Contents

Chapter 12

Elastodynamics

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Box 12.1
Reader's Guide

- This chapter is a companion to Chap. 11 (Elastostatics) and relies heavily on it.
- This chapter also relies rather heavily on geometric-optics concepts and formalism, as developed in Secs. 7.2 and 7.3, especially: phase velocity, group velocity, dispersion relation, rays and the propagation of waves and information and energy along them, the role of the dispersion relation as a Hamiltonian for the rays, and ray tracing.
- The discussion of continuum-mechanics wave equations in Box 12.2 underlies this book's treatment of waves in fluids (Chap. 16), especially in Plasmas (Part VI), and in general relativity (Chap. 27).
- The experience that the reader gains in this chapter with waves in solids will be useful when we encounter much more complicated waves in plasmas in Part VI.
- No other portions of this chapter are important for subsequent Parts of this book.

12.1 Overview

In the previous chapter we considered elastostatic equilibria, in which the forces acting on elements of an elastic solid were balanced, so the solid remained at rest. When this equilibrium is disturbed, the solid will undergo accelerations. This is the subject of this chapter — *elastodynamics*.

In Sec. 12.2, we derive the equations of motion for elastic media, paying particular attention to the underlying conservation laws and focusing especially on elastodynamic waves. We show that there are two distinct wave modes that propagate in a uniform, isotropic solid, *longitudinal* waves and *shear* waves, and both are nondispersive (their phase speeds are independent of frequency).

A major use of elastodynamics is in structural engineering, where one encounters vibrations (usually standing waves) on the beams that support buildings and bridges. In Sec. 12.3, we discuss the types of waves that propagate on bars, rods and beams and find that the boundary conditions at the free transverse surfaces make the waves dispersive. We also return briefly to the problem of bifurcation of equilibria (treated in Sec. 11.6) and show how, as the parameters controlling an equilibrium are changed, so it passes through a bifurcation point, the equilibrium becomes unstable; and the instability drives the system toward the new, stable equilibrium state.

A second application of elastodynamics is to seismology (Sec. 12.4). The earth is mostly a solid body through which waves can propagate. The waves can be excited naturally by earthquakes or artificially using man-made explosions. Understanding how waves propagate through the earth is important for locating the sources of earthquakes, for diagnosing the nature of an explosion (was it an illicit nuclear bomb test?) and for analyzing the structure of the earth. We briefly describe some of the wave modes that propagate through the earth and some of the inferences about the earth's structure that have been drawn from studying their propagation. In the process, we gain some experience in applying the tools of geometric optics to new types of waves, and we learn how rich can be the Green's function for elastodynamic waves, even when the medium is as simple as a homogeneous half space. We also briefly discuss how the methods by which geophysicists probe the earth's structure using seismic waves also find application in ultrasonic technology: imaging solid structures and also the human body using high-frequency sound waves.

Finally (Sec. 12.5), we return to physics to consider the quantum theory of elastodynamic waves. We compare the classical theory with the quantum theory, specializing to quantised vibrations in an elastic solid: phonons.

12.2 Basic Equations of Elastodynamics; Waves in a Homogeneous Medium

In Subsec. 12.2.1 of this section, we shall derive a vectorial equation that governs the dynamical displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ of a dynamically disturbed elastic medium. We shall then specialize to monochromatic plane waves in a homogeneous medium (Subsec. 12.2.2) and shall show how the monochromatic plane-wave equation can be converted into two wave equations, one for "longitudinal" waves (Subsec. 12.2.3) and the other for "transverse" waves (Subsec. 12.2.4). From those two wave equations we shall deduce the waves' dispersion relations, which act as Hamiltonians for geometric-optics wave propagation through inhomogeneous media. Our method of analysis is a special case of a very general approach to deriving wave equations in continuum mechanics. That general approach is sketched in Box 12.2. We shall follow that approach not only here, for elastic waves, but also in Chap. 16 for waves in fluids, Part VI for waves in plasmas and Chap. 27 for general relativistic gravitational waves. We shall conclude this section in Subsec. 12.2.5 with a discussion of the energy density and energy flux of these waves, and in Ex. 12.4 we shall explore the relationship of this energy density and flux to a Lagrangian for elastodynamic waves.

12.2.1 Equation of Motion for a Strained Elastic Medium

In Chap. 11, we learned that, when an elastic medium undergoes a displacement $\boldsymbol{\xi}(\mathbf{x})$, it builds up a strain $S_{ij} = \frac{1}{2}(\xi_{i;j} + \xi_{j;i})$, which in turn produces an internal stress $\mathbf{T} = -K\Theta\mathbf{g} - 2\mu\boldsymbol{\Sigma}$, where $\Theta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ is the expansion and $\boldsymbol{\Sigma} \equiv$(the trace-free part of $\mathbf{S}$) is the shear; see Eqs. (11.5) and (11.18). The stress $\mathbf{T}$ produces an elastic force per unit volume

$$\mathbf{f} = -\boldsymbol{\nabla}\cdot\mathbf{T} = \left(K + \frac{1}{3}\mu\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi} \tag{12.1}$$

[Eq. (11.30)], where K and μ are the bulk and shear moduli.

In Chap. 11, we restricted ourselves to elastic media that are in elastostatic equilibrium, so they are static. This equilibrium required that the net force per unit volume acting on the medium vanish. If the only force is elastic, then $\mathbf{f}$ must vanish. If the pull of gravity is also significant, then $\mathbf{f} + \rho\boldsymbol{g}$ vanishes, where ρ is the medium's mass density and $\boldsymbol{g}$ the acceleration of gravity.

In this chapter, we shall focus on dynamical situations, in which an unbalanced force per unit volume causes the medium to move — with the motion, in this chapter, taking the form of an elastodynamic wave. For simplicity, we shall assume, unless otherwise stated, that the only significant force is elastic; i.e., that the gravitational force is negligible by comparison. In Ex. 12.1, we shall show that this is the case for elastodynamic waves in most media on Earth whenever the wave frequency $\omega/2\pi$ is higher than about 0.001 Hz (which is usually the case in practice). Stated more precisely, in a homogeneous medium we can ignore the gravitational force whenever the elastodynamic wave's angular frequency ω is much larger than g/C, where g is the acceleration of gravity and C is the wave's propagation speed.

Consider, then, a dynamical, strained medium with elastic force per unit volume (12.1) and no other significant force (negligible gravity), and with velocity

$$\boxed{\mathbf{v} = \frac{\partial\boldsymbol{\xi}}{\partial t}}\,. \tag{12.2a}$$

The law of momentum conservation states that the force per unit volume $\mathbf{f}$, if nonzero, must produce a rate of change of momentum per unit volume $\rho\mathbf{v}$ according to the equation[1]

$$\frac{\partial(\rho\mathbf{v})}{\partial t} = \mathbf{f} = -\boldsymbol{\nabla}\cdot\mathbf{T} = \left(K + \frac{1}{3}\mu\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi}\ . \tag{12.2b}$$

[1] In Sec. 13.5 of the next chapter we shall learn that the motion of the medium produces a stress $\rho\mathbf{v}\otimes\mathbf{v}$ that must be included in this equation if the velocities are large. However, this subtle dynamical stress is always negligible in elastodynamic waves because the displacements and hence velocities $\mathbf{v}$ are tiny and $\rho\mathbf{v}\otimes\mathbf{v}$ is second order in the displacement. For this reason, we shall delay studying this subtle nonlinear effect until Chap. 13.

Notice that, when rewritten in the form

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} = 0\ ,$$

this is the version of the law of momentum conservation discussed in Chap. 1 [Eq. (1.36)]. It has the standard form for a conservation law (time derivative of density of something, plus divergence of flux of that something, vanishes (see end of Sec. 1.8); $\rho\mathbf{v}$ is the density of momentum, and the stress tensor $\mathbf{T}$ is by definition the flux of momentum. Equations (12.2a) and (12.2b), together with the law of mass conservation [the obvious analog of Eqs. (1.29) for conservation of charge and particle number],

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0} \tag{12.2c}$$

are a complete set of equations for the evolution of the displacement $\boldsymbol{\xi}(\mathbf{x},t)$, the velocity $\mathbf{v}(\mathbf{x},t)$ and the density $\rho(\mathbf{x},t)$.

The elastodynamic equations (12.2) are nonlinear because of the $\rho\mathbf{v}$ terms (see below). From them we shall derive a linear wave equation for the displacement vector $\boldsymbol{\xi}(\mathbf{x},t)$. Our derivation provides us with a simple (almost trivial) example of the general procedure discussed in Box 12.2.

To derive a linear wave equation, we must find some small parameter in which to expand. The obvious choice in elastodynamics is the magnitude of the components of the strain $S_{ij} = \frac{1}{2}(\xi_{i;j} + \xi_{j;i})$, which are less than about 10^{-3} so as to remain below the proportionality limit, i.e. to remain in the realm where stress is proportional to strain (Sec. 11.3.2). Equally well, we can regard the displacement $\boldsymbol{\xi}$ itself as our small parameter (or more precisely, $\xi/\lambda\!\!\!^{-}$, the magnitude of $\boldsymbol{\xi}$ divided by the reduced wavelength of its perturbations).

If the medium's equilibrium state were homogeneous, the linearization would be trivial. However, we wish to be able to treat perturbations of inhomogeneous equilibria such as seismic waves in the Earth, or perturbations of slowly changing equilibria such as vibrations of a pipe or mirror that is gradually changing temperature. In almost all situations the lengthscale $\mathcal{L}$ and timescale $\mathcal{T}$ on which the medium's equilibrium properties (ρ, K, μ) vary are extremely large compared to the lengthscale and timescale of the dynamical perturbations (their reduced wavelength $\lambda\!\!\!^{-}$ = wavelength$/2\pi$ and $1/\omega$ = period$/2\pi$). This permits us to perform a two-lengthscale expansion (like the one that underlies geometric optics, Sec. 7.3) alongside our small-strain expansion.

In analyzing a dynamical perturbation of an equilibrium state, we use $\boldsymbol{\xi}(\mathbf{x},t)$ to denote the dynamical displacement (i.e., we omit from it the equilibrium's static displacement, and similarly we omit from $\mathbf{S}$ the equilibrium strain). We write the density as $\rho+\delta\rho$, where $\rho(\mathbf{x})$ is the equilibrium density distribution and $\delta\rho(\mathbf{x},t)$ is the dynamical density perturbation, which is first-order in the dynamical displacement $\boldsymbol{\xi}$. Inserting these into the equation of mass conservation (12.2c), we obtain $\partial\delta\rho/\partial t + \boldsymbol{\nabla}\cdot[(\rho+\delta\rho)\mathbf{v}] = 0$. Because $\mathbf{v} = \partial\boldsymbol{\xi}/\partial t$ is first order, the term $(\delta\rho)\mathbf{v}$ is second order and can be dropped, resulting in the linearized equation $\partial\delta\rho/\partial t + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$. Because ρ varies on a much longer lengthscale than $\mathbf{v}$ ($\mathcal{L}$ vs. $\lambda\!\!\!^{-}$), we can pull ρ out of the derivative. Setting $\mathbf{v} = \partial\boldsymbol{\xi}/\partial t$ and interchanging the time

Box 12.2
Wave Equations in Continuum Mechanics

In this box, we make an investment for future chapters by considering wave equations in some generality.

Most wave equations arise as approximations to the full set of equations that govern a dynamical physical system. It is usually possible to arrange those full equations as a set of first order partial differential equations that describe the dynamical evolution of a set of n physical quantities, V_A, with $A = 1, 2, ..., n$: i.e.

$$\frac{\partial V_A}{\partial t} + F_A(V_B) = 0 \ . \tag{1}$$

[For elastodynamics there are $n = 7$ quantities V_A: $\{\rho, \rho v_x, \rho v_y, \rho v_z, \xi_x, \xi_y, \xi_z\}$ (in Cartesian coordinates); and the seven equations (1) are mass conservation, momentum conservation, and $\partial\xi_j/\partial t = v_j$; Eqs. (12.2).]

Now, most dynamical systems are intrinsically nonlinear (Maxwell's equations *in vacuo* being a conspicuous exception), and it is usually quite hard to find nonlinear solutions. However, it is generally possible to make a perturbation expansion in some small physical quantity about a time-independent equilibrium and retain only those terms that are linear in this quantity. We then have a set of n linear partial differential equations that are much easier to solve than the nonlinear ones—and that usually turn out to have the character of wave equations (i.e., to be *hyperbolic*). Of course, the solutions will only be a good approximation for small amplitude waves. [In elastodynamics, we justify linearization by requiring that the strains be below the proportionality limit, we linearize in the strain or displacement of the dynamical perturbation, and the resulting linear wave equation is $\rho\partial^2\boldsymbol{\xi}/\partial t^2 = (K + \frac{1}{3}\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi}$; Eq. (12.4b).]

Boundary Conditions

In some problems, e.g. determining the normal modes of vibration of a building during an earthquake, or analyzing the sound from a violin or the vibrations of a finite-length rod, the boundary conditions are intricate and have to be incorporated as well as possible, to have any hope of modeling the problem. The situation is rather similar to that familiar from elementary quantum mechanics. The waves are often localised within some region of space, like bound states, in such a way that the eigenfrequencies are discrete, for example, standing-wave modes of a plucked string. In other problems the volume in which the wave propagates is essentially infinite (e.g. waves on the surface of the ocean or seismic waves propagating through the earth), as happens with unbound quantum states. Then the only boundary condition is essentially that the wave amplitude remain finite at large distances. In this case, the wave spectrum is usually continuous.

Box 12.2, Continued

Geometric Optics Limit and Dispersion Relations

The solutions to the wave equation will reflect the properties of the medium through which the wave is propagating, as well as its boundaries. If the medium and boundaries have a finite number of discontinuities but are otherwise smoothly varying, there is a simple limiting case: waves of short enough wavelength and high enough frequency that they can be analyzed in the *geometric optics approximation* (Chap. 7).

The key to geometric optics is the *dispersion relation*, which (as we learned in Sec. 7.3) acts as a Hamiltonian for the propagation. Recall from Chap. 7 that, although the medium may actually be inhomogeneous and might even be changing with time, when deriving the dispersion relation we can approximate it as precisely homogeneous and time-independent, and can resolve the waves into plane-wave modes, i.e. modes in which the perturbations vary $\propto \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Here $\mathbf{k}$ is the *wave vector* and ω is the *angular frequency.* This allows us to remove all the temporal and spatial derivatives and converts our set of partial differential equations into a set of homogeneous, linear algebraic equations. When we do this, we say that our normal modes are *local.* If, instead, we were to go to the trouble of solving the partial differential wave equation with its attendant boundary conditions, the modes would be referred to as *global.*

The linear algebraic equations for a local problem can be written in the form $M_{AB}V_B = 0$, where V_A is the vector of n dependent variables and the elements M_{AB} of the $n \times n$ matrix $||M_{AB}||$ depend on $\mathbf{k}$ and ω as well as on parameters p_α that describe the local conditions of the medium. See, e.g., Eq. (12.6). This set of equations can be solved in the usual manner by requiring that the determinant of $\|M_{AB}\|$ vanish. Carrying through this procedure yields a polynomial, usually of n'th order, for $\omega(\mathbf{k}, p_\alpha)$. This polynomial is the dispersion relation. It can be solved (analytically in simple cases and numerically in general) to yield a number of complex values for ω (the eigenfrequencies), with $\mathbf{k}$ regarded as real. (Of course, we could treat $\mathbf{k}$ as complex and ω as real; but for concreteness, we shall take $\mathbf{k}$ to be real.) Armed with the complex eigenfrequencies, we can solve for the associated eigenvectors V_A. The eigenfrequencies and eigenvectors fully characterize the solution of the local problem, and can be used to solve for the waves' temporal evolution from some given initial conditions in the usual manner. (As we shall see several times, especially when we discuss Landau damping in Chap. 22, there are some subtleties that can arise.)

What does a complex value of the angular frequency ω mean? We have posited that all small quantities vary $\propto \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$. If ω has a positive imaginary part, then the small perturbation quantities will grow exponentially with time. Conversely, if it has a negative imaginary part, they will decay. Now, polynomial equations with real coefficients have complex conjugate solutions. Therefore if there is a decaying mode there must also be a growing mode. Growing modes correspond to *instability*, a topic that we shall encounter often.

derivative and divergence, we then obtain $\partial\delta\rho/\partial t + \rho\partial(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})/\partial t = 0$. Noting that ρ varies on a timescale $\mathcal{T}$ long compared to that $1/\omega$ of $\boldsymbol{\xi}$ and $\delta\rho$, we can integrate this to obtain the linear relation

$$\boxed{\frac{\delta\rho}{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}} \,. \tag{12.3}$$

This linearized equation for the fractional perturbation of density could equally well have been derived by considering a small volume V of the medium that contains mass $M = \rho V$, and by noting that the dynamical perturbations lead to a volume change $\delta V/V = \Theta = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ [Eq. (11.8)], so conservation of mass requires $0 = \delta M = \delta(\rho V) = V\delta\rho + \rho\delta V = V\delta\rho + \rho V\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, which implies $\delta\rho/\rho = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$. This is the same as Eq. (12.3).

The equation of momentum conservation (12.2b) can be handled similarly. By setting $\rho \to \rho(\mathbf{x}) + \delta\rho(\mathbf{x}, t)$, and then linearizing (i.e., dropping the $\delta\rho\,\mathbf{v}$ term) and pulling the slowly varying $\rho(\mathbf{x})$ out from under the time derivative, we convert $\partial(\rho\mathbf{v})/\partial t$ into $\rho\partial\mathbf{v}/\partial t = \rho\partial^2\boldsymbol{\xi}/\partial t^2$. Inserting this into Eq. (12.2b), we obtain the linear wave equation

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\boldsymbol{\nabla}\cdot\mathbf{T}_{el} \,, \tag{12.4a}$$

i.e.

$$\boxed{\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = (K + \frac{1}{3}\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \mu\nabla^2\boldsymbol{\xi}} \,. \tag{12.4b}$$

In this equation, terms involving a derivative of K or μ have been omitted because the two-lengthscale assumption $\mathcal{L} \gg \lambda\!\!\!^{-}$ makes them negligible compared to the terms we have kept.

Equation (12.4b) is the first of many wave equations we shall encounter in elastodynamics, fluid mechanics, plasma physics, and general relativity.

12.2.2 Elastodynamic Waves

Continuing to follow our general procedure for deriving and analyzing wave equations as outlined in Box 12.2, we next derive dispersion relations for two types of waves (longitudinal and transverse) that are jointly incorporated into the general elastodynamic wave equation (12.4b).

Recall from Sec. 7.3.1 that, although a dispersion relation can be used as a Hamiltonian for computing wave propagation through an *inhomogeneous medium*, one can derive the dispersion relation most easily by specializing to monochromatic plane waves propagating through a medium that is precisely homogeneous. Therefore, we seek a plane-wave solution, i.e. a solution of the form

$$\boldsymbol{\xi}(\mathbf{x}, t) \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \,, \tag{12.5}$$

to the wave equation (12.4b) with ρ, K and μ regarded as homogeneous (constant). (To deal with more complicated perturbations of a homogeneous medium, we can think of this wave as being an individual Fourier component and linearly superpose many such waves as a Fourier integral.) Since our wave is planar and monochromatic, we can remove the derivatives in Eq. (12.4b) by making the substitutions $\boldsymbol{\nabla} \to i\mathbf{k}$ and $\partial/\partial t \to -i\omega$ (the first of which implies $\nabla^2 \to -k^2$, $\boldsymbol{\nabla}\cdot \to i\mathbf{k}\cdot$, $\boldsymbol{\nabla}\times \to i\mathbf{k}\times$.) We thereby reduce the partial differential equation (12.4b) to a vectorial algebraic equation:

$$\rho\omega^2 \boldsymbol{\xi} = (K + \frac{1}{3}\mu)\mathbf{k}(\mathbf{k}\cdot\boldsymbol{\xi}) + \mu k^2 \boldsymbol{\xi}\ . \tag{12.6}$$

[This reduction is only possible because the medium is uniform, or in the geometric optics limit of near uniformity; otherwise, we must solve the second order partial differential equation (12.4b).]

How do we solve this equation? The sure way is to write it as a 3×3 matrix equation $M_{ij}\xi_j = 0$ for the vector $\boldsymbol{\xi}$ and set the determinant of M_{ij} to zero (Box 12.2 and Ex. 12.3). This is not hard for small or sparse matrices. However, some wave equations are more complicated and it often pays to think about the waves in a geometric, coordinate-independent way before resorting to brute force.

The quantity that oscillates in the elastodynamic waves (12.6) is the vector field $\boldsymbol{\xi}$. The nature of its oscillations is influenced by the scalar constants ρ, μ, K, ω and by just one quantity that has directionality: the constant vector $\mathbf{k}$. It seems reasonable to expect the description (12.6) of the oscillations to simplify, then, if we resolve the oscillations into a "longitudinal" component (or "mode") along $\mathbf{k}$ and a "transverse" component (or "mode") perpendicular to $\mathbf{k}$, as shown in Fig. 12.1:

$$\boxed{\boldsymbol{\xi} = \boldsymbol{\xi}_L + \boldsymbol{\xi}_T\ , \quad \boldsymbol{\xi}_L = \xi_L\hat{\mathbf{k}}\ , \quad \boldsymbol{\xi}_T\cdot\hat{\mathbf{k}} = 0}\ . \tag{12.7a}$$

Here $\hat{\mathbf{k}} \equiv \mathbf{k}/k$ is the unit vector along the propagation direction. It is easy to see that the longitudinal mode $\boldsymbol{\xi}_L$ has nonzero expansion $\Theta \equiv \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_L \neq 0$ but vanishing rotation $\boldsymbol{\phi} = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{\xi}_L = 0$, and can therefore be written as the gradient of a scalar potential,

$$\boxed{\boldsymbol{\xi}_L = \boldsymbol{\nabla}\psi}\ . \tag{12.7b}$$

By contrast, the transverse mode has zero expansion but nonzero rotation and can thus be written as the curl of a vector potential,

$$\boxed{\boldsymbol{\xi}_T = \boldsymbol{\nabla}\times\mathbf{A}}\ ; \tag{12.7c}$$

cf. Ex. 12.2.

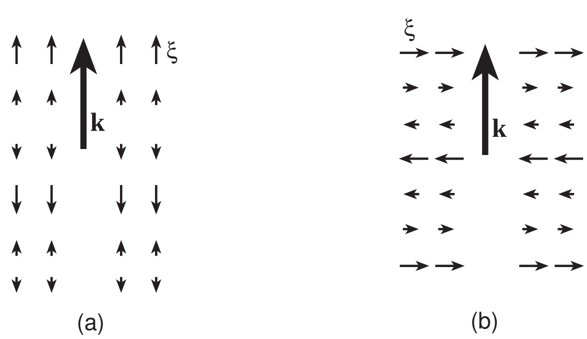

Fig. 12.1: Displacements in an isotropic, elastic solid, perturbed by a) a longitudinal mode, b) a transverse mode.

12.2.3 Longitudinal Sound Waves

For the longitudinal mode the algebraic wave equation (12.6) reduces to the following simple relation [as one can easily see by inserting $\boldsymbol{\xi} \equiv \boldsymbol{\xi}_L = \xi_L \hat{\mathbf{k}}$ into Eq. (12.6) , or, alternatively, by taking the divergence of (12.6), which is equivalent to taking the scalar product with $\mathbf{k}$]:

$$\omega^2 = \frac{K + \frac{4}{3}\mu}{\rho} k^2 \; ; \quad \text{i.e.} \quad \boxed{\omega = \Omega(\mathbf{k}) = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{1/2} k} \, . \tag{12.8}$$

Here $k = |\mathbf{k}|$ is the wave number (the magnitude of $\mathbf{k}$). This relation between ω and $\mathbf{k}$ is the longitudinal mode's dispersion relation.

From the geometric-optics analysis in Sec. 7.3 we infer that, if K, μ and ρ vary spatially on an inhomogeneity lengthscale $\mathcal{L}$ large compared to $1/k = \lambda\!\!\!^{-}$, and vary temporally on a timescale $\mathcal{T}$ large compared to $1/\omega$, then the dispersion relation (12.8), with Ω now depending on $\mathbf{x}$ and t through K, μ, and ρ, serves as a Hamiltonian for the wave propagation. In Sec. 12.4 and Fig. 12.5 below we shall use this to deduce details of the propagation of seismic waves through the Earth's inhomogeneous interior.

As we discussed in great detail in Sec. 7.2, associated with any wave mode is its phase velocity, $\mathbf{V}_{\rm ph} = (\omega/k)\hat{\mathbf{k}}$ and its phase speed $V_{\rm ph} = \omega/k$. The dispersion relation (12.8) implies that for longitudinal elastodynamic modes, the phase speed is

$$\boxed{C_L = \frac{\omega}{k} = \left(\frac{K + \frac{4}{3}\mu}{\rho}\right)^{1/2}} \, . \tag{12.9a}$$

As this does not depend on the wave number $k \equiv |\mathbf{k}|$, the mode is *non-dispersive*, and as it does not depend on the direction $\hat{\mathbf{k}}$ of propagation through the medium, the phase speed is also isotropic, naturally enough, and the group velocity $V_{g\,j} = \partial\Omega/\partial k_j$ is equal to the phase velocity:

$$\boxed{\mathbf{V}_g = \mathbf{V}_{\rm ph} = C_L \hat{\mathbf{k}}} \, . \tag{12.9b}$$

Elastodynamic longitudinal modes are similar to sound waves in a fluid. However, in a fluid, as we shall see in Eq. (16.48), the sound waves travel with phase speed $V_{\rm ph} = (K/\rho)^{1/2}$ [the limit of Eq. (12.9a) when the shear modulus vanishes].[2] This fluid sound speed is lower than the C_L of a solid with the same bulk modulus because the longitudinal displacement necessarily entails shear (note that in Fig. 12.1a the motions are not an isotropic expansion), and in a solid there is a restoring shear stress (proportional to μ) that is absent in a fluid.

Because the longitudinal phase velocity is independent of frequency, we can write down general planar longitudinal-wave solutions to the elastodynamic wave equation (12.4b) in the following form:

$$\boldsymbol{\xi} = \xi_L \hat{\mathbf{k}} = F(\hat{\mathbf{k}} \cdot \mathbf{x} - C_L t)\hat{\mathbf{k}} \, , \tag{12.10}$$

[2]Equation (16.48) says the fluid sound speed is $C = \sqrt{(\partial P/\partial \rho)_s}$, i.e. the square root of the derivative of the fluid pressure with respect to density at fixed entropy. In the language of elasticity theory, the fractional change of density is related to the expansion Θ by $\delta\rho/\rho = -\Theta$ [Eq. (12.3)], and the accompanying change of pressure is $\delta P = -K\Theta$ [paragraph preceding Eq. (11.18)], i.e. $\delta P = K(\delta\rho/\rho)$. Therefore the fluid mechanical sound speed is $C = \sqrt{\delta P/\delta\rho} = \sqrt{K/\rho}$.

where $F(x)$ is an arbitrary function. This describes a wave propagating in the (arbitrary) direction $\hat{\mathbf{k}}$ with an arbitrary profile determined by the function F.

12.2.4 Transverse Shear Waves

To derive the dispersion relation for a transverse wave we can simply make use of the transversality condition $\mathbf{k} \cdot \boldsymbol{\xi}_T = 0$ in Eq. (12.6); or, equally well, we can take the curl of Eq. (12.6) (multiply it by $i\mathbf{k}\times$), thereby projecting out the transverse piece, since the longitudinal part of $\boldsymbol{\xi}$ has vanishing curl. The result is

$$\omega^2 = \frac{\mu}{\rho} k^2 \; ; \quad \text{i.e.} \quad \boxed{\omega = \Omega(\mathbf{k}) \equiv \left(\frac{\mu}{\rho}\right)^{1/2} k} \, . \tag{12.11}$$

This dispersion relation $\omega = \Omega(\mathbf{k})$ serves as a geometric-optics Hamiltonian for wave propagation when μ and ρ vary slowly with $\mathbf{x}$ and/or t, and it also implies that the transverse waves propagate with a phase speed C_T and phase and group velocities given by

$$\boxed{C_T = \left(\frac{\mu}{\rho}\right)^{1/2}} \; ; \tag{12.12a}$$

$$\boxed{\mathbf{V}_{\rm ph} = \mathbf{V}_g = C_T \hat{\mathbf{k}}} \, . \tag{12.12b}$$

As $K > 0$, the shear wave speed C_T is always less than the speed C_L of longitudinal waves [Eq. (12.9a)].

These transverse modes are known as *shear waves* because they are driven by the shear stress; cf. Fig. 12.1b. There is no expansion and therefore no change in volume associated with shear waves. They do not exist in fluids, but they are close analogs of the transverse vibrations of a string.

Longitudinal waves can be thought of as scalar waves, since they are fully describable by a single component ξ_L of the displacement $\boldsymbol{\xi}$: that along $\hat{\mathbf{k}}$. Shear waves, by contrast, are inherently vectorial. Their displacement $\boldsymbol{\xi}_T$ can point in any direction orthogonal to $\mathbf{k}$. Since the directions orthogonal to $\mathbf{k}$ form a two-dimensional space, once $\mathbf{k}$ has been chosen, there are two independent states of polarization for the shear wave. These two polarization states, together with the single one for the scalar, longitudinal wave, make up the three independent degrees of freedom in the displacement $\boldsymbol{\xi}$.

In Ex. 12.3 we deduce these properties of $\boldsymbol{\xi}$ using matrix techniques.

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EXERCISES

Exercise 12.1 **** Problem: Influence of Gravity on Wave Speed*
Modify the wave equation (12.4b) to include the effect of gravity. Assume that the medium is homogeneous and the gravitational field is constant. By comparing the orders of magnitude

of the terms in the wave equation, verify that the gravitational terms can be ignored for high-enough frequency elastodynamic modes: $\omega \gg g/C_{L,T}$. For wave speeds ~ 3 km/s, this says $\omega/2\pi \gg 0.0005$ Hz. Seismic waves are generally in this regime.

Exercise 12.2 *Example: Scalar and Vector Potentials for Elastic Waves in a Homogeneous Solid*
Just as in electromagnetic theory, it is sometimes useful to write the displacement $\boldsymbol{\xi}$ in terms of scalar and vector potentials,

$$\boldsymbol{\xi} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla} \times \mathbf{A} . \tag{12.13}$$

(The vector potential $\mathbf{A}$ is, as usual, only defined up to a gauge transformation, $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\varphi$, where φ is an arbitrary scalar field.) By inserting Eq. (12.13) into the general elastodynamic wave equation (12.4b), show that the scalar and vector potentials satisfy the following wave equations in a homogeneous solid:

$$\frac{\partial^2\psi}{\partial t^2} = C_L^2 \nabla^2 \psi , \quad \frac{\partial^2 \mathbf{A}}{\partial t^2} = C_T^2 \nabla^2 \mathbf{A} . \tag{12.14}$$

Thus, the scalar potential ψ generates longitudinal waves, while the vector potential $\mathbf{A}$ generates transverse waves.

Exercise 12.3 *Example: Solving the Algebraic Wave Equation by Matrix Techniques*
By using the matrix techniques discussed in the next-to-the-last paragraph of Box 12.2, deduce that the general solution to the algebraic wave equation (12.6) is the sum of a longitudinal mode with the properties deduced in Sec. 12.2.3, and two transverse modes with the properties deduced in Sec. 12.2.4. [Note: This matrix technique is necessary and powerful when the algebraic dispersion relation is complicated, e.g. for plasma waves; Secs. 21.4.1 and 21.5.1. Elastodynamic waves are simple enough that we did not need this matrix technique in the text.] Guidelines for solution:

(a) Rewrite the algebraic wave equation in the matrix form $M_{ij}\xi_j = 0$, obtaining thereby an explicit form for the matrix $||M_{ij}||$ in terms of ρ, K, μ, ω and the components of $\mathbf{k}$.

(b) This matrix equation has a solution if and only if the determinant of the matrix $||M_{ij}||$ vanishes. (Why?) Show that $\det||M_{ij}|| = 0$ is a cubic equation for ω^2 in terms of k^2, and that one root of this cubic equation is $\omega = C_L k$, while the other two roots are $\omega = C_T k$ with C_L and C_T given by Eqs. (12.9a) and (12.12a).

(c) Orient Cartesian axes so that $\mathbf{k}$ points in the z direction. Then show that, when $\omega = C_L k$, the solution to $M_{ij}\xi_j = 0$ is a longitudinal wave, i.e., a wave with $\boldsymbol{\xi}$ pointing in the z direction, the same direction as $\mathbf{k}$.

(d) Show that, when $\omega = C_T k$, there are two linearly independent solutions to $M_{ij}\xi_j = 0$, one with $\boldsymbol{\xi}$ pointing in the x direction (transverse to $\mathbf{k}$) and the other in the y direction (also transverse to $\mathbf{k}$).

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12.2.5 Energy of Elastodynamic Waves

Elastodynamic waves transport energy, just like waves on a string. The waves' kinetic energy density is obviously $\frac{1}{2}\rho\mathbf{v}^2 = \frac{1}{2}\rho\dot{\boldsymbol{\xi}}^2$, where the dot means $\partial/\partial t$. The elastic energy density is given by Eq. (11.24), so the total energy density is

$$U = \frac{1}{2}\rho\dot{\boldsymbol{\xi}}^2 + \frac{1}{2}K\Theta^2 + \mu\Sigma_{ij}\Sigma_{ij} \ . \tag{12.15a}$$

In Ex. 12.4 we show that (as one might expect) the elastodynamic wave equation (12.4b) can be derived from an action whose Lagrangian density is the kinetic energy density minus the elastic energy density. We also show that associated with the waves is an energy flux $\mathbf{F}$ (not to be confused with a force for which we use the same notation) given by

$$F_i = -K\Theta\dot{\xi}_i - 2\mu\Sigma_{ij}\dot{\xi}_j \ . \tag{12.15b}$$

As the waves propagate, energy sloshes back and forth between the kinetic part and the elastic part, with the time averaged kinetic energy being equal to the time averaged elastic energy (equipartion of energy). For the planar, monochromatic, longitudinal mode, the time averaged energy density and flux are

$$\boxed{U_L = \rho\langle\dot{\xi}_L^2\rangle} \ , \quad \boxed{\mathbf{F}_L = U_L C_L \hat{\mathbf{k}}} \ , \tag{12.16}$$

where $\langle ...\rangle$ denotes an average over one period or wavelength of the wave. Similarly, for the planar, monochromatic, transverse mode, the time averaged density and flux of energy are

$$\boxed{U_T = \rho\langle\dot{\boldsymbol{\xi}}_T^2\rangle} \ , \quad \boxed{\mathbf{F}_T = U_T C_T \hat{\mathbf{k}}} \tag{12.17}$$

(Ex. 12.4). Thus, elastodynamic waves transport energy at the same speed $C_{L,T}$ as the waves propagate, and in the same direction $\hat{\mathbf{k}}$. This is the same behavior as electromagnetic waves in vacuum, whose Poynting flux and energy density are related by $\mathbf{F}_{EM} = U_{EM}c\hat{\mathbf{k}}$ with c the speed of light, and the same as all forms of dispersion-free scalar waves (e.g. sound waves in a medium), cf. Eq. (7.31). Actually, this is the dispersion-free limit of the more general result that the energy of any wave, in the geometric-optics limit, is transported with the wave's group velocity, $\mathbf{V}_g$; see Sec. 7.2.2.

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EXERCISES

Exercise 12.4 *Example: Lagrangian and Energy for Elastodynamic Waves*
Derive the energy-density, energy-flux, and Lagrangian properties of elastodynamic waves that are stated in Sec. 12.2.5. Guidelines:

(a) For ease of calculation (and for greater generality), consider an elastodynamic wave in a possibly anisotropic medium, for which

$$T_{ij} = -Y_{ijkl}\xi_{k;l} \tag{12.18}$$

with Y_{ijkl} the tensorial modulus of elasticity, which is symmetric under interchange of the first two indices ij, and under interchange of the last two indices kl, and under interchange of the first pair ij with the last pair kl [Eq. (11.17) and associated discussion]. Show that for an isotropic medium

$$Y_{ijkl} = \left(K - \frac{2}{3}\mu\right) g_{ij}g_{kl} + \mu(g_{ik}g_{jl} + g_{il}g_{jk}) \; . \tag{12.19}$$

(Recall that in the orthonormal bases to which we have confined ourselves, the components of the metric are $g_{ij} = \delta_{ij}$, i.e. the Kronecker delta.)

(b) For these waves the elastic energy density is $\frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l}$ [Eq. (11.25)]. Show that the kinetic energy density minus the elastic energy density

$$\mathcal{L} = \frac{1}{2}\rho\,\dot{\xi}_i\dot{\xi}_i - \frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l} \tag{12.20}$$

is a Lagrangian density for the waves; i.e., show that the vanishing of its variational derivative $\delta\mathcal{L}/\delta\xi_j \equiv \partial\mathcal{L}/\partial\xi_j - (\partial/\partial t)(\partial\mathcal{L}/\partial\dot{\xi}_j) = 0$ is equivalent to the elastodynamic equations $\rho\ddot{\boldsymbol{\xi}} = -\boldsymbol{\nabla}\cdot\mathbf{T}$.

(c) The waves' energy density and flux can be constructed by the vector-wave analog of the canonical procedure of Eq. (7.35c):

$$U = \frac{\partial\mathcal{L}}{\partial\dot{\xi}_i}\dot{\xi}_i - \mathcal{L} = \frac{1}{2}\rho\,\dot{\xi}_i\dot{\xi}_i + \frac{1}{2}Y_{ijkl}\xi_{i;j}\xi_{k;l} \; ,$$
$$F_j = \frac{\partial\mathcal{L}}{\partial\xi_{i;j}}\dot{\xi}_i = -Y_{ijkl}\dot{\xi}_i\xi_{k;l} \; . \tag{12.21}$$

Verify that these density and flux satisfy the energy conservation law, $\partial U/\partial t + \boldsymbol{\nabla}\cdot\mathbf{F} = 0$. Verify, using Eq. (12.19), that for an isotropic medium expressions (12.21) for the energy density and flux become the expressions (12.15) given in the text.

(d) Show that, in general (for an arbitrary mixture of wave modes), the time average of the total kinetic energy in some huge volume is equal to that of the total elastic energy. Show further that, for an individual longitudinal or transverse, planar, monochromatic, mode, the time averaged kinetic energy density and time averaged elastic energy density are both independent of spatial location. Combining these results, infer that for a single mode, the time averaged kinetic and elastic energy densities are equal, and therefore the time averaged total energy density is equal to twice the time averaged kinetic energy density. Show that this total time averaged energy density is given by the first of Eqs. (12.16) and (12.17).

(e) Show that the time average of the energy flux (12.15b) for the longitudinal and transverse modes is given by the second of Eqs. (12.16) and (12.17), so the energy propagates with the same speed and direction as the waves themselves.

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12.3 Waves in Rods, Strings and Beams

Before exploring applications of the above longitudinal and transverse waves (Sec. 12.4 below), we shall discuss how the wave equations and wave speeds get modified when the medium through which they propagate is bounded. Despite this situation being formally "global" in the sense of Box 12.2, elementary considerations enable us to derive the relevant dispersion relations without much effort.

12.3.1 Compression Waves in a Rod

First consider a longitudinal wave propagating along a light (negligible gravity), thin, unstressed rod. We shall call this a *compression wave*. Introduce a Cartesian coordinate system with the x-axis parallel to the rod. When there is a small displacement ξ_x independent of y and z, the restoring stress is given by $T_{xx} = -E\partial\xi_x/\partial x$, where E is Young's modulus (cf. end of Sec. 11.3). Hence the restoring force density $\mathbf{f} = -\boldsymbol{\nabla}\cdot\mathbf{T}$ is $f_x = E\partial^2\xi_x/\partial x^2$. The wave equation then becomes

$$\frac{\partial^2\xi_x}{\partial t^2} = \left(\frac{E}{\rho}\right)\frac{\partial^2\xi_x}{\partial x^2}\,, \tag{12.22}$$

and so the sound speed for compression waves in a long straight rod is

$$\boxed{C_C = \left(\frac{E}{\rho}\right)^{\frac{1}{2}}}\,. \tag{12.23}$$

Referring to Table 11.1 (in Chap. 11), we see that a typical value of Young's modulus in a solid is ~ 100 GPa. If we adopt a typical density $\sim 3\times10^3$ kg m^{-3}, then we estimate the compressional sound speed to be ~ 5 km s^{-1}. This is roughly 15 times the sound speed in air.

As this compressional wave propagates, in regions where the rod is compressed longitudinally, it bulges laterally by an amount given by Poisson's ratio; and where it is expanded longitudinally, the rod shrinks laterally. By contrast, for a longitudinal wave in a homogeneous medium, transverse forces prevent this lateral bulging and shrinking. This accounts for the different propagation speeds, $C_C \neq C_L$; see Ex. 12.6 below.

12.3.2 Torsion Waves in a Rod

Next consider a rod with circular cross section of radius a, subjected to a twisting force (Fig. 12.2). Let us introduce an angular displacement $\Delta\phi \equiv \varphi(x)$ that depends on distance x

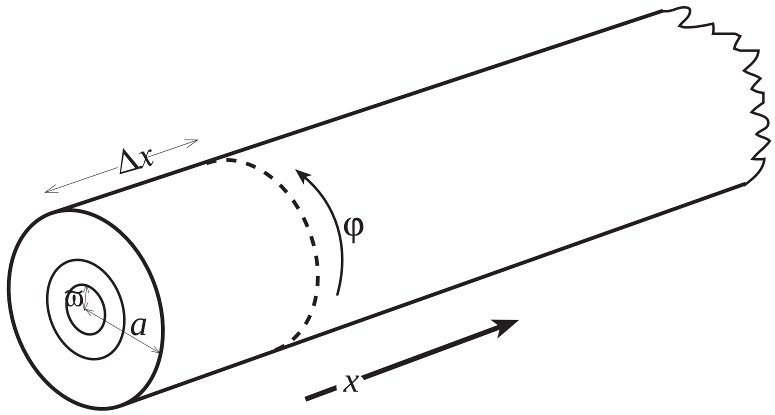

Fig. 12.2: When a rod with circular cross section is twisted, there will be a restoring torque.

down the rod. The only nonzero component of the displacement vector is then $\xi_\phi = \varpi\varphi(x)$, where $\varpi = \sqrt{y^2+z^2}$ is cylindrical radius. We can calculate the total torque, exerted by the portion of the rod to the left of x on the portion to the right, by integating over a circular cross section. For small twists, there is no expansion, and the only components of the shear tensor are

$$\Sigma_{\phi x} = \Sigma_{x\phi} = \frac{1}{2}\xi_{\phi,x} = \frac{\varpi}{2}\frac{\partial\varphi}{\partial x}\ . \tag{12.24}$$

The torque contributed by an annular ring of radius ϖ and thickness $d\varpi$ is $\varpi \cdot T_{\phi x} \cdot 2\pi\varpi d\varpi$, and we substitute $T_{\phi x} = -2\mu\Sigma_{\phi x}$ to obtain the total torque of the rod to the left of x on that to the right:

$$N = \int_0^a 2\pi\mu\varpi^3 d\varpi\frac{\partial\varphi}{\partial x} = \frac{\mu}{\rho}I\frac{\partial\varphi}{\partial x}\ , \quad \text{where} \quad I = \frac{\pi}{2}\rho a^4 \tag{12.25}$$

is the moment of inertia per unit length. Equating the net torque on a segment with length Δx to the rate of change of its angular momentum, we obtain

$$\frac{\partial N}{\partial x}\Delta x = I\frac{\partial^2\varphi}{\partial t^2}\Delta x\ , \tag{12.26}$$

which, using Eq. (12.25), becomes the wave equation for torsional waves:

$$\left(\frac{\mu}{\rho}\right)\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial t^2}\ . \tag{12.27}$$

The speed of torsional waves is thus

$$\boxed{C_T = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}}\ . \tag{12.28}$$

Note that this is the same speed as that of transverse shear waves in a uniform medium. This might have been anticipated as there is no change in volume in a torsional oscillation and so only the shear stress acts to produce a restoring force.

12.3.3 Waves on Strings

This example is surely all too familiar. When a string under a tension force T (*not* force per unit area) is plucked, there will be a restoring force proportional to the curvature of the string. If $\xi_z \equiv \eta$ is the transverse displacement (in the same notation as we used for rods in Secs. 11.5 and 11.6), then the wave equation will be

$$T\frac{\partial^2\eta}{\partial x^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} \,, \tag{12.29}$$

where Λ is the mass per unit length. The wave speed is thus

$$\boxed{C_S = \left(\frac{T}{\Lambda}\right)^{1/2}} \,. \tag{12.30}$$

12.3.4 Flexural Waves on a Beam

Now consider the small-amplitude, transverse displacement of a horizontal rod or beam that can be flexed. In Sec. 11.5 we showed that such a flexural displacement produces a net elastic restoring force per unit length given by $D\partial^4\eta/\partial x^4$, and we considered a situation where that force was balanced by the beam's weight per unit length, $W = \Lambda g$ [Eq. (11.45)]. Here

$$D = \frac{1}{12}Ewh^3 \tag{12.31}$$

is the flexural rigidity [Eq. (11.41b)], h is the beam's thickness in the direction of bend, w is its width, $\eta = \xi_z$ is the transverse displacement of the neutral surface from the horizontal, Λ is the mass per unit length, and g is the earth's acceleration of gravity. The solution of the resulting force-balance equation, $-D\partial^4\eta/\partial x^4 = W = \Lambda g$, was the quartic (11.46a), which described the equilibrium beam shape.

When gravity is absent and the beam is allowed to bend dynamically, the acceleration of gravity g gets replaced by the beam's dynamical acceleration, $\partial^2\eta/\partial t^2$; the result is a wave equation for flexural waves on the beam:

$$-D\frac{\partial^4\eta}{\partial x^4} = \Lambda\frac{\partial^2\eta}{\partial t^2} \,. \tag{12.32}$$

(This derivation of the wave equation is an elementary illustration of the *Principle of Equivalence*—the equivalence of gravitational and inertial forces, or gravitational and inertial accelerations—which underlies Einstein's general relativity theory; see Sec. 25.2.)

The wave equations we have encountered so far in this chapter have all described non-dispersive waves, for which the wave speed is independent of the frequency. Flexural waves, by contrast, are dispersive. We can see this by assuming that $\eta \propto \exp[i(kx-\omega t)]$ and thereby deducing from Eq. (12.32) the dispersion relation

$$\boxed{\omega = \sqrt{D/\Lambda}\; k^2 = \sqrt{Eh^2/12\rho}\; k^2} \,. \tag{12.33}$$

Here we have used Eq. (12.32) for the flexural rigidity D and $\Lambda = \rho w h$ for the mass per unit length.

Before considering the implications of this dispersion, we shall complicate the equilibrium a little. Let us suppose that, in addition to the net shearing force per unit length $-D\partial^4\eta/\partial x^4$, the beam is also held under a tension force T. We can then combine the two wave equations (12.29), (12.32) to obtain

$$-D\frac{\partial^4\eta}{\partial x^4} + T\frac{\partial^2\eta}{\partial x^2} = \Lambda\frac{\partial^2\eta}{\partial t^2} , \tag{12.34}$$

for which the dispersion relation is

$$\omega^2 = C_S^2 k^2\left(1 + \frac{k^2}{k_c^2}\right) , \tag{12.35}$$

where $C_S = \sqrt{T/\Lambda}$ is the wave speed when the flexural rigidity D is negligible so the beam is string-like, and

$$k_c = \sqrt{T/D} \tag{12.36}$$

is a critical wave number. If the average strain induced by the tension is $\epsilon = T/Ewh$, then $k_c = (12\epsilon)^{1/2}h^{-1}$, where h is the thickness of the beam, w is its width, and we have used Eq. (12.31). [Notice that k_c is equal to 1/(the lengthscale on which a pendulum's support wire—analog of our beam—bends), as discussed in Ex. 11.11.] For short wavelengths $k \gg k_c$, the shearing force dominates and the beam behaves like a tension-free beam; for long wavelengths $k \ll k_c$, it behaves like a string.

A consequence of dispersion is that waves with different wave numbers k propagate with different speeds, and correspondingly the group velocity $V_g = d\omega/dk$ with which wave packets propagate differs from the phase velocity $V_{\rm ph} = \omega/k$ with which a wave's crests and troughs move (see Sec. 7.2.2). For the dispersion relation (12.35), the phase and group velocities are

$$\begin{aligned} V_{\rm ph} &\equiv \omega/k = C_S(1 + k^2/k_c^2)^{1/2} , \\ V_g &\equiv d\omega/dk = C_S(1 + 2k^2/k_c^2)(1 + k^2/k_c^2)^{-1/2} . \end{aligned} \tag{12.37}$$

As we discussed in detail in Sec. 7.2.2 and Ex. 7.2, for dispersive waves such as this one, the fact that different Fourier components in the wave packet propagate with different speeds causes the packet to gradually spread; we explore this quantitatively for longitudinal waves on a beam in Ex. 12.5.

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EXERCISES

Exercise 12.5 *Derivation: Dispersion of Flexural Waves*
Verify Eqs. (12.35) and (12.37). Sketch the dispersion-induced evolution of a Gaussian wave packet as it propagates along a stretched beam.

Exercise 12.6 *Problem: Speeds of Elastic Waves*
Show that the sound speeds for the following types of elastic waves in an isotropic material are in the ratio $1 : (1-\nu^2)^{-1/2} : \left(\frac{1-\nu}{(1+\nu)(1-2\nu)}\right)^{1/2} : [2(1+\nu)]^{-1/2} : [2(1+\nu)]^{-1/2}$. Longitudinal waves along a rod, longitudinal waves along a sheet, longitudinal waves along a rod embedded in an incompressible fluid, shear waves in an extended solid, torsional waves along a rod. [*Note:* Here and elsewhere in this book, if you encounter grungy algebra (e.g. frequent conversions from $\{K, \mu\}$ to $\{E, \nu\}$), do not hesitate to use Mathematica or Maple or other symbolic manipulation software to do the algebra!]

Exercise 12.7 *Problem: Xylophones*
Consider a beam of length ℓ, whose weight is negligible in the elasticity equations, supported freely at both ends (so the slope of the beam is unconstrained at the ends). Show that the frequencies of standing flexural waves satisfy

$$\omega = \left(\frac{n\pi}{\ell}\right)^2 \left(\frac{D}{\rho A}\right)^{1/2} ,$$

where A is the cross-sectional area and n is an integer. Now repeat the exercise when the ends are clamped. Based on your result, explain why xylophones don't have clamped ends.

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12.3.5 Bifurcation of Equilibria and Buckling (once more)

We conclude this discussion by returning to the problem of buckling, which we introduced in Sec. 11.6. The example we discussed there was a playing card compressed until it wants to buckle. We can analyze small dynamical perturbations of the card, $\eta(x,t)$, by treating the tension T of the previous section as negative, $T = -F$ where F is the compressional force applied to the card's two ends in Fig. 11.11. Then the equation of motion (12.34) becomes

$$-D\frac{\partial^4 \eta}{\partial x^4} - F\frac{\partial^2 \eta}{\partial x^2} = \Lambda \frac{\partial^2 \eta}{\partial t^2} . \tag{12.38}$$

We seek solutions for which the ends of the playing card are held fixed (as shown in Fig. 11.11), $\eta = 0$ at $x = 0$ and $x = \ell$. Solving Eq. (12.38) by separation of variables, we see that

$$\eta = \eta_n \sin\left(\frac{n\pi}{\ell}x\right) e^{-i\omega_n t} . \tag{12.39}$$

Here $n = 1, 2, 3, ...$ labels the card's modes of oscillation, $n-1$ is the number of nodes in the card's sinusoidal shape for mode n, η_n is the amplitude of deformation for mode n, and the mode's eigenfrequency ω_n (of course) satisfies the same dispersion relation (12.35) as for waves on a long, stretched beam, with $T \to -F$ and $k \to n\pi/\ell$:

$$\omega_n^2 = \frac{1}{\Lambda}\left(\frac{n\pi}{\ell}\right)^2 \left[\left(\frac{n\pi}{\ell}\right)^2 D - F\right] = \frac{1}{\Lambda}\left(\frac{n\pi}{\ell}\right)^2 \left(n^2 F_{\rm crit} - F\right) , \tag{12.40}$$

where $F_{\rm crit} = \pi^2 D/\ell^2$ is the critical force that we introduced in Chap. 11 [Eq. (11.54)].

Consider the lowest normal mode, $n = 1$, for which the playing card is bent in the single-arch manner of Fig. 11.11 as it oscillates. When the compressional force F is small, ω_1^2 is positive, so ω_1 is real and the normal mode oscillates sinusoidally, stably. But for $F > F_{\rm crit} = \pi^2 D/\ell^2$, ω_1^2 is negative, so ω_1 is imaginary and there are two normal-mode solutions, one decaying exponentially with time, $\eta \propto \exp(-|\omega_1|t)$, and the other increasing exponentially with time, $\eta \propto \exp(+|\omega_1|t)$, signifying an *instability against buckling.*

Notice that the onset of instability occurs at identically the same compressional force, $F = F_{\rm crit}$, as the *bifurcation of equilibria*, [Eq. (11.54)], where a new, bent, equilibrium state for the playing card comes into existence. Notice, moreover, that the card's $n = 1$ normal mode has zero frequency, $\omega_1 = 0$, at this onset of instability and bifurcation of equilibria; the card can bend by an amount that grows linearly in time, $\eta = A \sin(\pi x/\ell)\, t$, with no restoring force or exponential growth. This zero-frequency motion leads the card from its original, straight equilibrium shape, to its new, bent equilibrium shape.

This is an example of a very general phenomenon, which we shall meet again in fluid mechanics (Sec. 15.6): For mechanical systems without dissipation (no energy losses to friction or viscosity or radiation or ...), as one gradually changes some "control parameter" (in this case the compressional force F), there can occur bifurcation of equilibria. At each bifurcation point, a normal mode of the original equilibrium becomes unstable, and at its onset of instability the mode has zero frequency and represents a motion from the original equilibrium (which is becoming unstable) to the new, stable equilibrium.

In our simple playing-card example, we see this phenomenon repeated again and again as the control parameter F is increased: One after another, at $F = n^2 F_{\rm crit}$, the modes $n = 1$, $n = 2$, $n = 3$, ... become unstable. At each onset of instability, ω_n vanishes, and the zero-frequency mode (with $n-1$ nodes in its eigenfunction) leads from the original, straight-card equilibrium to the new, stable, $(n-1)$-noded, bent equilibrium.

12.4 Body Waves and Surface Waves — Seismology and Ultrasound

In Sec. 12.2, we derived the dispersion relations $\omega = C_L k$ and $\omega = C_T k$ for longitudinal and transverse elastodynamic waves in uniform media. We now consider how the waves are modified in an inhomogeneous, finite body, the earth. The earth is well approximated as a sphere of radius $R \sim 6000$ km and mean density $\bar\rho \sim 6000$ kg m^{-3}. The outer crust comprising rocks of high tensile strength rests on a denser but more malleable mantle, the two regions being separated by the famous Moho discontinuity. Underlying the mantle is an outer core mainly comprised of liquid iron, which itself surrounds a denser, solid inner core; see Table 12.1 and Fig.12.5 below.

The pressure in the Earth's interior is much larger than atmospheric and the rocks are therefore quite compressed. Their atomic structure cannot be regarded as a small perturbation from their structure *in vacuo.* Nevertheless, we can still use linear elasticity theory to discuss small perturbations about their equilibrium. This is because the crystal lattice has had plenty of time to establish a new equilibrium with a much smaller lattice spacing

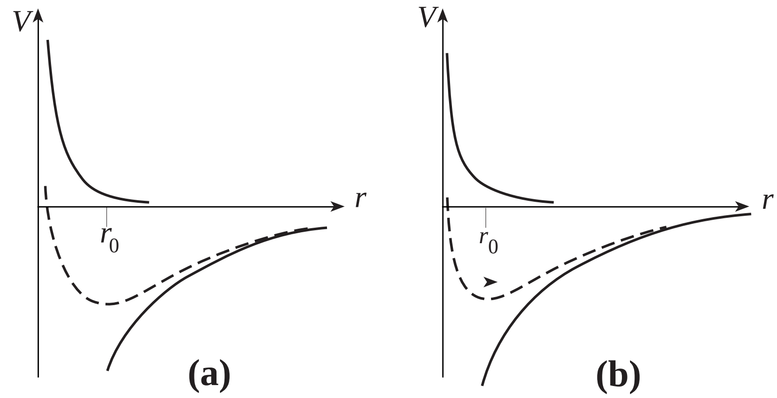

Fig. 12.3: Potential energy curves (dashed) for nearest neighbors in a crystal lattice. (a) At atmospheric (effectively zero) pressure, the equilibrium spacing is set by the minimum in the potential energy, which is a combination of hard electrostatic repulsion by the nearest neighbors (upper solid curve) and a softer overall attraction associated with all the nearby ions (lower solid curve). (b) At much higher pressure, the softer, attractive component is moved inward and the equilibrium spacing is greatly reduced. The bulk modulus is proportional to the curvature of the potential energy curve at its minimum, and is considerably increased.

than at atmospheric pressure (Figure 12.3). The density of lattice defects and dislocations will probably not differ appreciably from those at atmospheric pressure, so the proportionality limit and yield stress should be about the same as for rocks near the earth's surface. The linear stress-strain relation will still apply below the proportionality limit, although the elastic moduli are much greater than those measured at atmospheric pressure.

We can estimate the magnitude of the pressure P in the Earth's interior by idealizing the earth as an isotropic medium with negligible shear stress, so its stress tensor is like that of a fluid, $\mathbf{T} = P\mathbf{g}$ (where $\mathbf{g}$ is the metric tensor). Then the equation of static equilibrium takes the form

$$\frac{dP}{dr} = -g\rho \; , \tag{12.41}$$

where ρ is density and $g(r)$ is the acceleration of gravity at radius r. This equation can be approximated by

$$P \sim \bar{\rho} g R \sim 300\,\mathrm{GPa} \sim 3 \times 10^6\,\mathrm{atmospheres} \; , \tag{12.42}$$

where g is now the acceleration of gravity at the earth's surface $r = R$, and $\bar{\rho}$ is the earth's mean density. This agrees well numerically with the accurate value of 360GPa at the earth's center. The bulk modulus produces the isotropic pressure $P = -K\Theta$ [Eq. (12.28)]; and since $\Theta = -\delta\rho/\rho$ [Eq. (11.18)], the bulk modulus can be expressed as

$$\boxed{K = \frac{dP}{d\ln\rho}} \; . \tag{12.43}$$

[Strictly speaking, we should distinguish between adiabatic and isothermal variations in Eq. (12.43), but the distinction is small for solids; see Sec. **??** It is significant for gases.] Typically, the bulk modulus inside the earth is 4-5 times the pressure, and the shear modulus in the crust and mantle is about half the bulk modulus.

12.4.1 Body Waves

Virtually all our direct information about the internal structure of the earth comes from measurements of the propagation times of elastic waves generated by earthquakes. There are two fundamental kinds of body waves: the longitudinal and transverse modes of Sec. 12.2. These are known in seismology as P-modes and S-modes respectively. The two polarizations of the shear waves are designated SH and SV, where H and V stand for "horizontal" and "vertical" displacements, i.e., displacements orthogonal to $\mathbf{k}$ that are fully horizontal, or that are obtained by projecting the vertical direction $\mathbf{e}_z$ orthogonal to $\hat{\mathbf{k}}$.

We shall first be concerned with what seismologists call high-frequency (of order 1Hz) modes. This leads to three related simplifications. As typical wave speeds lie in the range 3–14 km s^{-1}, the wavelengths lie in the range $\sim 1-10$ km which is generally small compared with the distance over which gravity causes the pressure to change significantly – the *pressure scale height*. It turns out that we then can ignore the effects of gravity on the propagation of small perturbations. In addition, we can regard the medium as effectively homogeneous and infinite and use the local dispersion relations $\omega = C_{L,T}k$, Finally, as the wavelengths are short we can trace rays through the earth using geometrical optics (Sec. 7.3).

Zone	R	ρ	K	μ	C_P	C_S
	10^3km	10^3kg m^{-3}	GPa	GPa	km s^{-1}	km s^{-1}
Inner Core	1.2	13	1400	160	11	2
Outer Core	3.5	10-12	600-1300	-	8-10	-
Mantle	6.35	3-5	100-600	70-250	8-14	5-7
Crust	6.37	3	50	30	6-7	3-4
Ocean	6.37	1	2	-	1.5	-

Table 12.1: Typical outer radii (R), densities (ρ), bulk moduli (K), shear moduli(μ), P-wave speeds and S-wave speeds within different zones of the earth. Note the absence of shear waves in the fluid regions. (Adapted from Stacey 1977.)

Now, the earth is quite inhomogeneous and the sound speeds therefore vary significantly with radius; see Table 12.1. To a fair approximation, the earth is horizontally stratified below the outer crust (whose thickness is irregular). Two types of variation can be distinguished, the abrupt and the gradual. There are several abrupt changes in the crust and mantle (including the Moho discontinuity at the interface between crust and mantle), and also at the transitions between mantle and outer core, and between outer core and inner core. At these *surfaces of discontinuity*, the density, pressure and elastic constants apparently change over distances short compared with a wavelength. Seismic waves incident on these discontinuities behave like light incident on the surface of a glass plate; they can be reflected and refracted. In addition, as there are now two different waves with different phase speeds, it is possible to generate SV waves from pure P waves and *vice versa* at a discontinuity (Fig. 12.4). However, this wave-wave mixing is confined to SV and P; the SH waves do not mix with SV or P; see Ex. 12.9.

The junction conditions that control this wave mixing and all other details of the waves' behavior at a surface of discontinuity are: (i) the displacement $\boldsymbol{\xi}$ must be continuous across

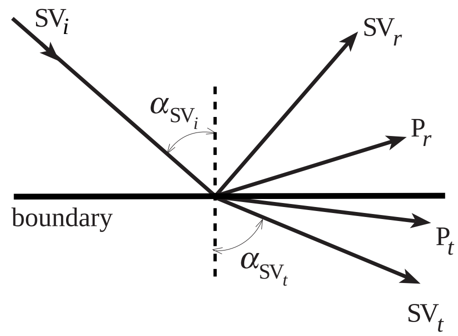

Fig. 12.4: An incident shear wave polarized in the vertical direction (SV$_i$), incident from above on a surface of discontinuity, produces both a longitudinal (P) wave and a SV wave in reflection and in transmission. If the wave speeds increase across the boundary (the case shown), then the transmitted waves, SV$_t$, P$_t$, will be refracted away from the vertical. A shear mode, SV$_r$, will be reflected at the same angle as the incident wave. However, the reflected P mode, P$_r$, will be reflected at a greater angle to the vertical as it is has a greater speed.

the discontinuity (otherwise there would be infinite strain and infinite stress there); and (ii) the net force acting on an element of surface must be zero (otherwise the surface, having no mass, would have infinite acceleration), so the force per unit area acting from the front face of the discontinuity to the back must be balanced by that acting from the back to the front. If we take the unit normal to the horizontal discontinuity to be $\mathbf{e}_z$, then these boundary conditions become

$$\boxed{[\boldsymbol{\xi}_j] = [\mathbf{T}_{jz}] = 0} \,, \tag{12.44}$$

where the notation $[X]$ signifies the difference in X across the boundary and the j is a vector index. (For an alternative, more formal derivation of $[T_{jz}] = 0$, see Ex. 12.8.)

One consequence of these boundary conditions is Snell's law for the directions of propagation of the waves: Since these continuity conditions must be satisfied all along the surface of discontinuity and at all times, the phase $\phi = \mathbf{k} \cdot \mathbf{x} - \omega t$ of the wave must be continuous across the suface at all locations $\mathbf{x}$ on it and all times, which means that the phase ϕ must be the same on the surface for all transmitted waves and all reflected waves as for the incident waves. This is possible only if the frequency ω, the horizontal wave number $k_H = k \sin\alpha$, and the horizontal phase speed $C_H = \omega/k_H = \omega/(k \sin\alpha)$, are the same for all the waves. (Here $k_H = k \sin\alpha$ is the magnitude of the horizontal component of a wave's propagation vector and α is the angle between its propagation direction and the vertical; cf. Fig. 12.4.) Thus, we arrive at Snell's law: For every reflected or transmitted wave J, the horizontal phase speed must be the same as for the incident wave:

$$\boxed{\frac{C_J}{\sin\alpha_J} = C_H \text{ is the same for all } J.} \tag{12.45}$$

It is straightforward though tedious to compute the reflection and transmission coefficients (e.g. the strength of transmitted P-wave produced by an incident SV wave) for the general case using the boundary conditions (12.44); see, e.g., Chap. 7 of Eringen and Suhubi (1975). For the very simplest of examples, see Ex. 12.10.

In the regions between the discontinuities, the pressures and consequently the elastic moduli increase steadily, over many wavelengths, with depth. The elastic moduli generally increase more rapidly than the density so the wave speeds generally also increase with depth, i.e. $dC/dr < 0$. This radial variation in C causes the rays along which the waves propagate to bend. The details of this bending are governed by Hamilton's equations, with the Hamiltonian $\Omega(\mathbf{x}, \mathbf{k})$ determined by the simple nondispersive dispersion relation $\Omega = C(\mathbf{x})k$ (Sec. 7.3.1). Hamilton's equations in this case reduce to the simple ray equation (7.47), which (since the index of refraction is $\propto 1/C$) can be rewritten as

$$\frac{d}{ds}\left(\frac{1}{C}\frac{d\mathbf{x}}{ds}\right) = \boldsymbol{\nabla}\left(\frac{1}{C}\right) . \tag{12.46}$$

Here s is distance along the ray, so $d\mathbf{x}/ds = \mathbf{n}$ is the unit vector tangent to the ray. This ray equation can be reexpressed in the following form:

$$d\mathbf{n}/ds = -(\boldsymbol{\nabla} \ln C)_\perp , \tag{12.47}$$

where the subscript $\perp$ means "projected perpendicular to the ray;" and this in turn means that the ray bends *away* from the direction in which C increases (i.e., it bends upward inside the earth since C increases downward) with the radius of curvature of the bend given by

$$\mathcal{R} = \frac{1}{|(\boldsymbol{\nabla} \ln C)_\perp|} = \frac{1}{|(d\ln C/dr)\sin\alpha|} . \tag{12.48}$$

Here α is the angle between the ray and the radial direction.

Figure 12.5 shows schematically the propagation of seismic waves through the earth. At each discontinuity in the earth's material, Snell's law governs the directions of the reflected and transmitted waves. As an example, note from Eq. (12.45) that an SV mode incident on a boundary cannot generate any P mode when its angle of incidence exceeds $\sin^{-1}(C_{Ti}/C_{Lt})$. (Here we use the standard notation C_T for the phase speed of an S wave and C_L for that of a P wave.) This is what happens at points b and c in Fig. 12.5.

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EXERCISES

Exercise 12.8 *Derivation: Junction Condition at a Discontinuity*
Derive the junction condition $[\mathbf{T}_{jz}] = 0$ at a horizontal discontinuity between two media by the same method as one uses in electrodynamics to show that the normal component of the magnetic field must be continuous: Integrate the equation of motion $\rho d\mathbf{v}/dt = -\boldsymbol{\nabla}\cdot\mathbf{T}$ over the volume of an infinitesimally thin "pill box" centered on the boundary, and convert the volume integral to a surface integral via Gauss's theorem.

Exercise 12.9 *Derivation: Wave Mixing at a Surface of Discontinuity*
Using the boundary conditions (12.44), show that at a surface of discontinuity inside the earth, SV and P waves mix, but SH waves do not mix with the other waves.

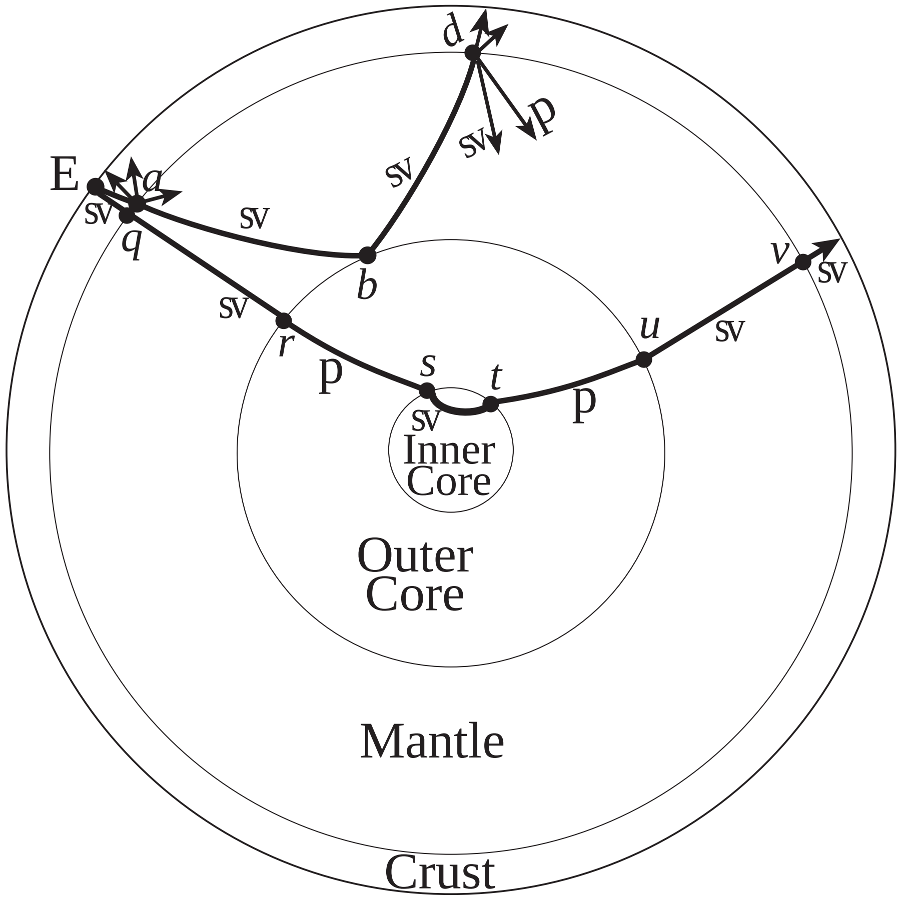

Fig. 12.5: Seismic wave propagation in a schematic earth model. A SV wave made by an earthquake, E, propagates to the crust-mantle boundary at a where it generates two transmitted waves (SV and P) and two reflected waves (SV and P). The transmitted SV wave propagates along rays that bend upward a bit (geometric optics bending) and hits the mantle-outer-core boundary at b. There can be no transmitted SV wave at b because the outer core is fluid; there can be no transmitted or reflected P wave because the angle of incidence of the SV wave is too great; so the SV wave is perfectly reflected. It then travels along an upward curving ray, to the crust-mantle interface at d, where it generates four waves, two of which hit the earth's surface. The earthquake E also generates an SV wave traveling almost radially inward, through the crust-mantle interface at q, to the mantle-outer-core interface at r. Because the outer core is liquid, it cannot support an SV wave, so only a P wave is transmitted into the outer core at r. That P wave propagates to the interface with the inner core at s, where it regenerates an SV wave (shown) along with the transmitted and reflected P waves. The SV wave refracts back upward in the inner core, and generates a P wave at the interface with the outer core t; that P wave propagates through the liquid outer core to u where it generates an SV wave along with its transmitted and reflected P waves; that SV wave travels nearly radially outward, through v to the earth's surface.

Exercise 12.10 *Example: Reflection and Transmission of Normal, Longitudinal Waves at a Boundary*
Consider a longitudinal elastic wave incident normally on the boundary between two media, labeled 1,2. By matching the displacement and the normal component of stress at the boundary, show that the ratio of the transmitted wave amplitude to the incident amplitude is given by

$$t = \frac{2Z_1}{Z_1 + Z_2}$$

where $Z_{1,2} = [\rho_{1,2}(K_{1,2} + 4\mu_{1,2}/3)]^{1/2}$ is known as the *acoustic impedance*. (The impedance is independent of frequency and just a characteristic of the material.) Likewise, evaluate the amplitude reflection coefficient and verify that wave energy flux is conserved.

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12.4.2 Edge Waves

One phenomenon that is important in seismology but is absent for many other types of wave motion is *edge waves*, i.e., waves that propagate along a discontinuity in the elastic medium. An important example is surface waves, which propagate along the surface of a medium (e.g., the earth's surface) and that decay exponentially with depth. Waves with such exponential decay are sometimes called *evanescent*.

The simplest type of surface wave is called a *Rayleigh* wave. We shall now analyze Rayleigh waves for the idealisation of a *plane semi-infinite solid* — which we shall also sometimes call a *homogeneous half space*. This discussion must be modified to allow for both the density stratification and the surface curvature when it is applied to the earth. However, the qualitative character of the mode is unchanged.

Rayleigh waves are an intertwined mixture of P and SV waves; and, in analyzing them, it is useful to resolve their displacement vector $\boldsymbol{\xi}$ into a sum of a (longitudinal) P-wave component, $\boldsymbol{\xi}^L$, and a (transverse) SV-wave component, $\boldsymbol{\xi}^T$.

Consider a semi-infinite elastic medium and introduce a local Cartesian coordinate system with $\mathbf{e}_z$ normal to the surface, with $\mathbf{e}_x$ lying in the surface, and with the propagation vector $\mathbf{k}$ in the $\mathbf{e}_z$-$\mathbf{e}_x$ plane. The propagation vector will have a real component along the horizontal, $\mathbf{e}_x$ direction, corresponding to true propagation, and an imaginary component along the $\mathbf{e}_z$ direction, corresponding to an exponential decay of the amplitude as one goes downward into the medium. In order for the longitudinal (P-wave) and transverse (SV-wave) parts of

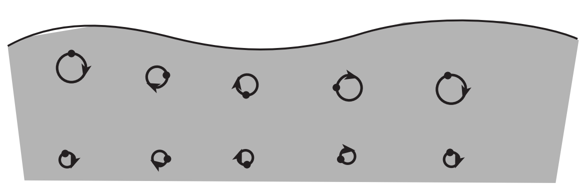

Fig. 12.6: Rayleigh waves in a semi-infinite elastic medium.

the wave to remain in phase with each other as they propagate along the boundary, they must have the same values of the frequency ω and horizontal wave number k_x. However, there is no reason why their vertical e-folding lengths should be the same, i.e. why their imaginary k_z's should be the same. We therefore shall denote their imaginary k_z's by $-iq_L$ for the longitudinal (P-wave) component and $-iq_T$ for the transverse (S-wave) component, and we shall denote their common k_x by k.

Focus attention, first, on the longitudinal part of the wave. Its displacement must have the form

$$\boldsymbol{\xi}_L = \mathbf{A} e^{q_L z + i(kx - \omega t)}, \quad z \leq 0 \, . \tag{12.49}$$

Substituting into the general dispersion relation $\omega^2 = C_L^2 \mathbf{k}^2$ for longitudinal waves, we obtain

$$q_L = (k^2 - \omega^2/c_L^2)^{1/2} \, . \tag{12.50}$$

Now, the longitudinal displacement field is irrotational (curl-free), so we can write

$$\xi^L_{x,z} = \xi^L_{z,x} \, , \tag{12.51}$$

or

$$\xi^L_z = \frac{-iq_L \xi^L_x}{k} \, . \tag{12.52}$$

As the transverse component is solenoidal (divergence-free), the expansion of the combined P-T wave is produced entirely by the P component:

$$\Theta = \boldsymbol{\nabla} \cdot \boldsymbol{\xi}^L = ik \left(1 - \frac{q_L^2}{k^2}\right) A \, . \tag{12.53}$$

Now turn to the transverse (SV-wave) component. We write

$$\boldsymbol{\xi}_T = \mathbf{B} \exp^{q_T z + i(kx - \omega t)}, \quad z \leq 0 \, , \tag{12.54}$$

where (by virtue of the transverse dispersion relation)

$$q_T = \left(k^2 - \frac{\omega^2}{C_T^2}\right)^{1/2} \, . \tag{12.55}$$

As the transverse mode is solenoidal, we obtain

$$\xi^T_z = \frac{-ik \xi^T_x}{q^T} \, , \tag{12.56}$$

and for the rotation

$$\boldsymbol{\phi}_y = \frac{1}{2} \mathbf{e}_y \cdot \boldsymbol{\nabla} \times \boldsymbol{\xi}^T = -\frac{1}{2} q_T \left(1 - \frac{k^2}{q_T^2}\right) B \, . \tag{12.57}$$

We must next impose boundary conditions at the surface. Now, as the surface is free, there will be no force acting upon it, so,

$$\mathbf{T} \cdot \mathbf{e}_z|_{z=0} = 0, \tag{12.58}$$

which is a special case of the general boundary condition (12.44). (Note that we can evaluate the stress at the unperturbed surface location rather than at the displaced surface as we are only working to linear order.) The normal stress is

$$-T_{zz} = K\Theta + 2\mu(\xi_{z,z} - \frac{1}{3}\Theta) = 0 \ , \tag{12.59}$$

and the tangential stress is

$$-T_{xz} = 2\mu(\xi_{z,x} + \xi_{x,z}) = 0 \ . \tag{12.60}$$

Combining Eqs. (12.53), (12.57), (12.59) and (12.60), we obtain

$$(k^2 + q_T^2)^2 = 4q_L q_T k^2 \ . \tag{12.61}$$

Next we substitute for q_L, q_T from (12.50) and (12.55) to obtain the dispersion relation

$$\zeta^3 - 8\zeta^2 + 8\left(\frac{2-\nu}{1-\nu}\right)\zeta - \frac{8}{(1-\nu)} = 0 \ , \tag{12.62}$$

where

$$\zeta = \left(\frac{\omega}{C_T k}\right)^2 = \left(\frac{C_R}{C_T}\right)^2 \ . \tag{12.63}$$

The dispersion relation (12.62) is a third order polynomial in ω^2 with generally just one positive real root. From Eqs. (12.62) and (12.63), we see that for a Poisson ratio characteristic of rocks, $0.2 \lesssim \nu \lesssim 0.3$, *the phase speed of a Rayleigh wave is roughly 0.9 times the speed of a pure shear wave;* cf. Fig. 12.7.

Rayleigh waves propagate around the surface of the earth rather than penetrate the interior. However, our treatment is inadequate because their wavelengths, typically 1–10 km if generated by an earthquake, are not necessarily small compared with the pressure scale heights in the outer crust. Our wave equation has to be modified to include these vertical gradients.

This vertical stratification has an important additional consequence. If, ignoring these gradients, we attempt to find an orthogonal surface mode just involving SH waves, we find

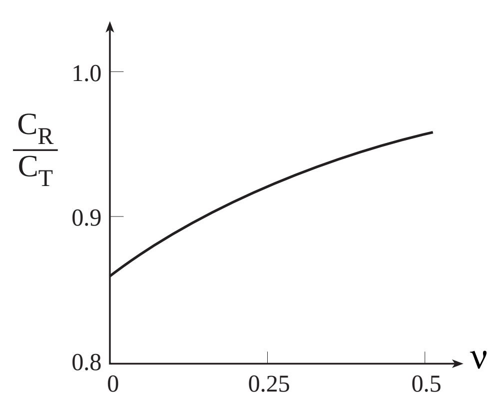

Fig. 12.7: Solution of the dispersion relation (12.62) for different values of Poisson's ratio, ν.

that we cannot simultaneously satisfy the surface boundary conditions on displacement and stress with a single evanescent wave. We need two modes to do this. However, when we allow for stratification, the strong refraction allows an SH surface wave to propagate. This is known as a *Love wave*. The reason for its practical importance is that seismic waves are also created by underground nuclear explosions and it is necessary to be able to distinguish explosion-generated waves from earthquake waves. Now, an earthquake is usually caused by the transverse slippage of two blocks of crust across a fault line. It is therefore an efficient generator of shear modes including Love waves. By contrast, explosions involve radial motions away from the point of explosion and are inefficient emitters of Love waves. This allows these two sources of seismic disturbance to be distinguished.

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EXERCISES

Exercise 12.11 *Example: Earthquakes*
The magnitude M of an earthquake is a quantitative measure of the strength of the seismic waves it creates. Roughly speaking, the elastic-wave energy release can be inferred semi-empirically from the magnitude using the formula

$$E = 10^{5.2+1.44M}\,\mathrm{J}$$

The largest earthquakes have magnitude ~ 8.5.
One type of earthquake is caused by slippage along a fault deep in the crust. Suppose that most of the seismic power in an earthquake with $M \sim 8.5$ is emitted at frequencies ~ 1Hz and that the quake lasts for a time $T \sim 100$s. If C is an average wave speed, then it is believed that the stress is relieved over an area of fault of length $\sim CT$ and a depth of order one wavelength. By comparing the stored elastic energy with the measured energy release make an estimate of the minimum strain prior to the earthquake. Is your answer reasonable? Hence estimate the typical displacement during the earthquake in the vicinity of the fault.
Make an order of magnitude estimate of the acceleration measurable by a seismometer in the next state and in the next continent. (Ignore the effects of density stratification, which are actually quite significant.)

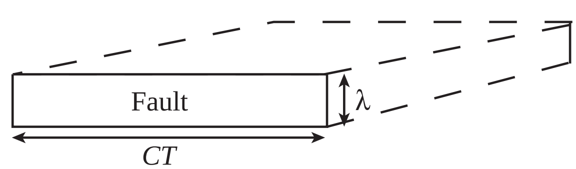

Fig. 12.8: Earthquake: The region of a fault that slips (solid rectangle), and the volume over which the strain is relieved, on one side of the fault (dashed region).

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12.4.3 Green's Function for a Homogeneous Half Space

To get insight into the combination of waves generated by a localized source, such as an explosion or earthquake, it is useful to examine the *Green's function* for excitations in a homogeneous half space. Physicists define the Green's function $G_{jk}(\mathbf{x}, t; \mathbf{x}', t')$ to be the displacement response $\xi_j(\mathbf{x}, t)$ to a unit delta-function force in the $\mathbf{e}_k$ direction at location $\mathbf{x}'$ and time t', $\mathbf{F} = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\mathbf{e}_k$. Geophysicists sometimes find it useful to work, instead, with the "Heaviside Green's function," $G^H_{jk}(\mathbf{x}, t; \mathbf{x}', t')$, which is the displacement response $\xi_j(\mathbf{x}, t)$ to a unit step-function force (one that turns on to unit strength and remains forever constant afterwards) at $\mathbf{x}'$ and t': $\mathbf{F} = \delta(\mathbf{x} - \mathbf{x}')H(t - t')\mathbf{e}_k$. Because $\delta(t - t')$ is the time derivative of the Heaviside step function $H(t - t')$, *the Heaviside Green's function is the time integral of the physicists' Green's function.* The Heaviside Green's function has the advantage that one can easily see, visually, the size of the step functions it contains, by contrast with the size of the delta functions contained in the physicists' Green's function.

It is a rather complicated task to compute the Heaviside Green's function, and geophysicists have devoted much effort to doing so. We shall not give details of such computations, but merely show the Green's function graphically in Fig. 12.9 for an instructive situation:

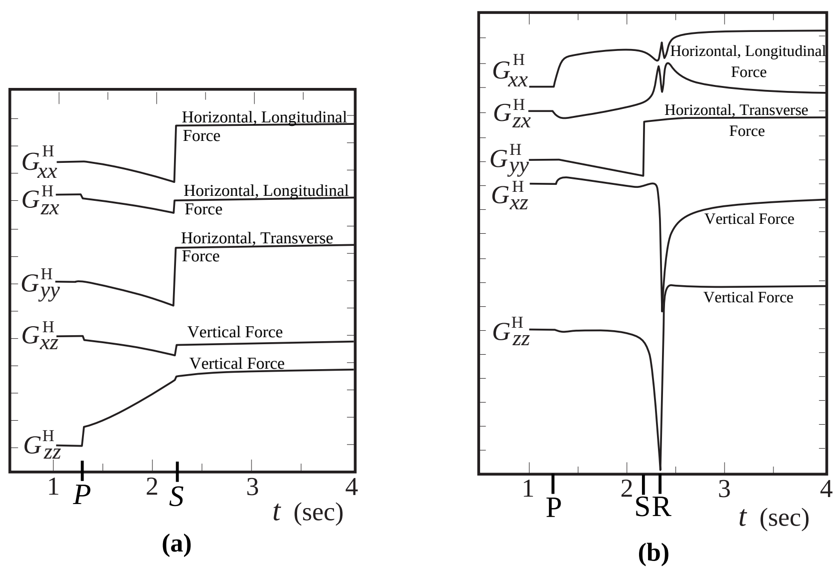

Fig. 12.9: The Heaviside Green's function (displacement response to a step-function force) in a homogeneous half space; adapted from Figs. 2 and 4 of Johnson (1974). The observer is at the surface. The force is applied at a point in the $x - z$ plane, with a direction given by the second index of G^H; the displacement direction is given by the first index of G^H. In (a), the source is nearly directly beneath the observer so the waves propagate nearly vertically upward; more specifically, the source is at 10 km depth and 2 km distance along the horizontal x direction. In (b), the source is close to the surface and the waves propagate nearly horizontally, in the x direction; more specifically, the source is at 2 km depth and is 10 km distance along the horizontal x direction. The longitudinal and transverse speeds are $C_L = 8$ km/s and $C_S = 4.62$ km/s, and the density is 3.30 g/cm^3. For a force of 1 Newton, a division on the vertical scale is 10^{-16} m. The moments of arrival of the P-wave, S-wave and Rayleigh wave from the moment of force turnon are indicated on the horizontal axis.

the displacement produced by a step-function force in a homogeneous half space with the observer at the surface and the force at two different locations: (a) a point nearly beneath the observer, and (b) a point close to the surface and some distance away in the x direction.

Several features of this Green's function deserve note:

- Because of their relative propagation speeds, the P-waves arrive at the observer first, then (about twice as long after the excitation) the S-waves, and shortly thereafter the Rayleigh waves. From the time interval ΔT between the first P-waves and first S-waves, one can estimate the distance to the source: $\ell \simeq (C_P - C_S)\Delta T \sim 3\text{km}(\Delta T/\text{s})$.
- For the source nearly beneath the observer [graphs (a)], there is no sign of any Rayleigh wave, whereas for the source close to the surface, the Rayleigh wave is the strongest feature in the x and z (longitudinal and vertical) displacements but is absent from the y (transverse) displacement. From this, one can infer the waves' propagation direction.
- The y (transverse) component of force produces a transverse displacement that is strongly concentrated in the S-wave.
- The x and z (longitudinal and vertical) components of force produce x and z displacements that include P-waves, S-waves, and (for the source near the surface) Rayleigh waves.
- The gradually changing displacements that occur between the arrival of the turn-on P-wave and turn-on S-wave are due to P-waves that hit the surface some distance from the observer, and from there diffract to the observer as a mixture of P- and S-waves, and similarly for gradual changes of displacement after the turn-on S-wave.

The complexity of seismic waves arises in part from the richness of features in this homogeneous-half-space Green's function, in part from the influences of the earth's inhomogeneities, and in part from the complexity of an earthquake's or explosion's forces.

12.4.4 Free Oscillations of Solid Bodies

In computing the dispersion relations for body (P- and S-wave) modes and surface (Rayleigh-wave) modes, we have assumed that the wavelength is small compared with the earth's radius and therefore the modes can have a continuous frequency spectrum. However, it is also possible to excite global modes in which the whole earth "rings", with a discrete spectrum. If we approximate the earth as spherically symmetric and ignore its rotation (whose period is long compared to a normal-mode period), then we can isolate three types of global modes: *radial*, *torsional*, and *spheroidal*.

To compute the details of these modes, we can solve the equations of elastodynamics for the displacement vector in spherical polar coordinates, by separation of variables; this is much like solving the Schrödinger equation for a central potential. See Ex. 12.12 for a simple example. In general, as in that example, each of the three types of modes has a displacement vector $\boldsymbol{\xi}$ characterized by its own type of spherical harmonic.

The radial modes have spherically symmetric, purely radial displacements, $\xi_r(r)$, and so have spherical harmonic order $l = 0$; see Fig. 12.10a and Ex. 12.12a.

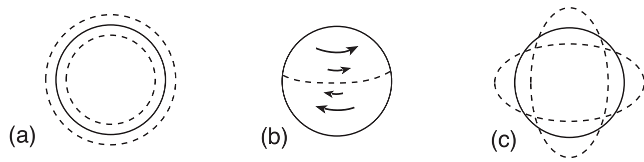

Fig. 12.10: Displacements associated with three types of global modes for an elastic sphere such as the earth. The polar axis points upward. (a) A radial mode shown on a cross section through the sphere's center; matter originally on the solid circle moves in and out between the two dashed circles. (b) A $l = 1, m = 0$ torsional mode; the arrows are proportional to the displacement vector on the sphere's surface at one moment of time. (c) A $l = 2$, $m = 0$ spheroidal mode shown on a cross section through the sphere's center; matter originally on the solid circle moves between the two ellipses.

The torsional modes have vanishing radial displacements, and their nonradial displacements are proportional to the vector spherical harmonic $\hat{\mathbf{L}}\, Y_l^m(\theta, \phi)$, where θ, ϕ are spherical coordinates, Y_l^m is the scalar spherical harmonic, and $\hat{\mathbf{L}} = \mathbf{x} \times \boldsymbol{\nabla}$ is the angular momentum operator (aside from an omitted factor $\hbar/i$), which plays a major role in quantum theory. Spherical symmetry guarantees that these modes' eigenfrequencies are independent of m (because by reorienting one's polar axis, one mixes the various m among each other); so $m = 0$ is representative. In this case, $\boldsymbol{\nabla} Y_l^0 \propto \boldsymbol{\nabla} P_l(\cos\theta)$ is in the θ direction, so $\hat{\mathbf{L}} Y_l^0$ is in the the ϕ direction, whence the only nonzero component of the displacement vector is

$$\xi_\phi \propto \partial P_l(\cos\theta)/\partial\theta = \sin\theta P_l'(\cos\theta) \; . \tag{12.64}$$

(Here P_l is the Legendre polynomial and P_l' is the derivative of P_l with respect to its argument.) Therefore, in these modes, alternate zones of different latitude oscillate in opposite directions (clockwise or counterclockwise at some chosen moment of time), in such a way as to conserve total angular momentum. See Fig. 12.10b and Ex. 12.12b. In the high-frequency limit, the torsional modes become SH waves (since their displacements are horizontal).

The spheroidal modes have radial displacements proportional to $Y_l^m(\theta, \phi)\mathbf{e}_r$, and they have nonradial components proportional to $\boldsymbol{\nabla} Y_l^m$. These two displacements can combine into a single mode because they have the same parity (and opposite parity from the torsional modes) as well as the same spherical-harmonic orders l, m. The eigenfrequencies again are independent of m, and thus can be studied by specializing to $m = 0$, in which case the displacements become

$$\xi_r \propto P_l(\cos\theta) \; , \quad \xi_\theta \propto \sin\theta P_l'(\cos\theta). \tag{12.65}$$

These displacements deform the sphere in a spheroidal manner for the special case $l = 2$ (Fig. 12.10c and Ex. 12.12b), whence their spheroidal name. The radial modes are the special case $l = 0$ of these spheroidal modes. It is sometimes mistakenly asserted that there are no $l = 1$ spheroidal modes because of conservation of momentum. In fact, $l = 1$ modes do exist: for example, the central regions of the sphere can move up, while the outer regions move down. For the earth, the lowest-frequency $l = 2$ spheroidal mode has a period of 53 minutes and can ring for about 1000 periods; i.e., its quality factor is $Q \sim 1000$.) This is typical for solid planets. In the high-frequency limit, the spheroidal modes become a mixture of P and SV waves.

When one writes the displacement vector $\boldsymbol{\xi}$ for a general vibration of the earth as a sum over these various types of global modes, and inserts that sum into the wave equation (12.4b) (augmented, for greater realism, by gravitational forces), spherical symmetry of the unperturbed earth guarantees that the various modes will separate from each other, and for each mode, the wave equation will give a radial wave equation analogous to that for a hydrogen atom in quantum mechanics. The boundary condition $\mathbf{T} \cdot \mathbf{e}_r = 0$ at the earth's surface constrains the solutions of the radial wave equation, for each mode, to be a discrete set, which one can label by the number n of radial nodes that they possess (just as for the hydrogen atom). The frequencies of the modes increase with both n and l. See Ex. 12.12 for details in a simple case.

For small values of the quantum numbers, the modes are quite sensitive to the model assumed for the earth's structure. For example, they are sensitive to whether one correctly includes the gravitational restoring force in the wave equation. However, for large l and n, the spheroidal and toroidal modes become standing combinations of P waves, SV waves, SH waves, Rayleigh and Love waves, and therefore they are rather insensitive to one's ignoring the effects of gravity.

12.4.5 Seismic Tomography

Observations of all of these types of seismic waves clearly code much information about the earth's structure. Inverting the measurements to infer this structure has become a highly sophisticated and numerically intensive branch of geophysics. The travel times of the P and S body waves can be measured between various pairs of points over the earth's surface and essentially allow C_L and C_T and hence K/ρ and μ/ρ to be determined as functions of radius inside the earth. Travel times are $\lesssim 1$ hour. Using this type of analysis, seismologists can infer the presence of hot and cold regions within the mantle and then infer how the rocks are circulating under the crust.

It is also possible to combine the observed travel times with the earth's equation of elastostic equilibrium

$$\frac{dP}{dr} = \frac{K}{\rho}\frac{d\rho}{dr} = -g(r)\rho \; , \quad \text{where } g(r) = \frac{4\pi G}{r^2}\int_0^r r'^2 \rho(r')dr' \text{ is the gravitational acceleration,} \tag{12.66}$$

to determine the distributions of density, pressure and elastic constants. Measurements of Rayleigh and Love waves can be used to probe the surface layers. The results of this procedure are then input to obtain free oscillation frequencies (global mode frequencies), which compare well with the observations. The damping rates for the free oscillations furnish information on the interior viscosity.

12.4.6 Ultrasound; Shock Waves in Solids

Sound waves at frequencies above 20,000 Hz (where human hearing ends) are widely used in modern technology, especially for imaging and tomography. This is much like exploring the earth with seismic waves.

Just as seismic waves can travel from the earth's solid mantle through its liquid outer core and on into its solid inner core, so these *ultrasonic waves* can travel through both solids and liquids, with reflection, refraction, and transmission similar to that in the earth.

Applications of ultrasound include, among others, medical imaging (e.g. of structures in the eye and of a human fetus during pregnancy), inspection of materials (e.g. cracks, voids and welding joints), acoustic microscopy at frequencies up to ~ 3 GHz (e.g. scanning acoustic microscopes), ultrasonic cleaning (e.g. of jewelery), and ultrasonic welding (with sonic vibratrions creating heat at the interface of the materials to be joined).

When an ultrasonic wave (or any other sound wave) in an elastic medium reaches a sufficiently high amplitude, its high-amplitude regions propagate faster than its than its low-amplitude regions; this causes it to develop a *shock front*, in which the compression increases almost discontinuously. This nonlinear mechanism for shock formation in a solid is essentially the same as in a fluid, where we shall study it in depth, in Sec. 17.4.1; and the theory of the elastodynamic shock itself (the *jump conditions across the shock* is essentially the same as in a fluid (Sec. 17.5.1). For details of the theory in both solids and fluids, see, e.g., Davison (2010). Ultrasonic shock waves are used to break up kidney stones in the human body. In laboratories, shocks from the impact of a rapidly moving projectile on a material specimen, or from intense pulsed laser-induced ablation, are used to compress the specimen to pressures similar to the earth's core or higher, and thereby explore the specimen's high-density properties. Strong shock waves from meteorite impacts alter the properties of the rock through which they travel.

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EXERCISES

Exercise 12.12 ***Example: Normal Modes of a Homogeneous, Elastic Sphere*[3]

Show that, for frequency-ω oscillations of a homogeneous elastic sphere with negligible gravity, the displacement vector can everywhere be written as $\boldsymbol{\xi} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\times\mathbf{A}$, where ψ is a scalar potential that satisfies the longitudinal-wave Helmholtz equation and $\mathbf{A}$ is a divergence-free vector potential that satisfies the transverse-wave Helmholtz equation:

$$\boldsymbol{\xi} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\times\mathbf{A}\ , \quad (\nabla^2 + k_L^2)\psi = 0\ , \quad (\nabla^2 + k_T^2)\mathbf{A} = 0\ , \quad \boldsymbol{\nabla}\cdot\mathbf{A} = 0\ ; \quad k_L = \frac{\omega}{C_L}\ ,\ k_T = \frac{\omega}{C_T}\ . \tag{12.67}$$

[Hint: See Ex. 12.2 and make use of gauge freedom in the vector potential.]

(a) ***Radial Modes***: The radial modes are purely longitudinal; their scalar potential [general solution of $(\nabla^2 + k_L^2)\psi = 0$ that is regular at the origin] is the spherical Bessel

[3] Our approach to the mathematics in this exercise is patterned after that of Sec. III of Ashby and Dreitlein (1975), and our numerical evaluations are for the same cases as Secs. 195 and 196 of Love (1927) and of the classic paper on this topic, Lamb (1882). It is interesting to compare the mathematics of our analysis with the nineteenth century mathematics of Lamb and Love, which uses radial functions $\psi_l(r) \sim r^{-l} j_l(r)$ and *solid harmonics* that are sums over m of $r^l Y_l^m(\theta, \phi)$ and satisfy Laplace's equation. Here j_l is the spherical Bessel function of order l and Y_l^m is the spherical harmonic. Solid harmonics can be written as $\mathcal{F}_{ij\ldots q} x_i x_j \ldots x_q$ with $\boldsymbol{\mathcal{F}}$ a symmetric, trace-free tensor of order l; and a variant of them is widely used today in multipolar expansions of gravitational radiation; see, e.g., Sec. II of Thorne (1980).

function of order zero, $\psi = j_0(k_L r) = \sin(k_L r)/k_L r$. The corresponding displacement vector $\boldsymbol{\xi} = \boldsymbol{\nabla}\psi$ has as its only nonzero component

$$\xi_r = j_0'(k_L r) \ , \tag{12.68}$$

where the prime on j_0 and on any other function below means derivative with respect to its argument. (Here we have dropped a multiplicative factor k_L.) Explain why the only boundary condition that need be imposed is $T_{rr} = 0$ at the sphere's surface. By computing T_{rr} for the displacement vector (12.68) and setting it to zero, deduce the following eigenequation for the wave numbers k_L and thence frequencies $\omega = C_L k_L$ of the radial modes:

$$\frac{\tan x_L}{x_L} = \frac{1}{1 - (x_T/2)^2} \ , \tag{12.69}$$

where

$$x_L \equiv k_L R = \frac{\omega R}{C_L} = \omega R \sqrt{\frac{\rho}{K + 4\mu/3}} \ , \quad x_T \equiv k_T R = \frac{\omega R}{C_T} = \omega R \sqrt{\frac{\rho}{\mu}} \ . \tag{12.70}$$

with R the radius of the sphere. For $(x_T/x_L)^2 = 3$, which corresponds to a Poisson ratio $\nu = 1/4$ (about that of glass and typical rock; see Table 11.1), numerical solution of this eigenequation gives the spectrum for modes $\{l = 0\,; n = 0, 1, 2, 3, ...\}$:

$$\frac{x_L}{\pi} = \frac{\omega}{\pi C_L/R} = \{0.8160,\ 1.9285,\ 2.9539,\ 3.9658,\ ...\} \tag{12.71}$$

Note that these eigenvalues get closer and closer to integers as one ascends the spectrum — and this will be true also for any other physically reasonable value of x_T/x_L.

(b) **T2** ***Torsional Modes***: The general solution to the Helmholtz equation $(\nabla^2 + k^2)\psi$ that is regular at the origin is a sum over eigenfuctions of the form $j_l(kr)Y_l^m(\theta,\phi)$. Show that the angular momentum operator $\hat{\mathbf{L}} = \mathbf{x}\times\boldsymbol{\nabla}$ commutes with the Laplacian ∇^2, and thence infer that $\hat{\mathbf{L}}\,[j_l Y_l^m]$ satisfies the vector Helmholtz equation. Verify, further, that it is divergence-free, which means it must be expressible as the curl of a vector potential that is also divergence free and satisfies the vector Helmholtz equation. This means that $\boldsymbol{\xi} = \hat{\mathbf{L}}\,[j_l(k_T r)Y_l^m(\theta,\phi)]$ is a displacement vector that satisfies the elastodynamic equation for transverse waves. Since $\hat{\mathbf{L}}$ differentiates only transversely (in the θ and ϕ directions), we can rewrite this as $\boldsymbol{\xi} = j_l(k_T r)\hat{\mathbf{L}}\,Y_l^m(\theta,\phi)$. To simplify computing the eigenfrequencies (which are independent of m), specialize to $m = 0$, and show that the only nonzero component of the displacement is

$$\xi_\phi \propto j_l(k_T r) \sin\theta P_l'(\theta) \ . \tag{12.72}$$

This, for our homogeneous sphere, is the torsional mode discussed in the text [Eq. (12.64)]. Show that the only boundary condition that must be imposed on this displacement function is $T_{\phi r} = 0$ at the sphere's surface, $r = R$. Compute this component of the stress tensor (with the aid of Box 11.4 for the shear), and by setting it to zero derive the following eigenequation for the torsional-mode frequencies:

$$x_T\, j_l'(x_T) - j_l(x_T) = 0 \ . \tag{12.73}$$

For $l = 1$ (the case illustrated in Fig. 12.10b above), this eigenequation reduces to $(3 - x_T^2)\tan x_T = 3x_T$, and the lowest few eigenfrequencies are

$$\frac{x_T}{\pi} = \frac{\omega}{\pi C_T/R} = \{1.8346,\ 2.8950,\ 3.9225,\ 4.9385,\ ...\}\ . \qquad (12.74)$$

As for the radial modes, these eigenvalues get closer and closer to integers as one ascends the spectrum.

(c) ***Ellipsoidal Modes***: The displacement vector for the ellipsoidal modes is the sum of a longitudinal piece $\boldsymbol{\nabla}\psi$, with $\psi = j_l(k_L r)Y_l^m(\theta,\phi)$ satisfying the longitudinal Helmholtz equation; and a transverse piece $\boldsymbol{\nabla}\times\mathbf{A}$, with $\mathbf{A} = j_l(k_T r)\hat{\mathbf{L}}\,Y_l^m(\theta,\phi)$, which as we saw in part (b) satisfies the Helmholtz equation and is divergence free. Specializing to $m = 0$ to derive the eiqenequation (which is independent of m), show that the components of the displacement vector are

$$\begin{aligned} \xi_r &= \left[\frac{\alpha}{k_L} j_l'(k_L r) + \frac{\beta}{k_T} l(l+1)\frac{j_l(k_T r)}{k_T r}\right] P_l(\cos\theta)\ , \\ \xi_\theta &= -\left[\frac{\alpha}{k_L}\frac{j_l(k_L r)}{k_L r} + \frac{\beta}{k_T}\left(j'(k_T r) + \frac{j(k_T r)}{k_T r}\right)\right] \sin\theta P_l'(\cos\theta);\ , \end{aligned} \qquad (12.75)$$

where α/k_L and β/k_T are constants that determine the weightings of the longitudinal and transverse pieces, and we have included the k_L and k_T to simplify the stress tensor, below. (To get the β term in ξ_r, you will need to use a differential equation satisfied by P_l.) Show that the boundary conditions we must impose on these eigenfuctions are $T_{rr} = 0$ and $T_{r\theta} = 0$ at the sphere's surface $r = R$. By evaluating these [with the help of the shear components in Box 11.4 and the differential equation satisfied by $j_l(x)$ and using $(K + \frac{4}{3}\mu)/\mu = (x_T/x_L)^2$], obtain the following expressions for these components of the stress tensor at the surface:

$$\begin{aligned} T_{rr} &= -\mu P_l(\cos\theta)\Big[\alpha\left\{2j_l''(x_L) - [(x_T/x_L)^2 - 2]j_l(x_L)\right\} + \beta\, 2l(l+1)f_1(x_T)\Big] = 0\ , \\ T_{r\theta} &= \mu\sin\theta P_l(\cos\theta)\Big[\alpha\, 2f_1(x_L) + \beta\left\{j_l''(x_T) + [l(l+1) - 2]f_0(x_T)\right\}\Big] = 0\ , \\ &\text{where}\quad f_0(x) \equiv \frac{j_l(x)}{x^2}\ ,\quad f_1(x) \equiv \left(\frac{j_l(x)}{x}\right)'\ . \end{aligned} \qquad (12.76)$$

These simultaneous equations for the ratio α/β have a solution if and only if the determinant of their coefficients vanishes:

$$\begin{aligned} &\Big\{2j_l''(x_L) - [(x_T/x_L)^2 - 2]j_l(x_L)\Big\}\Big\{j_l''(x_T) + [l(l+1) - 2]f_0(x_T)\Big\} \\ &-4l(l+1)f_1(x_L)f_1(x_T) = 0\ . \end{aligned} \qquad (12.77)$$

This is the eigenequation for the ellipsoidal modes. For $l = 2$ and $x_T/x_L = 3$, it predicts for the lowest four ellipsoidal eigenfrequencies

$$\frac{x_T}{\pi} = \frac{\omega}{\pi C_T/R} = \{0.8403,\ 1.5487,\ 2.6513,\ 3.1131\,...\}\ . \qquad (12.78)$$

Notice that these are significantly smaller than the torsional frequencies. Show that, in the limit of an incompressible sphere ($K \to \infty$) and for $l = 2$, the eigenequation becomes $(4 - x_T^2)[j_2''(x_T) + 4f_0(x_T)] - 24f_1(x_T) = 0$, which predicts for the lowest four eigenfrequencies

$$\frac{x_T}{\pi} = \frac{\omega}{\pi C_T/R} = \{0.8485,\ 1.7421,\ 2.8257,\ 3.8709\,...\}\ . \tag{12.79}$$

These are modestly larger than the compressible case, Eq. (12.78).

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12.5 T2 The Relationship of Classical Waves to Quantum Mechanical Excitations

In the previous chapter, we identified impacts of atomic structure on the continuum approximation for elastic solids. Specifically, we showed that the atomic structure accounts for the magnitude of the elastic moduli and explains why most solids yield under comparatively small strain. A quite different connection of the continuum theory to atomic structure is provided by the normal modes of vibration of a finite sized solid body—e.g., the sphere treated in Sec. 12.4.4 and Ex. 12.12.

For any such body, one can solve the vector wave equation (12.4b) [subject to the vanishing-surface-force boundary condition $\mathbf{T} \cdot \mathbf{n} = 0$, Eq. (11.32)] to find the body's normal modes, as we did in Ex. 12.12 for the sphere. In this section, we shall label the normal modes by a single index N (encompassing $\{l, m, n\}$ in the case of a sphere), and shall denote the eigenfrequency of mode N by ω_N and its (typically complex) eigenfunction by $\boldsymbol{\xi}_N$. Then any general, small-amplitude disturbance in the body can be decomposed into a linear superposition of these normal modes:

$$\boxed{\boldsymbol{\xi}(\mathbf{x}, t) = \Re \sum_N a_N(t) \boldsymbol{\xi}_N(\mathbf{x})\ , \quad a_N = A_N \exp(-i\omega_N t)}\ . \tag{12.80}$$

Here $\Re$ means to take the real part, a_N is the *complex generalized coordinate* of mode N, and A_N is its complex amplitude. It is convenient to normalize the eigenfunctions so that

$$\int \rho |\boldsymbol{\xi}_N|^2 dV = M\ , \tag{12.81}$$

where M is the mass of the body; A_N then measures the mean physical displacement in mode N.

Classical electromagnetic waves *in vacuo* are described by linear Maxwell equations, and, after they have been excited, will essentially propagate forever. This is not so for elastic waves, where the linear wave equation is only an approximation. Nonlinearities, and most especially impurities and defects in the structure of the body's material, will cause the different modes to interact weakly so that their complex amplitudes A_N change slowly with time according to a damped simple harmonic oscillator differential equation of the form

$$\boxed{\ddot{a}_N + (2/\tau_N)\dot{a}_N + \omega_N^2 a_N = F'_N/M} \,. \tag{12.82}$$

Here the second term on the left hand side is a damping term (due to weak coupling with other modes), that will cause the mode to decay as long as $\tau_N > 0$; and F'_N is a fluctuating or *stochastic* force on mode N (also caused by weak coupling to the other modes). Equation (12.82) is the *Langevin equation* that we studied in Sec. 6.8.1, and the spectral density of the fluctuating force F'_N is proportional to $1/\tau_N$ and determined by the fluctuation-dissipation theorem, Eq. (6.87). If the modes are thermalized at temperature T, then the fluctuating forces maintain an average energy of kT in each one.

Now, what happens quantum mechanically? The ions and electrons in an elastic solid interact so strongly that it is very difficult to analyze them directly. A quantum mechanical treatment is much easier if one makes a canonical transformation from the coordinates and momenta of the individual ions or atoms to new, generalized coordinates $\hat{x}_N$ and momenta $\hat{p}_N$ that represent weakly interacting normal modes. These coordinates and momenta are Hermitian operators, and they are related to the quantum mechanical complex generalized coordinate $\hat{a}_n$ by

$$\boxed{\hat{x}_N = \frac{1}{2}(\hat{a}_N + \hat{a}_N^\dagger)} \,, \tag{12.83a}$$

$$\boxed{\hat{p}_N = \frac{M\omega_N}{2i}(\hat{a}_N - \hat{a}_N^\dagger)} \,, \tag{12.83b}$$

where the dagger denotes the Hermitean adjoint. We can transform back to obtain an expression for the displacement of the i'th ion

$$\hat{\mathbf{x}}_i = \frac{1}{2}\Sigma_N[\hat{a}_N \xi_N(\mathbf{x}_i) + \hat{a}_N^\dagger \xi_N^*(\mathbf{x}_i)] \tag{12.84}$$

[a quantum version of Eq. (12.80)].

The Hamiltonian can be written in terms of these coordinates as

$$\boxed{\hat{H} = \Sigma_N \left(\frac{\hat{p}_N^2}{2M} + \frac{1}{2}M\omega_N^2 \hat{x}_N^2\right) + \hat{H}_{\rm int}} \,, \tag{12.85}$$

where the first term is a sum of simple harmonic oscillator Hamiltonians for individual modes and $\hat{H}_{\rm int}$ is the perturbative interaction Hamiltonian which takes the place of the combined damping and stochastic forcing terms in the classical Langevin equation (12.82). When the various modes are thermalized, the mean energy in mode N takes on the standard Bose-Einstein form

$$\bar{E}_N = \hbar\omega_N \left[\frac{1}{2} + \frac{1}{\exp(\hbar\omega_N/kT) - 1}\right] \tag{12.86}$$

[Eq. (4.27b) with vanishing chemical potential and augmented by a "zero-point energy" of $\frac{1}{2}\hbar\omega$], which reduces to kT in the classical limit $\hbar \to 0$.

As the unperturbed Hamiltonian for each mode is identical to that for a particle in a harmonic oscillator potential well, it is sensible to think of each wave mode as analogous to such a particle-in-well. Just as the particle-in-well can reside in any one of a series of discrete energy levels lying above the "zero point" energy of $\hbar\omega/2$, and separated by $\hbar\omega$, so each wave mode with frequency ω_N must have an energy $(n + 1/2)\hbar\omega_N$, where n is an integer. The operator that causes the energy of the mode to decrease by $\hbar\omega_N$ is the *annihilation operator* for mode n

$$\boxed{\hat{\alpha}_N = \left(\frac{M\omega_N}{\hbar}\right)^{1/2} \hat{a}_N}\,, \tag{12.87}$$

and the operator that causes an increase in the energy by $\hbar\omega_N$ is its Hermitian conjugate, the *creation operator* $\hat{\alpha}_N^\dagger$. It is useful to think of each increase or decrease of a mode's energy as the creation or annihilation of an individual quantum or "particle" of energy, so that when the energy in mode N is $(n + 1/2)\hbar\omega_N$, there are n quanta (particles) present. These particles are called *phonons*. Because phonons can co-exist in the same state (the same mode), they are *bosons*. They have individual energies and momenta which must be conserved in their interactions with each other and with other types of particles, e.g. electrons. This conservation law shows up classically as resonance conditions in mode-mode mixing; cf. the discussion in nonlinear optics, Sec. 10.6.1.

The important question is now, given an elastic solid at finite temperature, do we think of its thermal agitation as a superposition of classical modes or do we regard it as a gas of quanta? The answer depends on what we want to do. From a purely fundamental viewpoint, the quantum mechanical description takes precedence. However, for many problems where the number of phonons per mode (the mode's mean occupation number) $\eta_N \sim kT/\hbar\omega_N$ is large compared to one, the classical description is amply adequate and much easier to handle. We do not need a quantum treatment when computing the normal modes of a vibrating building excited by an earthquake or when trying to understand how to improve the sound quality of a violin. Here the difficulty is in accommodating the boundary conditions so as to determine the normal modes. All this was expected. What comes as more of a surprise is that often, for purely classical problems, where $\hbar$ is quantitatively irrelevant, the fastest way to analyze a practical problem formally is to follow the quantum route and then take the limit $\hbar \to 0$. We shall see this graphically demonstrated when we discuss nonlinear plasma physics in Chap. 23.

Bibliographic Note

The classic textbook treatments of elastodynamics from a physicist's viewpoint are Landau and Lifshitz (1970) and — in nineteenth-century language — Love (1927). For a lovely and very readable introduction to the basic concepts, with a focus on elastodynamic waves, see Kolsky (1963). Our favorite textbook and treatise from an engineer's viewpoint is Eringen and Suhubi (1975).

By contrast with elastostatics, where there are a number of good, twenty-first-century engineering-oriented textbooks, there are none that we know of for elastodynamics: Engineers are more concerned with designing and analyzing structures that don't break, than

Box 12.3
Important Concepts in Chapter 12

- Elastodynamic conservation of mass and momentum, Sec. 12.2.1
- Methods of deriving and solving wave equations in continuum mechanics, Box 12.2
- Decomposition of elastodynamic waves into longitudinal and transverse components, Sec. 12.2.2 and Ex. 12.2
- Dispersion relation and propagation speeds for longitudinal and transverse waves, Secs. 12.2.3 and 12.2.4
- Energy density and energy flux of elastodynamic waves, Sec. 12.2.5
- Waves on rods: compression waves, torsion waves, string waves, flexural waves, Secs. 12.3.1 – 12.3.4
- Edge waves, and Rayleigh waves as an example, Sec. 12.4.2
- Wave mixing in reflections off boundaries, Sec. 12.4.1
  - Conservation of tangential phase speed and its implications for directions of wave propagation, Sec. 12.4.1
  - Boundary conditions on stress and displacement, Sec. 12.4.1
- Greens functions for elastodynamic waves; Heaviside vs. physicists' Greens functions Sec. 12.4.3
- Elastodynamic free oscillations (normal modes), Secs. 12.4.4 and 12.5
- Relation of classical waves to quantum mechanical excitations, Sec. 12.5
- Onset of instability and zero-frequency modes related to bifurcation of equilibria, Sec. 12.3.5
- Seismic tomography for studying the earth's interior, Sec. 12.4.5
- Imaging and tomography with ultrasound, Sec. 12.4.6

with studying their oscillatory dynamics.

However, we do know two good engineering-oriented books on methods to solve the elastodynamic equations in nontrivial situations: Kausel (2006) and Poruchikov (1993). And for a compendium of practical engineering lore about vibrations of engineering structures, from building foundations to bell towers to suspension bridges, see Bachman (1994).

For seismic waves and their geophysical applications, we recommend the textbooks by

Shearer (2009) and by Stein, Seth and Wysession (2003).

Bibliography

Ashby, Neil and Dreitlein, Joseph. 1975. "Gravitational Wave Reception by a Sphere", *Phys. Rev. D* **12**, 336–349.

Bachman, Hugo, ed. 1994. *Vibration Problems in Structures*, Basel: Birkhauser.

Davison, Lee. 2010. *Fundamental os Shock Wave Propagation in Solids*, Berlin: Springer-Verlag.

Eringen, A. C. & Suhubi, E. S. 1975. Chap. 7 of *Elastodynamics, Vol. II, Linear Theory*, New York: Academic Press.

Kausel, Eduardo. 2006. *Fundamental Solutions in Elastodynamics*, Cambridge: Cambridge University Press.

Kolsky, H. 1963. *Stress Waves in Solids*, New York: Dover.

Lamb, Horace. 1882. "On the Vibrations of an Elastic Sphere", *Proc. London Math. Soc.* **13**, 189–212.

Johnson, L. R. 1974. "Green's function for Lamb's problem," *Geophys. J. Roy. astron. Soc.* **37**, 99–131.

Landau, L. D. and Lifshitz, E. M. 1970. *Elasticity*, Oxford: Pergamon.

Poruchikov, V. B. 1993. *Methods of the Classical Theory of Elastodynamics*, Berlin: Springer-Verlag.

Shearer, Peter M. 20. *Introduction to Seismology*, second edition, Cambridge: Cambridge University Press

Stacey, F. D. 1977. *Physics of the Earth*, New York: Wiley.

Stein, Seth and Wysession, Michael. 2003. *An Introduction to Seismology, Earthquakes and Earth Structure*, Oxford: Blackwell.