boldsymbol{\xi}_L = \mathbf{A} e^{q_L z + i(kx-\omega t)}, \quad z \le 0\,. \tag{12.54}$$

Substituting into the general dispersion relation $\omega^2 = C_L^2 \mathbf{k}^2$ for longitudinal waves, we obtain

$$q_L = (k^2 - \omega^2/c_L^2)^{1/2}\,. \tag{12.55}$$

Now, the longitudinal displacement field is irrotational (curl-free), so we can write

$$\xi^L_{x,z} = \xi^L_{z,x} \tag{12.56}$$

or

$$\xi^L_z = \frac{-iq_L \xi^L_x}{k} \tag{12.57}$$

As the transverse component is solenoidal (divergence-free), the expansion of the combined P-T wave is produced entirely by the P component:

$$\Theta = \boldsymbol{\nabla} \cdot \boldsymbol{\xi}^L = ik\left(1 - \frac{q_L^2}{k^2}\right) A\,. \tag{12.58}$$

Now turn to the transverse (SV-wave) component. We write

$$\boldsymbol{\xi}_T = \mathbf{B} \exp^{q_T z + i(kx-\omega t)}, \quad z \le 0\,, \tag{12.59}$$

where (by virtue of the transverse dispersion relation)

$$q_T = \left(k^2 - \frac{\omega^2}{c_T^2}\right)^{1/2}\,. \tag{12.60}$$

As the transverse mode is solenoidal, we obtain

$$\xi^T_z = \frac{-ik\xi^T_x}{q^T} \tag{12.61}$$

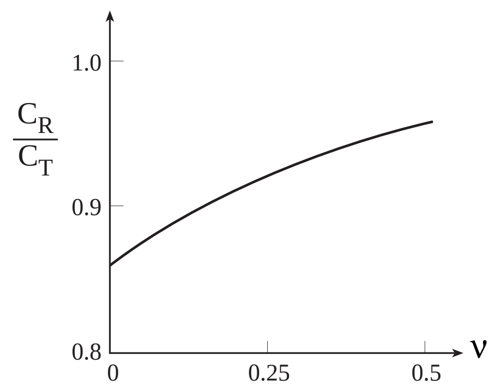



**Fig. 12.8:** Solution of the dispersion relation (12.67) for different values of Poisson's ratio, $\nu$.

and for the rotation

$$\phi_y = \frac{1}{2}\mathbf{e}_y \cdot \nabla \times \boldsymbol{\xi}^T = -\frac{1}{2} q_T \left(1 - \frac{k^2}{q_T^2}\right) B \ . \tag{12.62}$$

We must next impose boundary conditions at the surface. Now, as the surface is free, there will be no force acting upon it, so,

$$\mathbf{T} \cdot \mathbf{e}_z|_{z=0} = 0, \tag{12.63}$$

which is a special case of the general boundary condition (12.49). (Note that we can evaluate the stress at the unperturbed surface location rather than at the displaced surface as we are only working to linear order.) The normal stress is

$$-T_{zz} = K\Theta + 2\mu(\xi_{z,z} - \frac{1}{3}\Theta) = 0 \ , \tag{12.64}$$

and the tangential stress is

$$-T_{xz} = 2\mu(\xi_{z,x} + \xi_{x,z}) = 0 \ . \tag{12.65}$$

Combining Eqs. (12.58), (12.62), (12.64) and (12.65), we obtain

$$(k^2 + q_T^2)^2 = 4 q_L q_T k^2 \ . \tag{12.66}$$

Next we substitute for $q_L, q_T$ from (12.55) and (12.60) to obtain the dispersion relation

$$\zeta^3 - 8\zeta^2 + 8\left(\frac{2-\nu}{1-\nu}\right)\zeta - \frac{8}{(1-\nu)} = 0 \ , \tag{12.67}$$

where

$$\zeta = \left(\frac{\omega}{C_T k}\right)^2 \ . \tag{12.68}$$

The dispersion relation (12.67) is a third order polynomial in $\omega^2$ with generally just one positive real root. From Eqs. (12.67) and (12.68), we see that for a Poisson ratio characteristic of rocks, $0.2 \lesssim \nu \lesssim 0.3$, the phase speed of a Rayleigh wave is roughly 0.9 times the speed of a pure shear wave; cf. Fig. 12.8.

Rayleigh waves propagate around the surface of the earth rather than penetrate the interior. However, our treatment is inadequate because their wavelengths, typically 1–10 km if generated by an earthquake, are not necessarily small compared with the pressure scale heights in the outer crust. Our wave equation has to be modified to include these vertical gradients.

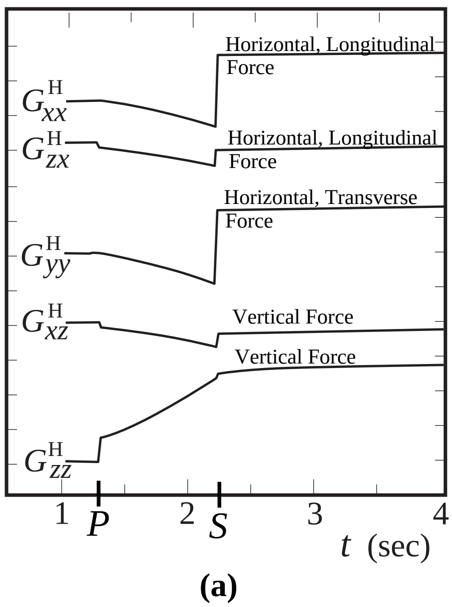
This vertical stratification has an important additional consequence. If, ignoring these gradients, we attempt to find an orthogonal surface mode just involving SH waves, we find that we cannot simultaneously satisfy the surface boundary conditions on displacement and stress with a single evanescent wave. We need two modes to do this. However, when we allow for stratification, the strong refraction allows an SH surface wave to propagate. This is known as a *Love wave*. The reason for its practical importance is that seismic waves are also created by underground nuclear explosions and it is necessary to be able to distinguish explosion-generated waves from earthquake waves. Now, an earthquake is usually caused by the transverse slippage of two blocks of crust across a fault line. It is therefore an efficient generator of shear modes including Love waves. By contrast, explosions involve radial motions away from the point of explosion and are inefficient emitters of Love waves. This allows these two sources of seismic disturbance to be distinguished.

### 12.4.3 Green's Function for a Homogeneous Half Space

To get insight into the combination of waves generated by a localized source, such as an explosion or earthquake, it is useful to examine the *Green's function* for excitations in a homogeneous half space. Physicists define the Green's function $G_{jk}(\mathbf{x}, t; \mathbf{x}', t')$ to be the displacement response $\xi_j(\mathbf{x}, t)$ to a unit delta-function force in the $\mathbf{e}_k$ direction at location $\mathbf{x}'$ and time $t'$, $\mathbf{F} = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\mathbf{e}_k$. Geophysicists sometimes find it useful to work, instead, with the "Heaviside Green's function," $G^H_{jk}(\mathbf{x}, t; \mathbf{x}', t')$, which is the displacement response $\xi_j(\mathbf{x}, t)$ to a unit step-function force (one that turns on to unit strength and remains forever constant afterwards) at $\mathbf{x}'$ and $t'$: $\mathbf{F} = \delta(\mathbf{x} - \mathbf{x}')H(t - t')\mathbf{e}_k$. Because $\delta(t - t')$ is the time derivative of the Heaviside step function $H(t - t')$, the Heaviside Green's function is the time integral of the physicists' Green's function. The Heaviside Green's function has the advantage that one can easily see, visually, the size of the step functions it contains, by contrast with the size of the delta functions contained in the physicists' Green's function.

It is a rather complicated task to compute the Heaviside Green's function, and geophysicists have devoted much effort to doing so. We shall not give details of such computations, but merely show the Green's function graphically in Fig. 12.9 for an instructive situation: the displacement produced by a step-function force in a homogeneous half space with the observer at the surface and the force at two different locations: (a) a point nearly beneath the observer, and (b) a point close to the surface and some distance away in the $x$ direction.

Several features of this Green's function deserve note: (i) For the source nearly beneath the observer [graphs (a)], there is no sign of any Rayleigh wave, whereas for the source close to the surface, the Rayleigh wave is the strongest feature in the $x$ and $z$ (longitudinal and vertical) displacements but is absent from the $y$ (transverse) displacement. (ii) The $y$ (transverse) component of force produces a transverse displacement that is strongly concentrated in the S-wave. (iii) The $x$ and $z$ (longitudinal and vertical) components of force produce $x$



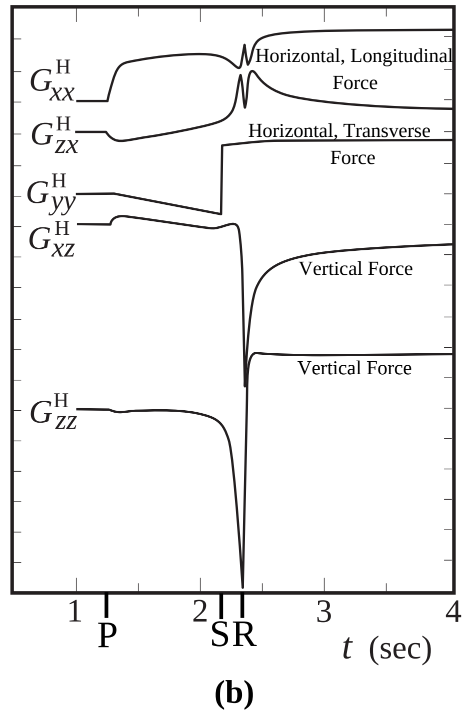




**Fig. 12.9:** The Heaviside Green's function (displacement response to a step-function force) in a homogeneous half space; adapted from Figs. 2 and 4 of Johnson (1974). The observer is at the surface. The force is applied at a point in the $x - z$ plane, with a direction given by the second index of $G^H$; the displacement direction is given by the first index of $G^H$. In (a), the source is nearly directly beneath the observer so the waves propagate nearly vertically upward; more specifically, the source is at 10 km depth and 2 km distance along the horizontal $x$ direction. In (b), the source is close to the surface and the waves propagate nearly horizontally, in the $x$ direction; more specifically, the source is at 2 km depth and is 10 km distance along the horizontal $x$ direction. The longitudinal and transverse speeds are $c_H = 8$ km/s and $C_S = 4.62$ km/s, and the density is 3.30 g/cm$^3$. For a force of 1 dyne, a division on the vertical scale is $10^{-19}$ cm. The moments of arrival of the P-wave, S-wave and Rayleigh wave from the moment of force turnon are indicated on the horizontal axis.

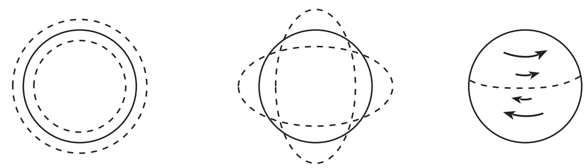


**Fig. 12.10:** Surface displacements associated with three simple classes of free oscillation. a)Radial modes. b) l=2 spheroidal mode. c) Torsional mode.

and $z$ displacements that include P-waves, S-waves, and (for the source near the surface) Rayleigh waves. (iv) The gradually changing displacements that occur between the arrival of the turn-on P-wave and turn-on S-wave are due to P-waves that hit the surface some distance from the observer, and from there diffract to the observer as a mixture of P- and S-waves, and similarly for gradual changes of displacement after the turn-on S-wave.

The complexity of seismic waves arises in part from the richness of features in this homogeneous-half-space Green's function, in part from the influences of the earth's inhomogeneities, and in part from the complexity of an earthquake's or explosion's forces.

### 12.4.4 Free Oscillations of Solid Bodies

In computing the dispersion relations for body (P- and S-wave) and surface (Rayleigh-wave) modes, we have assumed that the wavelength is small compared with the earth's radius and therefore can have a continuous frequency spectrum. However, it is also possible to excite global wave modes in which the whole earth "rings". If we regard the earth as spherically symmetric, then we can isolate three fundamental types of oscillation, *radial*, *spheroidal* and *torsional*.

If we introduce spherical polar coordinates for the displacement, then it is possible to separate and solve the equations of elastodynamics to find the normal modes just like solving the Schrodinger equation for a central potential. Each of the three types of modes has a displacement vector $\boldsymbol{\xi}$ characterized by its own type of spherical harmonic.

The spheroidal modes have radial displacements proportional to $Y_l^m(\theta,\phi)\mathbf{e}_r$ (where $\theta$, $\phi$ are spherical coordinates, $Y_l^m$ is the scalar spherical harmonic of order $l$ and $m$, and $\mathbf{e}_r$ is the unit radial vector; and they have nonradial components proportional to $\boldsymbol{\nabla} Y_l^m$). These modes are called "spheroidal" because (when one ignores the tiny nonsphericity of the earth and ignores Coriolis and centrifugal forces due to the earth's rotation), their eigenfrequencies are independent of $m$, and thus can be studied by specializing to $m = 0$, in which case the displacements become

$$\xi_r \propto P_l(\cos\theta)\ , \quad \xi_\theta \propto \sin\theta P_l'(\cos\theta). \tag{12.69}$$

These displacements deform the earth in a spheroidal manner for the special case $l = 2$. [In Eq. (12.69) $P_l$ is the Legendre polynomial and $P_l'$ is the derivative of $P_l$ with respect to its argument.] The radial modes are the special case $l = 0$ of these spheroidal modes. It is often mistakenly asserted that there are no $l = 1$ modes because of conservation of momentum. In fact, $l = 1$ modes do exist: for example, the central regions of the earth can move up, while the outer regions move down. The $l = 2$ spheroidal mode has a period of 53 min. and can ring for about 1000 periods. (We say that its quality factor is $Q \sim 1000$.) This is typical for solid planets.

Toroidal modes have vanishing radial displacements, and their nonradial displacements are proportional to the vector spherical harmonic $\mathbf{e}_r \times \boldsymbol{\nabla} Y_l^m$. As for spheroidal modes, spherical symmetry of the unperturbed earth guarantees that the eigenfrequencies will be independent of the azimuthal quantum number $m$, so $m = 0$ is representative. For $m = 0$ the only nonzero component of the vector spherical harmonic $\mathbf{e}_r \times \boldsymbol{\nabla} Y_l^m$ is in the $\phi$ direction, and it gives

$$\xi_\phi \propto \sin\theta P_l'(\cos\theta) \; . \tag{12.70}$$

In these modes alternate zones of different latitude oscillate in opposite directions (clockwise or counterclockwise at some chosen moment of time), in such a way as to conserve total angular momentum.

When one writes the displacement vector $\boldsymbol{\xi}$ for a general vibration of the earth as a sum over these various types of normal modes, and inserts that sum into the wave equation (12.4b) (augmented, for greater realism, by gravitational forces), spherical symmetry of the unperturbed earth guarantees that the various modes will separate from each other, and for each mode the wave equation will give a radial wave equation analogous to that for a hydrogen atom in quantum mechanics. The boundary condition $\mathbf{T} \cdot \mathbf{e}_r = 0$ at the earth's surface constrains the solutions of the radial wave equation, for each mode, to be a discrete set, which one can label by the number $n$ of radial nodes that they possess (just as for the hydrogen atom). The frequencies of the modes increase with both $n$ and $l$.

For small values of the quantum numbers, the modes are quite sensitive to the model assumed for the earth's structure. For example, they are sensitive to whether one correctly includes the gravitational restoring force in the wave equation. However, for large $l$ and $n$, the spheroidal and toroidal modes become standing combinations of $P$ waves, $SV$ waves, $SH$ waves, Rayleigh and Love waves, and therefore they are rather insensitive to one's ignoring the effects of gravity.

### 12.4.5 Seismic tomography

Observations of all of these types of seismic waves clearly code much information about the earth's structure and inverting the measurements to infer this structure has become a highly sophisticated and numerically intensive branch of geophysics. The travel times of the P and S body waves can be measured at various points over the earth's surface and essentially allow $C_L$ and $C_T$ and hence $K/\rho$ and $\mu/\rho$ to be determined as functions of radius inside the earth. Travel times are $\lesssim 1$ hour. Using this type of analysis, seismologists can infer the presence of hot and cold regions within the mantle and then show how the rocks are circulating under the crust.

It is also possible to combine the observed travel times with the the earth's equation of elastostic equilibrium

$$\frac{dP}{dr} = \frac{K}{\rho}\frac{d\rho}{dr} = -g(r)\rho \; , \tag{12.71}$$

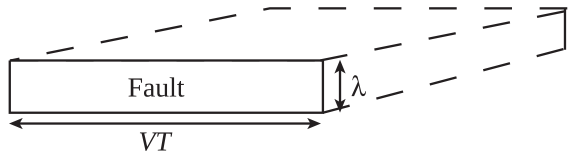



**Fig. 12.11:** Earthquake: The region of the fault that slips (solid rectangle), and the volume over which the strain is relieved, on one side of the fault (dashed region).

where the local gravity is given by

$$g = \frac{4\pi G}{r^2}\int_0^r r'^2 \rho(r') dr' \ , \tag{12.72}$$

to determine the distributions of density, pressure and elastic constants. Measurements of Rayleigh and Love waves can be used to probe the surface layers. The results of this procedure are then input to obtain free oscillation frequencies which compare well with the observations. The damping rates for the free oscillations furnish information on the interior viscosity.

****************************

## EXERCISES

**Exercise 12.9** *Derivation: Junction Condition at a Discontinuity*
Derive the junction condition $[\mathbf{T}_{jz}] = 0$ at a horizontal discontinuity between two media by the same method as one uses in electrodynamics to show that the normal component of the magnetic field must be continuous: Integrate the equation of motion $\rho d\mathbf{v}/dt = -\boldsymbol{\nabla} \cdot \mathbf{T}$ over the volume of an infinitesimally thin "pill box" centered on the boundary, and convert the volume integral to a surface integral via Gauss's theorem.

**Exercise 12.10** *Example: Reflection and Transmission of Normal, Longitudinal Waves at a Boundary*
Consider a longitudinal elastic wave incident normally on the boundary between two media, labeled 1,2. By matching the displacement and the normal component of stress at the boundary, show that the ratio of the transmitted wave amplitude to the incident amplitude is given by

$$t = \frac{2Z_1}{Z_1 + Z_2}$$

where $Z_{1,2} = [\rho_{1,2}(K_{1,2} + 4\mu_{1,2}/3)]^{1/2}$ is known as the *acoustic impedance.* (The impedance is independent of frequency and just a characteristic of the material.) Likewise, evaluate the amplitude reflection coefficent and verify that wave energy flux is conserved.

**Exercise 12.11** *Example: Earthquakes*

The magnitude $M$ of an earthquake is a quantitative measure of the strength of the seismic waves it creates. Roughly speaking, the elastic wave energy release can be inferred semi-empirically from the magnitude using the formula

$$E = 10^{5.2+1.44M}\,\mathrm{J}$$

The largest earthquakes have magnitude $\sim 8.5$.
One type of earthquake is caused by slippage along a fault deep in the crust. Suppose that most of the seismic power in an earthquake with $M \sim 8.5$ is emitted at frequencies $\sim$ 1Hz and that the quake lasts for a time $T \sim$ 100s. If $V$ is an average wave speed, then it is believed that the stress is relieved over an area of fault of length $\sim VT$ and a depth of order one wavelength. By comparing the stored elastic energy with the measured energy release make an estimate of the minimum strain prior to the earthquake. Is this reasonable? Hence estimate the typical displacement during the earthquake in the vicinity of the fault.
Make an order of magnitude estimate of the acceleration measurable by a seismometer in the next state and in the next continent. (Ignore the effects of density stratification, which are actually quite significant.)

**Exercise 12.12** *Example: Normal Modes of an Elastic, Homogeneous Sphere*
*EXERCISE NOT YET WRITTEN.*

****************************

## 12.5 The Relationship of Classical Waves to Quantum Mechanical Excitations.

In the previous chapter, we explored the limits of the continuum approximation and showed how we must acknowledge that solids are composed of atoms in order to account for the magnitude of the elastic constants and to explain why most solids yield under comparatively small strain. A quite different demonstration of the limits of the continuum approximation is provided by the normal modes of vibration of a finite sized solid body—e.g., the sphere treated in Sec. 12.4.4 and Ex. 12.12.

For any such body, one can solve the wave equation (12.4b) [subject to the vanishing-surface-force boundary condition $\mathbf{T} \cdot \mathbf{n} = 0$, Eq. (11.21)] to find the body's normal modes, as we did in Ex. 12.12 for the sphere. We shall label the normal modes by a single index $N$, and shall denote the eigenfrequency of mode $N$ by $\omega_N$ and its (typically complex) eigenfunction by $\boldsymbol{\xi}_N$. Then any general, small-amplitude disturbance in the body can be decomposed into a linear superposition of these normal modes:

$$\boxed{\boldsymbol{\xi}(\mathbf{x}, t) = \Re \sum_N a_N(t) \boldsymbol{\xi}_N(\mathbf{x}) \ , \quad a_N = A_N \exp(-i\omega_N t) \ .} \tag{12.73}$$

Here $\Re$ means to take the real part, $a_N$ is the *complex generalized coordinate* of mode $N$, and $A_N$ is its complex amplitude. It is convenient to normalize the eigenfunctions so that

$$\int \rho |\boldsymbol{\xi}_N|^2 dV = M \ , \tag{12.74}$$

where $M$ is the mass of the body; $A_N$ then measures the mean physical displacement in mode $N$.

Classical electromagnetic waves *in vacuo* are described by linear Maxwell equations, and, after they have been excited, will essentially propagate forever. This is not so for elastic waves, where the linear wave equation is only an approximation. Nonlinearities, and most especially impurities and defects in the homogeneous structure of the body's material, will cause the different modes to interact weakly so that their complex amplitudes $A_N$ change slowly with time according to a damped simple harmonic oscillator differential equation of the form

$$\boxed{\ddot{a}_N + (2/\tau_N)\dot{a}_N + \omega_N^2 a_N = F'_N/M \ .} \tag{12.75}$$

Here the second term on the left hand side is a damping term that will cause the mode to decay as long as $\tau_N > 0$, and $F'_N$ is a fluctuating or *stochastic* force on mode $N$ caused by weak coupling to the other modes. Equation (12.75) is the *Langevin equation* that we studied in Chap. 5, and the strength and spectrum of the fluctuating force $F'_N$ is determined by the fluctuation-dissipation theorem, Eq. (5.111). If the modes are thermalized at temperature $T$, then the fluctuating forces maintain an average energy of $kT$ in each one.

Now, what happens quantum mechanically? The ions and electrons in an elastic solid interact so strongly that it is very difficult to analyze them directly. A quantum mechanical treatment is much easier if one makes a canonical transformation from the coordinates and momenta of the individual ions or atoms to new, generalized coordinates $\hat{x}_N$ and momenta $\hat{p}_N$ that represent weakly interacting normal modes. These coordinates and momenta are Hermitian operators, and they are related to the quantum mechanical complex generalized coordinate $\hat{a}_n$ by

$$\boxed{\hat{x}_N = \frac{1}{2}(\hat{a}_N + \hat{a}_N^\dagger),} \tag{12.76a}$$

$$\boxed{\hat{p}_N = \frac{M\omega_N}{2i}(\hat{a}_N - \hat{a}_N^\dagger),} \tag{12.76b}$$

where the dagger denotes the Hermitean adjoint. We can transform back to obtain an expression for the displacement of the i'th ion

$$\hat{\mathbf{x}}_i = \frac{1}{2}\Sigma_N[\hat{a}_N \xi_N(\mathbf{x}_i) + \hat{a}_N^\dagger \xi_N^*(\mathbf{x}_i)] \tag{12.77}$$

[a quantum version of Eq. (12.73)].

The Hamiltonian can be written in terms of these coordinates as

$$\boxed{\hat{H} = \Sigma_N \left( \frac{\hat{p}_N^2}{2M} + \frac{1}{2}M\omega_N^2 \hat{x}_N^2 \right) + \hat{H}_{\rm int} \ ,} \tag{12.78}$$

where the first term is a sum of simple harmonic oscillator Hamiltonians for individual modes and $\hat{H}_{\rm int}$ is the perturbative interaction Hamiltonian which takes the place of the combined damping and stochastic forcing terms in the classical Langevin equation (12.75). When the various modes are thermalized, the mean energy in mode $N$ takes on the standard Bose-Einstein form

$$\bar{E}_N = \hbar\omega_N \left[\frac{1}{2} + \frac{1}{\exp(\hbar\omega_N/kT) - 1}\right] \tag{12.79}$$

[Eq. (4.27b) with vanishing chemical potential and augmented by a "zero-point energy" of $\frac{1}{2}\hbar\omega$], which reduces to $kT$ in the classical limit $\hbar \to 0$.

As the unperturbed Hamiltonian for each mode is identical to that for a particle in a harmonic oscillator potential well, it is sensible to think of each wave mode in a manner analogous to such a particle-in-well. Just as the particle-in-well can reside in any one of a series of discrete energy levels lying above the "zero point" energy of $\hbar\omega/2$, and separated by $\hbar\omega$, so each wave mode with frequency $\omega_N$ must have an energy $(n + 1/2)\hbar\omega_N$, where $n$ is an integer. The operator which causes the energy of the mode to decrease by $\hbar\omega_N$ is the *annihilation operator* for mode $n$

$$\boxed{\hat{\alpha}_N = \left(\frac{M\omega_N}{\hbar}\right)^{1/2} \hat{a}_N,} \tag{12.80}$$

and the operator which causes an increase in the energy by $\hbar\omega_N$ is its Hermitian conjugate, the *creation operator* $\hat{\alpha}_N^\dagger$. In the case of wave modes, it is useful to think of each increase or decrease in the energy as the creation or annihilation of an individual quantum or "particle" of energy, so that when the energy in mode $N$ is $(n+1/2)\hbar\omega_N$, there are $n$ quanta (particles) present. These particles are called *phonons*. Phonons are not conserved, and because they can co-exist in the same state (the same mode), they are *bosons*. They have individual energies and momenta which must be conserved in their interactions with each other and with other types of particles, e.g. electrons.

The important question is now, given an elastic solid at finite temperature, do we think of its thermal agitation as a superposition of classical wave modes or do we regard it as a gas of quanta? The answer depends upon what we want to do. From a purely fundamental viewpoint, the quantum mechanical description takes precedence. However, for many problems where the number of phonons per mode $n_N \sim kT/\hbar\omega_N$ is large compared to one, the classical description is amply adequate and much easier to handle. We do not need a quantum treatment when computing the normal modes of a vibrating building excited by an earthquake or when trying to understand how to improve the sound quality of a violin. Here the difficulty is in accommodating the boundary conditions so as to determine the normal modes. All this was expected. What comes as more of a surprise is that often, for purely classical problems, where $\hbar$ is quantitatively irrelevant, the fastest way to procede formally is to follow the quantum route and then take the limit $\hbar \to 0$. We shall see this graphically demonstrated when we discuss nonlinear plasma physics in Chap. 22.

**Box 12.3**
**Important Concepts in Chapter 12**

- Elastodynamic conservation of mass and momentum, Sec. 12.2.1
- Methods of deriving and solving wave equations in continuum mechanics, Box 12.2
- Decomposition of elastodynamic waves into longitudinal and transverse components, Sec. 12.2.2 and Ex. 12.1
- Dispersion relation and propagation speeds for longitudinal and transverse waves, Secs. 12.2.3 and 12.2.4
- Energy density and energy flux of elastodynamic waves, Sec. 12.2.5
- Waves on rods: compression waves, torsion waves, string waves, flexural waves, Secs. 12.3.1 – 12.3.4
- Edge waves, and Rayleigh waves as an example, Sec. 12.4.2
- Wave mixing in reflections off boundaries, Sec. 12.4.1
  - Conservation of tangential phase speed and its implications for directions of wave propagation, Sec. 12.4.1
  - Boundary conditions on stress and displacement, Sec. 12.4.1
- Greens functions for elastodynamic waves; Heaviside vs. physicists' Greens functions Sec. 12.4.3
- Elastodynamic free oscillations (normal modes), Secs. 12.4.4 and 12.5
- Relation of classical waves to quantum mechanical excitations, Sec. 12.5
- Onset of instability and zero-frequency modes related to bifurcation of equilibria, Sec. 12.3.5
- Free energy for a system on which a force is acting, and its use to diagnose stability, Ex. 12.8

# Bibliographic Note

For a discussion of textbooks on elasticity theory, including both elastostatics and elastodynamics, see the Bibliographic note at the end of Chap. 10.

# Bibliography


Anderson, D. L. & Dziewanski, A. M. 1984. *Scientific American.*

Eringen, A. C. & Suhubi, E. S. 1975. Chap. 7 of *Elastodynamics, Vol. II, Linear Theory*, New York: Academic Press.

Kolsky, H. 1963. *Stress Waves in Solids*, New York: Dover.

Johnson, L. R. 1974. "Green's function for Lamb's problem," *Geophys. J. Roy. astron. Soc.* **37**, 99–131.

Lifshitz, E. M. and Pitaevskii, L. P. 1980. *Statistical Physics. Part 1 (Third Edition)*, Oxford: Pergamon.

Landau, L. D. and Lifshitz, E. M. 1970. *Elasticity*, Oxford: Pergamon.

Stacey, F. D. 1977. *Physics of the Earth*, New York: Wiley.

Turcotte, D. L. and Schubert, G. 1982. *Geodynamics*, New York: Wiley.