Contents

Chapter 22

Kinetic Theory of Warm Plasmas

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Box 22.1
Reader's Guide

- This chapter relies significantly on:
  - Portions of Chap. 3 on kinetic theory: Secs. 3.2.1 and 3.2.3 on the distribution function, and Sec. 3.6 on Liouville's theorem and the collisionless Boltzmann equation.
  - Section 20.3 on Debye shielding, collective behavior of plasmas and plasma oscillations.
  - Portions of Chap. 21: Sec. 21.2 on the wave equation and dispersion relation for dielectrics, Sec. 21.3 on the two-fluid formalism, Sec. 21.4 on Langmuir and ion acoustic waves, and Sec. 21.6 on the two-stream instability.
- Chapter 23 on nonlinear dynamics of plasmas relies heavily on this chapter.

22.1 Overview

At the end of Chap. 21, we showed how to generalize cold-plasma two-fluid theory so as to accommodate several distinct plasma beams, and thereby we discovered an instability. If the beams are not individually monoenergetic (i.e. cold), as we assumed there, but instead have broad velocity dispersions that overlap in velocity space (i.e. if the beams are *warm*), then the two-fluid approach of Chap. 21 cannot be used, and a more powerful, kinetic-theory description of the plasma is required.

Chapter 21's approach entailed specifying the positions and velocities of specific groups of particles (the "fluids"); this is an example of a *Lagrangian* description. It turns out that the most robust and powerful method for developing the kinetic theory of warm plasmas is an *Eulerian* one in which we specify how many particles are to be found in a fixed volume of one-particle phase space.

In this chapter, using this Eulerian approach, we develop the kinetic theory of plasmas. We begin in Sec. 22.2 by introducing kinetic theory's one-particle *distribution function*, $f(\mathbf{x}, \mathbf{v}, t)$ and recovering its evolution equation (the *collisionless Boltzmann* equation, also called the *Vlasov equation*), which we have met previously in Chap. 3. We then use this Vlasov equation to derive the two-fluid formalism used in Chap. 21 and to deduce some physical approximations that underlie the two-fluid description of plasmas.

In Sec. 22.3, we explore the application of the Vlasov equation to Langmuir waves—the one-dimensional electrostatic modes in an unmagnetized plasma that we explored in Chap. 21 using the two-fluid formalism. Using kinetic theory, we rederive Sec. 21.4.3's Bohm-Gross dispersion relation for Langmuir waves, and as a bonus we uncover a physical damping mechanism, called *Landau damping*, that did not and cannot emerge from the two-fluid analysis. This subtle process leads to the transfer of energy from a wave to those particles that can "surf" or "phase-ride" the wave (i.e. those whose velocity projected parallel to the wave vector is slightly less than the wave's phase speed). We show that Landau damping works because there are usually fewer particles traveling faster than the wave and losing energy to it than those traveling slower and extracting energy from it. However, in a collisionless plasma, the particle distributions need not be Maxwellian. In particular, it is possible for a plasma to possess an "inverted" particle distribution with more fast ones than slow ones; then there is a net injection of particle energy into the waves, which creates an instability. In Sec. 22.4, we use kinetic theory to derive a necessary and sufficient criterion for this instability.

In Sec. 22.5, we examine in greater detail the physics of Landau damping and show that it is an intrinsically nonlinear phenomenon; and we give a semi-quantitative discussion of *nonlinear Landau damping*, prefatory to a more detailed treatment of some other nonlinear plasma effects in the following chapter.

Although the kinetic-theory, Vlasov description of a plasma that is developed and used in this chapter is a great improvement on the two-fluid description of Chap. 21, it is still an approximation; and some situations require more accurate descriptions. We conclude this chapter in Sec. 22.6 by introducing greater accuracy via *N-particle distribution functions*, and as applications we use them (i) to explore the approximations underlying the Vlasov description, and (ii) to explore two-particle correlations that are induced in a plasma by Coulomb interactions, and the influence of those correlations on a plasma's equation of state.

22.2 Basic Concepts of Kinetic Theory and its Relationship to Two-Fluid Theory

22.2.1 Distribution Function and Vlasov Equation

In Chap. 3, we introduced the number density of particles in phase space, called the *distribution function* $\mathcal{N}(\mathbf{p}, \mathbf{x}, t)$. We showed that this quantity is Lorentz invariant and that it satisfies the collisionless Boltzmann equation (3.62) and (3.63); and we interpreted this equation as $\mathcal{N}$ being constant along the phase-space trajectory of any freely moving particle.

In order to comply with the conventions of the plasma-physics community, we shall use the name *Vlasov equation* in place of collisionless Boltzmann equation,[1] and we shall change notation in a manner described in Sec. 3.2.3: We use a particle's velocity $\mathbf{v}$ rather than its momentum $\mathbf{p}$ as an independent variable, and we define the distribution function f to be the number density of particles in physical and velocity space

$$f(\mathbf{v}, \mathbf{x}, t) = \frac{dN}{d\mathcal{V}_x d\mathcal{V}_v} = \frac{dN}{dxdydzdv_xdv_ydv_z} \; . \tag{22.1}$$

Note that the integral of f over velocity space is the number density $n(\mathbf{x}, t)$ of particles in physical space:

$$\boxed{\int f(\mathbf{v}, \mathbf{x}, t) d\mathcal{V}_v = n(\mathbf{x}, t)} \; , \tag{22.2}$$

where $d\mathcal{V}_v \equiv dv_xdv_ydv_z$ is the three-dimensional volume element of velocity space. (For simplicity, we shall also restrict ourselves to nonrelativistic speeds; the generalization to relativistic plasma theory is straightforward, though seldom used.)

This one-particle distribution function $f(\mathbf{v}, \mathbf{x}, t)$ and its resulting kinetic theory give a good description of a plasma in the regime of large Debye number, $N_D \gg 1$—which includes almost all plasmas that occur in the universe; cf. Sec. 20.3.2 and Fig. 20.1. The reason is that, when $N_D \gg 1$, we can define $f(\mathbf{v}, \mathbf{x}, t)$ by averaging over a physical-space volume that is large compared to the average interparticle spacing and that thus contains many particles, but is still small compared to the Debye length. By such an average—the starting point of kinetic theory—, the electric fields of individual particles are made unimportant, and the Coulomb-interaction-induced correlations between pairs of particles are made unimportant. We shall explore this issue in detail in Sec. 22.6.2, using a 2-particle distribution function.

In Chap. 3, we showed that, in the absence of collisions (a good assumption for plasmas!), the distribution function evolves in accord with the Vlasov equation (3.62), (3.63). We shall now rederive that Vlasov equation beginning with the law of conservation of particles for each species $s = e$ (electrons) and p (protons):

$$\frac{\partial f_s}{\partial t} + \boldsymbol{\nabla} \cdot (f_s \mathbf{v}) + \boldsymbol{\nabla}_v \cdot (f_s \mathbf{a}) \equiv \frac{\partial f_s}{\partial t} + \frac{\partial (f_s v_j)}{\partial x_j} + \frac{\partial (f_s a_j)}{\partial v_j} = 0 \; . \tag{22.3}$$

[1] This equation was introduced and explored in 1913 by James Jeans in the context of stellar dynamics, and then rediscovered and explored by Anatoly Alexandrovich Vlasov in 1938 in the context of plasma physics. Plasma physicists have honored Vlasov by naming the equation after him. For details of this history, see Hénon (1982).

Here

$$\boxed{\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B})} \tag{22.4}$$

is the electromagnetic acceleration of a particle of species s, which has mass m_s and charge q_s, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields averaged over the same volume as is used in constructing f. Equation (22.3) has the standard form for a conservation law: the time derivative of a density (in this case density of particles in phase space, not just physical space), plus the divergence of a flux (in this case the spatial divergence of the particle flux, $f\mathbf{v} = f d\mathbf{x}/dt$, in the physical part of phase space, plus the velocity divergence of the particle flux, $f\mathbf{a} = f d\mathbf{v}/dt$, in velocity space) is equal to zero.

Now $\mathbf{x}, \mathbf{v}$ are independent variables, so that $\partial x_i/\partial v_j = 0$ and $\partial v_i/\partial x_j = 0$. In addition, $\mathbf{E}$ and $\mathbf{B}$ are functions of $\mathbf{x}, t$ and not $\mathbf{v}$, and the term $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$. Therefore,

$$\boldsymbol{\nabla}_v \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 \, . \tag{22.5}$$

These facts permit us to pull $\mathbf{v}$ and $\mathbf{a}$ out of the derivatives in Eq. (22.3), thereby obtaining

$$\boxed{\frac{\partial f_s}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) f_s + (\mathbf{a} \cdot \boldsymbol{\nabla}_v) f_s \equiv \frac{\partial f_s}{\partial t} + \frac{dx_j}{dt}\frac{\partial f_s}{\partial x_j} + \frac{dv_j}{dt}\frac{\partial f_s}{\partial v_j} = 0} \, . \tag{22.6}$$

We recognize this as the statement that f_s is a constant along the trajectory of a particle in phase space,

$$\boxed{\frac{df_s}{dt} = 0} \, , \tag{22.7}$$

which is the *Vlasov equation* for species s.

Equation (22.7) tells us that, when the space density near a given particle increases, the velocity-space density must decrease, and vice versa. Of course, if we find that other forces or collisions are important in some situation, we can represent them by extra terms added to the right hand side of the Vlasov equation (22.7) in the manner of the Boltzmann transport equation (3.64); cf. Sec. 3.6.

So far, we have treated the electromagnetic field as being somehow externally imposed. However, it is actually produced by the net charge and current densities associated with the two particle species. These are expressed in terms of the distribution functions by

$$\boxed{\rho_e = \sum_s q_s \int f_s \, d\mathcal{V}_v \, , \quad \mathbf{j} = \sum_s q_s \int f_s \mathbf{v} \, d\mathcal{V}_v} \, . \tag{22.8}$$

Equations (22.8), together with Maxwell's equations and the Vlasov equation (22.6), with $\mathbf{a} = d\mathbf{v}/dt$ given by the Lorentz force law (22.4), form a complete set of equations for the structure and dynamics of a plasma. They constitute the kinetic theory of plasmas.

22.2.2 Relation of Kinetic Theory to Two-Fluid Theory

Before developing techniques to solve the Vlasov equation, we shall first relate it to the two-fluid approach used in the previous chapter. We begin by constructing the moments of the distribution function f_s, defined by

$$
\begin{aligned}
n_s &= \int f_s \, d\mathcal{V}_v \ , \\
\mathbf{u}_s &= \frac{1}{n_s} \int f_s \mathbf{v} \, d\mathcal{V}_v \ , \\
\mathbf{P}_s &= m_s \int f_s(\mathbf{v} - \mathbf{u}_s) \otimes (\mathbf{v} - \mathbf{u}_s) \, \mathcal{V}_v \ .
\end{aligned} \tag{22.9}
$$

These are the density, the mean fluid velocity and the pressure tensor for species s. (Of course, $\mathbf{P}_s$ is just the three-dimensional stress tensor $\mathbf{T}_s$ [Eq. (3.30d)] evaluated in the rest frame of the fluid.)

By integrating the Vlasov equation (22.6) over velocity space and using

$$
\begin{aligned}
\int (\mathbf{v} \cdot \boldsymbol{\nabla}) f_s \, d\mathcal{V}_v &= \int \boldsymbol{\nabla} \cdot (f_s \mathbf{v}) \, d\mathcal{V}_v = \boldsymbol{\nabla} \cdot \int f_s \mathbf{v} \, d\mathcal{V}_v \ , \\
\int (\mathbf{a} \cdot \boldsymbol{\nabla}_v) f_s \, d\mathcal{V}_v &= -\int (\boldsymbol{\nabla}_v \cdot \mathbf{a}) f_s \, d\mathcal{V}_v = 0 \ ,
\end{aligned} \tag{22.10}
$$

together with Eq. (22.9), we obtain the continuity equation

$$
\frac{\partial n_s}{\partial t} + \boldsymbol{\nabla} \cdot (n_s \mathbf{u}_s) = 0 \tag{22.11}
$$

for each particle species s. [It should not be surprising that the Vlasov equation implies the continuity equation, since the Vlasov equation is equivalent to the conservation of particles in phase space (22.3), while the continuity equation is just the conservation of particles in physical space.]

The continuity equation is the first of the two fundamental equations of two-fluid theory. The second is the equation of motion, i.e. the evolution equation for the fluid velocity $\mathbf{u}_s$. To derive this, we multiply the Vlasov equation (22.6) by the particle velocity $\mathbf{v}$ and then integrate over velocity space, i.e. we compute the Vlasov equation's first moment. The details are a useful exercise for the reader (Ex. 22.1); the result is

$$
n_s m_s \left(\frac{\partial \mathbf{u}_s}{\partial t} + (\mathbf{u}_s \cdot \boldsymbol{\nabla}) \mathbf{u}_s \right) = -\boldsymbol{\nabla} \cdot \mathbf{P}_s + n_s q_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) \ , \tag{22.12}
$$

which is identical with Eq. (21.13b).

A difficulty now presents itself, in the two-fluid approximation to kinetic theory. We can use Eqs. (22.8) to compute the charge and current densities from n_s and $\mathbf{u}_s$, which are evolved via the fluid equations (22.11) and (22.12). However, we do not yet know how to compute the pressure tensor $\mathbf{P}_s$ within the two-fluid approximation. We could derive a fluid equation for its evolution by taking the second moment of the Vlasov equation (i.e. multiplying it by $\mathbf{v} \otimes \mathbf{v}$ and integrating over velocity space), but that evolution equation would involve an unknown third moment of f_s on the right hand side, $\mathbf{M}_3 = \int f_s \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{v} \, d\mathcal{V}_v$ which is related to the heat-flux tensor. In order to determine the evolution of this $\mathbf{M}_3$, we would have to construct the third moment of the Vlasov equation, which would involve the fourth moment of f_s as a driving term, and so on. Clearly, this procedure will never terminate unless we introduce some additional relationship between the moments. Such a relationship, called a *closure relation*, permits us to build a self-contained theory involving only a finite number of moments.

For the two-fluid theory of Chap. 21, the closure relation that we implicitly used was the same idealization that one makes when regarding a fluid as perfect, namely that the heat-flux tensor vanishes. This idealization is less well justified in a collisionless plasma, with its long mean free paths, than in a normal gas or liquid with its short mean free paths.

An example of an alternative closure relation is one that is appropriate if radiative processes thermostat the plasma to a particular temperature so T_s =constant; then we can set $\mathbf{P}_s = n_s k_B T_s \mathbf{g} \propto n_s$ where $\mathbf{g}$ is the metric tensor. Clearly, a fluid theory of plasmas can be no more accurate than its closure relation.

22.2.3 Jeans' Theorem.

Let us now turn to the difficult task of finding solutions to the Vlasov equation. There is an elementary (and, after the fact, obvious) method to write down a class of solutions that are often useful. This is based on Jeans' theorem (named after the astronomer who first drew attention to it in the context of stellar dynamics; Jeans 1926).

Suppose that we know the particle acceleration $\mathbf{a}$ as a function of $\mathbf{v}$, $\mathbf{x}$, and t. (We assume this for pedagogical purposes; it is not necessary for our final conclusion). Then, for any particle with phase space coordinates $(\mathbf{x}_0, \mathbf{v}_0)$ specified at time t_0, we can (at least in principle) compute the particle's future motion, $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, \mathbf{v}_0, t), \mathbf{v} = \mathbf{v}(\mathbf{x}_0, \mathbf{v}_0, t)$. These particle trajectories are the *characteristics* of the Vlasov equation, analogous to the characteristics of the equations for one-dimensional supersonic fluid flow which we studied in Sec. 17.4 (see Fig. 17.7). Now, for many choices of the acceleration $\mathbf{a}(\mathbf{v}, \mathbf{x}, t)$, there are *constants of the motion*, also known as *integrals of the motion*, that are preserved along the particle trajectories. Simple examples, familiar from elementary mechanics, include the energy (for a time-independent plasma) and the angular momentum (for a spherically symmetric plasma). These integrals can be expressed in terms of the initial coordinates $(\mathbf{x}_0, \mathbf{v}_0)$. If we know n constants of the motion, then only $6 - n$ additional variables need be chosen from $(\mathbf{x}_0, \mathbf{v}_0)$ to completely specify the motion of the particle.

Now, the Vlasov equation tells us that f_s is constant along a trajectory in $\mathbf{x} - \mathbf{v}$ space. Therefore, f_s must, in general be expressible as a function of $(\mathbf{x}_0, \mathbf{v}_0)$. Equivalently, it can be rewritten as a function of the n constants of the motion and the remaining $6-n$ initial phase-space coordinates. However, there is no requirement that it actually depend on *all* of these variables. In particular, any function of the integrals of motion alone that is independent of the remaining initial coordinates will satisfy the Vlasov equation (22.6). This is *Jeans' Theorem*. In words, *functions of constants of the motion take constant values along actual dynamical trajectories in phase space and therefore satisfy the Vlasov equation.*

Of course, a situation may be so complex that no integrals of the particles' equation of motion can be found, in which case, Jeans' theorem is useless. Alternatively, there may be integrals but the initial conditions may be sufficiently complex that extra variables are required to determine f_s. However, it turns out that in a wide variety of applications, particularly those with symmetries such as time independence $\partial f_s/\partial t = 0$, simple functions of simple integrals of the motion suffice to describe a plasma's distribution functions.

We have already met and used a variant of Jeans' theorem in our analysis of statistical equilibrium in Sec. 4.4. There the statistical mechanics distribution function ρ was found to depend only on the integrals of the motion.

We have also, unknowingly, used Jeans' theorem in our discussion of Debye shielding in a plasma (Sec. 20.3.1). To understand this, let us suppose that we have a single isolated positive charge at rest in a stationary plasma ($\partial f_s/\partial t = 0$), and we want to know the electron distribution function in its vicinity. Let us further suppose that the electron distribution at large distances from the charge is known to be Maxwellian with temperature T, i.e. $f_e(\mathbf{v}, \mathbf{x}, t) \propto \exp(-\frac{1}{2}m_e v^2/k_B T)$. Now, the electrons have an energy integral, $E = \frac{1}{2}m_e v^2 - e\Phi$, where Φ is the electrostatic potential. As Φ becomes constant at large distance from the charge, we can therefore write $f_e \propto \exp(-E/k_B T)$ at large distance. However, the particles near the charge must have traveled there from large distance and so must have this same distribution function. Therefore, close to the charge,

$$f_e \propto e^{-E/k_B T} = e^{-[(m_e v^2/2 - e\Phi)/k_B T]} \,, \tag{22.13}$$

and the electron density is obtained by integration over velocity

$$n_e = \int f_e \, d\mathcal{V}_v \propto e^{(e\Phi/k_B T)} \,. \tag{22.14}$$

This is just the Boltzmann distribution that we asserted to be appropriate in Sec. 20.3.1.

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EXERCISES

Exercise 22.1 *Derivation: Two-Fluid Equation of Motion*
Derive the two-fluid equation of motion (22.12) by multiplying the Vlasov equation (22.6) by $\mathbf{v}$ and integrating over velocity space.

Exercise 22.2 *Example: Positivity of Distribution Function*
The one-particle distribution function $f(\mathbf{v}, \mathbf{x}, t)$ ought not to become negative if it is to remain physical. Show that this is guaranteed if it initially is everywhere nonnegative and it evolves by the collisionless Vlasov equation.

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22.3 Electrostatic Waves in an Unmagnetized Plasma: Landau Damping

As our principal application of the kinetic theory of plasmas, we shall explore its predictions for the dispersion relations, stability, and damping of longitudinal, electrostatic waves in an unmagnetized plasma—Langmuir waves and ion acoustic waves. When studying these waves in Sec. 21.4 using two-fluid theory, we alluded time and again to properties of the waves that could not be derived by fluid techniques. Our goal, now, is to elucidate those properties using kinetic theory. As we shall see, their origin lies in the plasma's velocity-space dynamics.

22.3.1 Formal Dispersion Relation

Consider an electrostatic wave propagating in the z direction. Such a wave is one dimensional in that the electric field points in the z direction, $\mathbf{E} = E\mathbf{e}_z$, and varies as $e^{i(kz-\omega t)}$ so it depends only on z and not on x or y; the distribution function similarly varies as $e^{i(kz-\omega t)}$ and is independent of x, y; and the Vlasov, Maxwell, and Lorentz force equations produce no coupling of particle velocities v_x, v_y into the z direction. This suggests the introduction of one-dimensional distribution functions, obtained by integration over v_x and v_y:

$$\boxed{F_s(v,z,t) \equiv \int f_s(v_x, v_y, v = v_z, z, t)\, dv_x dv_y} \,. \tag{22.15}$$

Here and throughout we suppress the subscript z on v_z.

Restricting ourselves to weak waves so nonlinearities can be neglected, we linearize the one-dimensional distribution functions:

$$\boxed{F_s(v,z,t) \simeq F_{s0}(v) + F_{s1}(v,z,t)} \,. \tag{22.16}$$

Here $F_{s0}(v)$ is the distribution function of the unperturbed particles ($s = e$ for electrons and $s = p$ for protons) in the absence of the wave, and F_{s1} is the perturbation induced by and linearly proportional to the electric field E. The evolution of F_{s1} is governed by the linear approximation to the Vlasov equation (22.6):

$$\frac{\partial F_{s1}}{\partial t} + v\frac{\partial F_{s1}}{\partial z} + \frac{q_s E}{m_s}\frac{dF_{s0}}{dv} = 0 \,. \tag{22.17}$$

Here E is a first-order quantity, so in its term we keep only the zero-order dF_{s0}/dv.

We seek a monochromatic, plane-wave solution to this Vlasov equation, so $\partial/\partial t \to -i\omega$ and $\partial/\partial z \to ik$ in Eq. (22.17). Solving the resulting equation for F_{s1}, we obtain

$$\boxed{F_{s1} = \frac{-iq_s}{m_s(\omega - kv)}\frac{dF_{s0}}{dv}E} \,. \tag{22.18}$$

This equation implies that the charge density associated with the wave is related to the electric field by

$$\rho_e = \sum_s q_s \int_{-\infty}^{+\infty} F_{s1} dv = \left(\sum_s \frac{-iq_s^2}{m_s} \int_{-\infty}^{+\infty} \frac{F'_{s0}\, dv}{\omega - kv} \right) E \,, \tag{22.19}$$

where the prime denotes a derivative with respect to v: $F'_{s0} = dF_{s0}/dv$.

A quick route from here to the waves' dispersion relation is to insert this charge density into Poisson's equation $\boldsymbol{\nabla} \cdot \mathbf{E} = ikE = \rho_e/\epsilon_0$ and note that both sides are proportional to E, so a solution is possible only if

$$1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'_{s0}\, dv}{\omega - kv} = 0\ . \tag{22.20}$$

An alternative route, which makes contact with the general analysis of waves in a dielectric medium (Sec. 21.2), is developed in Ex. 22.3 below. This route reveals that the dispersion relation is given by the vanishing of the zz component of the dielectric tensor, which we denoted ϵ_3 in Chap. 21 [Eq. (21.43)], and it shows that ϵ_3 is given by expression (22.20):

$$\epsilon_3(\omega, k) = 1 + \sum_s \frac{q_s^2}{m_s \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'_{s0}\, dv}{\omega - kv} = 0\ . \tag{22.21}$$

Since $\epsilon_3 = \epsilon_{zz}$ is the only component of the dielectric tensor that we shall meet in this chapter, we shall simplify notation henceforth by omitting the subscript 3, i.e. by denoting $\epsilon_{zz} = \epsilon$.

The form of the dispersion relation (22.21) suggests that we combine the unperturbed electron and proton distribution functions $F_{e0}(v)$ and $F_{p0}(v)$ to produce a single, unified distribution function

$$\boxed{F(v) \equiv F_{e0}(v) + \frac{m_e}{m_p} F_{p0}(v)}\ , \tag{22.22}$$

in terms of which *the dispersion relation takes the form*

$$\boxed{\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'\, dv}{\omega - kv} = 0}\ . \tag{22.23}$$

Note that each proton is weighted less heavily than each electron by a factor $m_e/m_p = 1/1836$ in the unified distribution function (22.22) and the dispersion relation (22.23). This is due to the protons' greater inertia and corresponding weaker response to an applied electric field, and it causes the protons to be of no importance in Langmuir waves (Sec. 22.3.5 below). However, in ion-acoustic waves (Sec. 22.3.6), the protons can play an important role because large numbers of them may move with thermal speeds that are close to the waves' phase velocity and thereby can interact resonantly with the waves.

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EXERCISES

Exercise 22.3 *Example: Dielectric Tensor and Dispersion Relation for Longitudinal, Electrostatic Waves*
Derive expression (22.21) for the zz component of the dielectric tensor in a plasma excited by a weak electrostatic wave, and show that the wave's dispersion relation is $\epsilon_3 = 0$. [*Hints:* Notice that the z component of the plasma's electric polarization P_z is related to the charge

density by $\nabla \cdot \mathbf{P} = ikP_z = -\rho_e$ [Eq. (21.1)]; combine this with Eq. (22.19) to get a linear relationship between P_z and $E_z = E$; argue that the only nonzero component of the plasma's electric susceptibility is χ_{zz} and deduce its value by comparing the above result with Eq. (21.3); then construct the dielectric tensor ϵ_{ij} from Eq. (21.5) and the algebratized wave operator L_{ij} from Eq. (21.9), and deduce that the dispersion relation $\det||L_{ij}|| = 0$ takes the form $\epsilon_{zz} \equiv \epsilon_3 = 0$, where ϵ_3 is given by Eq. (22.21).]

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22.3.2 Two-Stream Instability

As a first application of the general dispersion relation (22.23), we use it to rederive the dispersion relation (21.70) associated with the cold-plasma two-stream instability of Sec. 21.6.

We begin by performing an integration by parts on the general dispersion relation (22.23), obtaining:

$$\frac{e^2}{m_e \epsilon_0} \int_{-\infty}^{+\infty} \frac{F dv}{(\omega - kv)^2} = 1 \ . \tag{22.24}$$

We then presume, as in Sec. 21.6, that the fluid consists of two or more streams of cold particles (protons or electrons) moving in the z direction with different fluid speeds u_1, u_2, ..., so $F(v) = n_1 \delta(v - u_1) + n_2 \delta(v - u_2) + \ldots$. Here n_j is the number density of particles in stream j if the particles are electrons, and m_e/m_p times the number density if they are protons. Inserting this $F(v)$ into Eq. (22.24) and noting that $n_j e^2/m_e \epsilon_0$ is the squared plasma frequency ω_{pj}^2 of stream j, we obtain the dispersion relation

$$\frac{\omega_{p1}^2}{(\omega - ku_1)^2} + \frac{\omega_{p2}^2}{(\omega - ku_2)^2} + \ldots = 1 \ , \tag{22.25}$$

which is identical to the dispersion relation (21.73) used in our analysis of the two-stream instability.

It should be evident that the general dispersion relation (22.24) [or equally well (22.23)] provides us with a tool for exploring how the two-stream instability is influenced by a warming of the plasma, i.e. by a spread of particle velocities around the mean, fluid velocity of each stream. We shall explore this in Sec. 22.4 below.

22.3.3 The Landau Contour

The general dispersion relation (22.23) has a troubling feature: for real ω and k its integrand becomes singular at $v = \omega/k =$ (the waves' phase velocity) unless dF/dv vanishes there, which is generically unlikely. This tells us that if, as we shall assume, k is real, then ω cannot be real, except, perhaps, for a non-generic mode whose phase velocity happens to coincide with a velocity for which $dF/dv = 0$.

With ω/k complex, we must face the possibility of some subtlety in how the integral over v in the dispersion relation (22.23) is performed—the possibility that we may have to make v complex in the integral and follow some special route in the complex velocity plane from $v = -\infty$ to $v = +\infty$. Indeed, there is such a subtlety, as Lev Landau (1946) has shown. Our simple derivation of the dispersion relation, above, cannot reveal this subtlety—and, indeed, is suspicious, since in going from Eq. (22.17) to (22.18) our derivation entailed dividing by $\omega - kv$ which vanishes when $v = \omega/k$, and dividing by zero is always a suspicious practice. Faced by this conundrum, Landau developed a more sophisticated derivation of the dispersion relation, one based on posing generic initial data for electrostatic waves, then evolving those data forward in time and identifying the plasma's electrostatic modes by their late-time sinusoidal behaviors, and finally reading off the dispersion relation for the modes from the equations for the late-time evolution. In the remainder of this section, we shall present a variant of Landau's analysis. *Note:* This analysis is very important, including the portion (Ex. 22.4) assigned for the reader to work out. The reader is encouraged to read through this section slowly, with care, so as to understand clearly what is going on.

For simplicity, from the outset we restrict ourselves to plane waves propagating in the z direction with some fixed, real wave number k, so the linearized one-dimensional distribution function and the electric field have the forms

$$F_s(v,z,t) = F_{s0}(v) + F_{s1}(v,t)e^{ikz} \ , \quad E(z,t) = E(t)e^{ikz} \ . \tag{22.26}$$

At $t = 0$ we pose initial data $F_{s1}(v,0)$ for the electron and proton velocity distributions; these data determine the initial electric field $E(0)$ via Poisson's equation. We presume that these initial distributions [and also the unperturbed plasma's velocity distribution $F_{s0}(v)$] are analytic functions of velocity v, but aside from this constraint, the $F_{s1}(v,0)$ are generic. (A Maxwellian distribution is analytic, and most any physically reasonable initial distribution can be well approximated by an analytic function.)

We then evolve these initial data forward in time. The ideal tool for such evolution is the Laplace transform, and *not* the Fourier transform. The power of the Laplace transform is much appreciated by engineers, and under-appreciated by many physicists. Those readers who are not intimately familiar with evolution via Laplace transforms should work carefully through Ex. 22.4. That exercise uses Laplace transforms, followed by conversion of the final answer into Fourier language, to derive the following formula for the time-evolving electric field in terms of the initial velocity distributions $F_{s1}(v,0)$:

$$E(t) = \int_{i\sigma-\infty}^{i\sigma+\infty} \frac{e^{-i\omega t}}{\epsilon(\omega,k)} \left[\sum_s \frac{q_s}{2\pi\epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F_{s1}(v,0)}{\omega - kv} dv \right] d\omega \ . \tag{22.27}$$

Here the integral in frequency space is along the solid horizontal line at the top of Fig. 22.1, with the imaginary part of ω held fixed at $\omega_i = \sigma$ and the real part ω_r varying from $-\infty$ to $+\infty$. The Laplace techniques used to derive this formula are carefully designed to avoid any divergences and any division by zero. This careful design leads to the requirement that the height σ of the integration line above the real frequency axis be *larger* than the e-folding rate $\Im(\omega)$ of the plasma's most rapidly growing mode (or, if none grow, still larger than zero and thus larger than $\Im(\omega)$ for the most slowly decaying mode):

$$\sigma > p_o \equiv \max_n \Im(\omega_n) \ , \quad \text{and } \sigma > 0 \ . \tag{22.28}$$

Here $n = 1, 2, \ldots$ labels the modes, ω_n is the complex frequency of mode n, and $\Im$ means "imaginary part of". [Note: We shall see below that the modes are the zeroes of the integrand of the frequency integral, i.e. its poles.]

Equation (22.27) also entails a velocity integral. In the Laplace-based analysis (Ex. 22.4) that leads to this formula, there is never any question about the nature of the velocity v: it is always real, so the integral is over real v running from $-\infty$ to $+\infty$. However, because all the frequencies ω appearing in Eq. (22.27) have imaginary parts $\omega_i = \sigma > 0$, there is no possibility in the velocity integral of any divergence of the integrand.

In Eq. (22.27) for the evolving field, $\epsilon(\omega, k)$ is the same dielectric function (22.23) as we deduced in our previous analysis (Sec. 22.3.1):

$$\epsilon(\omega, k) = 1 + \frac{e^2}{m_e \epsilon_0 k} \int_{-\infty}^{+\infty} \frac{F'\, dv}{\omega - kv} \,, \quad \text{where} \quad F(v) = F_{e0}(v) + \frac{m_e}{m_p} F_{p0} \,. \tag{22.29}$$

However, here by contrast with there, our derivation has dictated unequivocally how to handle the v integration—the same way as in Eq. (22.27): v is strictly real and the only frequencies appearing in the evolution equations have $\omega_i = \sigma > 0$, so the v integral, running along the real velocity axis, passes under the integrand's pole at $v = \omega/k$ as shown in Fig. 22.2a.

To read off the modal frequencies from the evolving field $E(t)$ at times $t > 0$, we use techniques from complex-variable theory. It can be shown that, because (by hypothesis) $F_{s1}(v, 0)$ and $F_{s0}(v)$ are analytic functions of v, the integrand of the ω integral in Eq. (22.27) is meromorphic—i.e., when the integrand is analytically continued throughout the complex frequency plane, its only singularities are poles. This permits us to evaluate the frequency integral, at times $t > 0$, by closing the integration contour in the lower-half frequency plane as shown by the dashed curve in Fig. 22.1. Because of the exponential factor $e^{-i\omega t}$, the contribution from the dashed part of the contour vanishes, which means that the integral around the contour is equal to $E(t)$ (the contribution from the solid horizontal part). Complex-variable

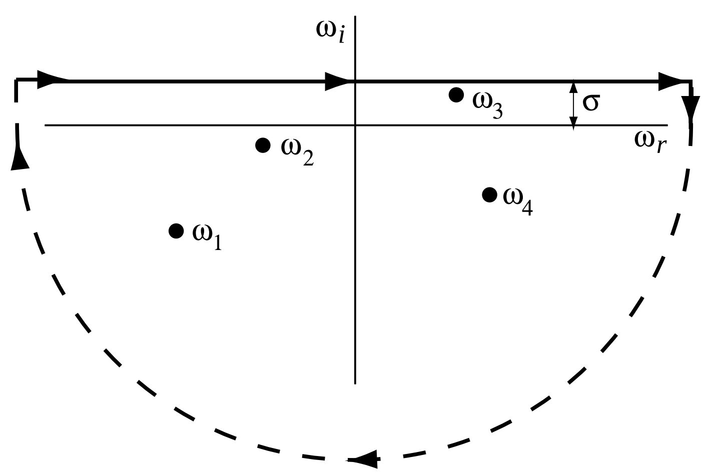

Fig. 22.1: Contour of integration for evaluating $E(t)$ [Eq. (22.27)] as a sum over residues of the integrand's poles—the modes of the plasma.

theory tells us that this integral is given by a sum over the residues R_n of the integrand at the poles (labeled $n = 1, 2, \ldots$):

$$E(t) = 2\pi i \sum_n R_n = \sum_n A_n e^{-i\omega_n t} \,. \tag{22.30}$$

Here ω_n is the frequency at pole n, and A_n is $2\pi i R_n$ with its time dependence $e^{-i\omega_n t}$ factored out. *It is evident, then, that each pole of the analytically continued integrand of Eq. (22.27) corresponds to a mode of the plasma and the pole's complex frequency is the mode's frequency.*

Now, for very special choices of the initial data $F_{s1}(v, 0)$, there may be poles in the square-bracket term in Eq. (22.27), but for generic initial data there will be none, and the only poles will be the zeroes of $\epsilon(\omega, k)$. *Therefore, generically, the modes' frequencies are the zeroes of $\epsilon(\omega, k)$—when that function (which was originally defined only for ω along the line $\omega_i = \sigma$) has been analytically extended throughout the complex frequency plane.*

So how do we compute the analytically extended dielectric function $\epsilon(\omega, k)$? Imagine holding k fixed and real, and exploring the (complex) value of ϵ, thought of as a function of ω/k, by moving around the complex ω/k plane (same as complex velocity plane). In particular, imagine computing ϵ from Eq. (22.29) at one point after another along the arrowed path shown in Fig. 22.2b,c. This path begins at an initial location ω/k where $\omega_i/k = \sigma/k > 0$ and travels downward to some other location below the real axis. At the starting point, the discussion following Eq. (21.28) tells us how to handle the velocity integral: just integrate v along the real axis. As ω/k is moved continuously (with k held fixed), $\epsilon(\omega, k)$ being analytic must vary continuously. If, when ω/k crosses the real velocity axis, the integration contour in Eq. (22.29) were to remain on the velocity axis, then the contour would jump over the integral's moving pole $v = \omega/k$, and there would be a discontinuous jump of the function $\epsilon(\omega, k)$ at the moment of crossing, which is not possible. To avoid such a discontinuous jump, *it is necessary that the contour of integration be kept below the pole, $v = \omega/k$, as that pole moves into the lower half velocity plane;* cf. Fig. 22.2b,c.

The rule that the integration contour must always pass beneath the pole $v = \omega/k$ as shown in Fig. 22.2 is called the *Landau prescription*; the contour is called the *Landau contour* and

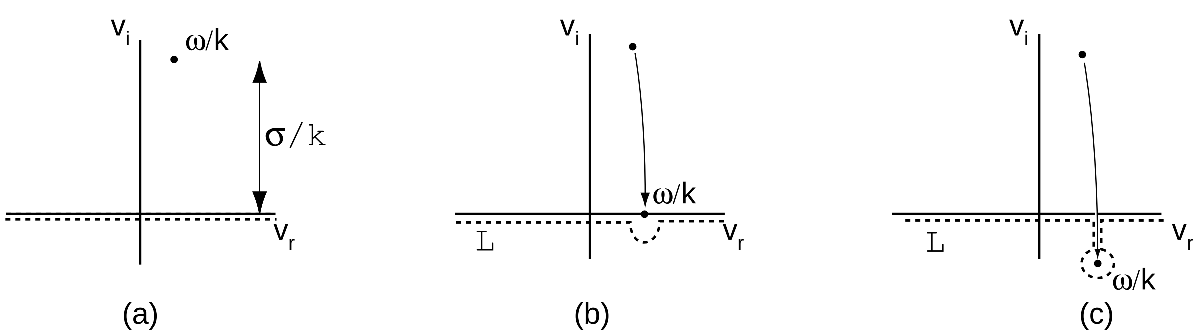

Fig. 22.2: Derivation of the Landau contour $\mathcal{L}$: The dielectric function $\epsilon(\omega, k)$ is originally defined, in Eqs. (22.27) and (22.29), solely for $\omega_i/k = \sigma/k > 0$, the point in diagram (a). Since $\epsilon(\omega, k)$ must be an analytic function of ω at fixed k and thus must vary continuously as ω is continuously changed, the dashed contour of integration in Eq. (22.29) must be kept always below the pole $v = \omega/k$, as shown in (b) and (c).

is denoted $\mathcal{L}$; and our final formula for the dielectric function (and for its vanishing at the modal frequencies—the dispersion relation) is

$$\boxed{\epsilon(\omega,k) = 1 + \frac{e^2}{m_e\epsilon_0 k}\int_{\mathcal{L}} \frac{F' dv}{\omega - kv} = 0}\,, \quad \text{where} \quad \boxed{F(v) = F_e(v) + \frac{m_e}{m_p}F_p(v)}\,. \tag{22.31}$$

For future use we have omitted the subscript 0 from the unperturbed distribution functions F_s as there should be no confusion in future contexts. We shall refer to this as *the general dispersion relation for electrostatic waves in an unmagnetized plasma.*

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EXERCISES

Exercise 22.4 **** Example: Electric Field for Electrostatic Wave Deduced Using Laplace Transforms*

Use Laplace-transform techniques to derive Eqs. (22.27)–(22.29) for the time-evolving electric field of electrostatic waves with fixed wave number k and initial velocity perturbations $F_{s1}(v,0)$. A sketch of the solution follows.

(a) When the initial data are evolved forward in time, they produce $F_{s1}(v,t)$ and $E(t)$. Construct the Laplace transforms of these evolving quantities:[2]

$$\tilde{F}_{s1}(v,p) = \int_0^\infty dt\, e^{-pt} F_{s1}(v,t)\ , \quad \tilde{E}(p) = \int_0^\infty dt\, e^{-pt} E(t)\ . \tag{22.32}$$

To ensure that the time integral is convergent, insist that $\Re(p)$ be greater than $p_0 \equiv \max_n \Im(\omega_n) \equiv$ (the e-folding rate of the most strongly growing mode—or, if none grow, then the most weakly damped mode). Also, for simplicity of the subsequent analysis, insist that $\Re(p) > 0$. Below, in particular, we will need the Laplace transforms for $\Re(p) =$ some fixed value σ that satisfies $\sigma > p_o$ and $\sigma > 0$.

(b) By constructing the Laplace transform of the one-dimensional Vlasov equation (22.17) and integrating by parts the term involving $\partial F_{s1}/\partial t$, obtain an equation for a linear combination of $\tilde{F}_{s1}(v,p)$ and $\tilde{E}(p)$ in terms of the initial data $F_{s1}(v, t=0)$. By then combining with the Laplace transform of Poisson's equation, show that

$$\tilde{E}(p) = \frac{1}{\epsilon(ip,k)} \sum_s \frac{q_s}{k\epsilon_0} \int_{-\infty}^{\infty} \frac{F_{s1}(v,0)}{ip - kv} dv\ . \tag{22.33}$$

Here $\epsilon(ip,k)$ is the dielectric function (22.23) evaluated for frequency $\omega = ip$, with the integral running along the real v axis, and [as we noted in part (a)] $\Re(p)$ must be greater than p_0, the largest ω_i of any mode, and greater than 0. This situation for the dielectric function is the one depicted in Fig. 22.2a.

[2] For useful insights into the Laplace transform, see, e.g., Sec. 4-2 of Mathews and Walker (1964) or Chap. 20 of Arfken, Weber and Harris (2013).

(c) Laplace-transform theory tells us that the time-evolving electric field (with wave number k) can be expressed in terms of its Laplace transform (22.33) by

$$E(t) = \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{E}(p)\, e^{pt} \, \frac{dp}{2\pi i} \ , \tag{22.34}$$

where σ [as introduced in part (a)] is any real number larger than p_0 and larger than 0. Combine this equation with expression (22.33) for $\tilde{E}(p)$, and set $p = -i\omega$. Thereby arrive at the desired result, Eq. (22.27).

****

22.3.4 Dispersion Relation For Weakly Damped or Growing Waves

In most practical situations, electrostatic waves are weakly damped or weakly unstable, i.e. $|\omega_i| \ll \omega_r$ (where ω_r and ω_i are the real and imaginary parts of the wave frequency ω), so the amplitude changes little in one wave period. In this case, the dielectric function (22.31) can be evaluated at $\omega = \omega_r + i\omega_i$ using the first term in a Taylor series expansion away from the real axis:

$$\begin{aligned} \epsilon(k, \omega_r + i\omega_i) &\simeq \epsilon(k, \omega_r) + \omega_i \left(\frac{\partial \epsilon_r}{\partial \omega_i} + i \frac{\partial \epsilon_i}{\partial \omega_i} \right)_{\omega_i = 0} \\ &= \epsilon(k, \omega_r) + \omega_i \left(-\frac{\partial \epsilon_i}{\partial \omega_r} + i \frac{\partial \epsilon_r}{\partial \omega_r} \right)_{\omega_i = 0} \\ &\simeq \epsilon(k, \omega_r) + i\omega_i \left(\frac{\partial \epsilon_r}{\partial \omega_r} \right)_{\omega_i = 0} \ . \end{aligned} \tag{22.35}$$

Here ϵ_r and ϵ_i are the real and imaginary parts of ϵ; in going from the first line to the second we have assumed that $\epsilon(k, \omega)$ is an analytic function of ω near the real axis and thence have used the Cauchy-Riemann equations for the derivatives; and in going from the second line to the third we have used the fact that $\epsilon_i \to 0$ when the velocity distribution is one that produces $\omega_i \to 0$ [cf. Eq. (22.47) below], so the middle term on the second line is second order in ω_i and can be neglected.

Equation (22.35) expresses the dielectric function slightly away from the real axis in terms of its value and derivative on and along the real axis. The on-axis value can be computed from Eq. (22.31) by breaking the Landau contour depicted in Fig. 22.2b into three pieces—two lines from $\pm\infty$ to a small distance δ from the pole, plus a semicircle of radius δ under and around the pole—and by then taking the limit $\delta \to 0$. The first two terms (the two straight lines) together produce the Cauchy principal value of the integral (denoted $\int_{\rm P}$ below), and the third produces $2\pi i$ times half the residue of the pole at $v = \omega_r / k$, so Eq. (22.31) becomes:

$$\epsilon(k, \omega_r) = 1 - \frac{e^2}{m_e \epsilon_0 k^2} \left[\int_{\rm P} \frac{F' \, dv}{v - \omega_r / k} dv + i\pi F'(v = \omega_r / k) \right] \ . \tag{22.36}$$

Inserting this equation and its derivative with respect to ω_r into Eq. (22.35), and setting the result to zero, we obtain

$$\epsilon(k,\omega_r+i\omega_i) \simeq 1-\frac{e^2}{m_e\epsilon_0 k^2}\left[\int_{\rm P}\frac{F'\,dv}{v-\omega_r/k}+i\pi F'(\omega_r/k)+i\omega_i\frac{\partial}{\partial\omega_r}\int_{\rm P}\frac{F'\,dv}{v-\omega_r/k}\right]=0\ . \quad (22.37)$$

Notice that the vanishing of ϵ_r determines the real part of the frequency

$$\boxed{1-\frac{e^2}{m_e\epsilon_0 k^2}\int_{\rm P}\frac{F'}{v-\omega_r/k}dv=0 \quad \text{determines } \omega_r}\ , \quad (22.38a)$$

and the vanishing of ϵ_i determines the imaginary part

$$\boxed{\omega_i=\frac{\pi F'(\omega_r/k)}{-\frac{\partial}{\partial\omega_r}\int_{\rm P}\frac{F'}{v-\omega_r/k}dv}}\ . \quad (22.38b)$$

Equations (22.38) are the dispersion relation in the limit $|\omega_i| \ll \omega_r$. We shall refer to this as *the small-$|\omega_i|$ dispersion relation for electrostatic waves in an unmagnetized plasma.*

Notice that the sign of ω_i is influenced by the sign of $F' = dF/dv$ at $v = \omega_r/k = V_\phi =$ (the waves' phase velocity). As we shall see, this has a simple physical origin and important physical consequences. Usually, but not always, the denominator of Eq. (22.38b) is positive, so the sign of ω_i is the same as the sign of $F'(\omega_r/k)$.

22.3.5 Langmuir Waves and their Landau Damping

We shall now apply the small-$|\omega_i|$ dispersion relation (22.38) to Langmuir waves in a thermalized plasma. Langmuir waves typically move so fast that the slow ions cannot interact with them, so their dispersion relation is influenced significantly only by the electrons. We therefore shall ignore the ions and include only the electrons in $F(v)$. We obtain $F(v)$ by integrating out v_y and v_z in the 3-dimensional Boltzmann distribution [Eq. (3.22d) with $E = \frac{1}{2}m_e(v_x^2+v_y^2+v_z^2)$]; the result, correctly normalized so that $\int F(v)dv = n$, is

$$F \simeq F_e = n\left(\frac{m_e}{2\pi k_B T}\right)^{1/2} e^{-(m_e v^2/2k_B T)}\ , \quad (22.39)$$

where T is the electron temperature.

Now, as we saw in Eq. (22.38b), ω_i is proportional to $F'(v=\omega_r/k)$ with a proportionality constant that is usually positive. Physically, this proportionality arises from the manner in which electrons surf on the waves: Those electrons moving slightly faster than the waves' phase velocity $V_\phi = \omega_r/k$ (usually) lose energy to the waves on average, while those moving slightly slower (usually) extract energy from the waves on average. Therefore, (i) if there are more slightly slower particles than slightly faster [$F'(v=\omega_r/k) < 0$], then the particles on average gain energy from the waves and the waves are damped [$\omega_i < 0$]; (ii) if there are more slightly faster particles than slightly slower [$F'(v=\omega_r/k) > 0$], then the particles on average lose energy to the waves and the waves are amplified [$\omega_i > 0$]; and (iii) the bigger the disparity between the number of slightly faster electrons and the number of slightly slower electrons, i.e. the bigger $|F'(\omega_r/k)|$, the larger will be the damping or growth of wave energy, i.e. the larger will be $|\omega_i|$. It will turn out, quantitatively, that, if the waves' phase velocity ω_r/k is anywhere near the steepest point on the side of the electron velocity distribution, *i.e.* if ω_r/k is of order the electron thermal velocity $\sqrt{k_BT/m_e}$, then the waves will be strongly damped, $\omega_i \sim -\omega_r$. Since our dispersion relation (22.39) is valid only when the waves are weakly damped, we must restrict ourselves to waves with $\omega_r/k \gg \sqrt{k_BT/m_e}$ (a physically allowed regime) or to $\omega_r/k \ll \sqrt{k_BT/m_e}$ (a regime that does not occur in Langmuir waves; cf. Fig. 21.1).

Requiring, then, that $\omega_r/k \gg \sqrt{k_BT/m_e}$ and noting that the integral in Eq. (22.47) gets its dominant contribution from velocities $v \lesssim \sqrt{k_BT/m_e}$, we can expand $1/(v-\omega_r/k)$ in the integrand as a power series in vk/ω_r, obtaining

$$\begin{aligned}
\int_{\mathrm{P}} \frac{F'\,dv}{v-\omega_r/k} &= -\int_{-\infty}^{\infty} dvF'\left[\frac{k}{\omega_r}+\frac{k^2v}{\omega_r^2}+\frac{k^3v^2}{\omega_r^3}+\frac{k^4v^3}{\omega_r^4}+\ldots\right] \\
&= \frac{nk^2}{\omega_r^2}+\frac{3n\langle v^2\rangle k^4}{\omega_r^4}+\ldots \\
&= \frac{nk^2}{\omega_r^2}\left(1+\frac{3k_BTk^2}{m_e\omega_r^2}+\ldots\right) \\
&\simeq \frac{nk^2}{\omega_r^2}\left(1+3k^2\lambda_D^2\frac{\omega_p^2}{\omega_r^2}\right)\;.
\end{aligned} \tag{22.40}$$

Substituting Eqs. (22.39) and (22.40) into Eqs. (22.38a) and (22.38b), and noting that $\omega_r/k \gg \sqrt{k_BT/m_e} \equiv \omega_p\lambda_D$ implies $k\lambda_D \ll 1$ and $\omega_r \simeq \omega_p$, we obtain

$$\boxed{\omega_r = \omega_p(1+3k^2\lambda_D^2)^{1/2}}\,, \tag{22.41a}$$

$$\boxed{\omega_i = -\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_p}{k^3\lambda_D^3}\exp\left(-\frac{1}{2k^2\lambda_D^2}-\frac{3}{2}\right)}\,. \tag{22.41b}$$

The real part of this dispersion relation, $\omega_r = \omega_p\sqrt{1+3k^2\lambda_D^2}$, reproduces the Bohm-Gross result that we derived using the two-fluid theory in Sec. 21.4.3 and plotted in Fig. 21.1. The imaginary part reveals the damping of these Langmuir waves by surfing electrons—so-called *Landau damping*. The two-fluid theory could not predict this Landau damping, because it is a result of internal dynamics in the electrons' velocity space, of which that theory is oblivious.

Notice that, as the waves' wavelength is decreased, i.e. as k increases, the waves' phase velocity decreases toward the electron thermal velocity and the damping becomes stronger, as is expected from our discussion of the number of electrons that can surf on the waves. In the limit $k \to 1/\lambda_D$ (where our dispersion relation has broken down and so is only an order-of-magnitude guide), the dispersion relation predicts that $\omega_r/k \sim \sqrt{k_BT}$ and $\omega_i/\omega_r \sim 1/10$.

In the opposite regime of large wavelength $k\lambda_D \ll 1$ (where our dispersion relation should be quite accurate), the Landau damping is very weak—so weak that ω_i decreases to zero with increasing k faster than any power of k.

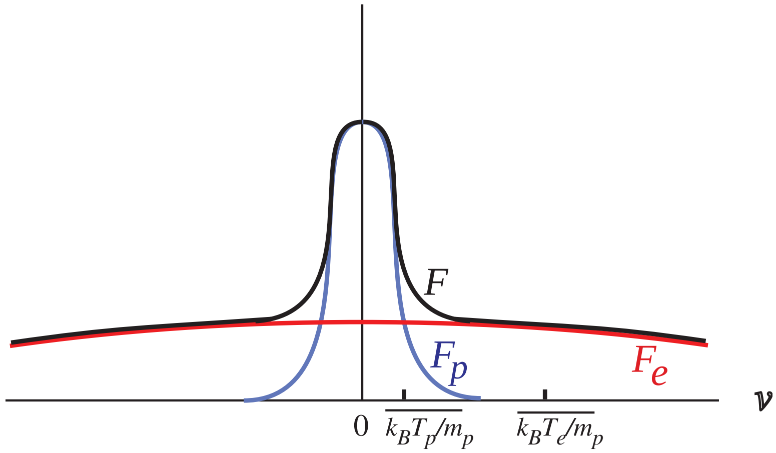

Fig. 22.3: Electron and ion contributions to the net distribution function $F(v)$ in a thermalized plasma. When $T_e \sim T_p$, the phase speed of ion acoustic waves is near the left tick mark on the horizontal axis—a speed at which surfing protons have a maximal ability to Landau-damp the waves, and the waves are strongly damped. When $T_e \gg T_p$, the phase speed is near the right tick mark—far out on the tail of the proton velocity distribution—, so few protons can surf and damp the waves; and the phase speed is near the peak of the electron distribution, so the number of electrons moving slightly slower than the waves is nearly the same as the number moving slightly faster and there is little net damping by the electrons. In this case the waves can propagate.

22.3.6 Ion Acoustic Waves and Conditions for their Landau Damping to be Weak

As we saw in Sec. 21.4.3, ion acoustic waves are the analog of ordinary sound waves in a fluid: They occur at low frequencies where the mean (fluid) electron velocity is very nearly locked to the mean (fluid) proton velocity so the electric polarization is small; the restoring force is due to thermal pressure and not to the electrostatic field; and the inertia is provided by the heavy protons. It was asserted in Sec. 21.4.3 that to avoid these waves being strongly damped, the electron temperature must be much higher than the proton temperature, $T_e \gg T_p$. We can now understand this in terms of particle surfing and Landau damping:

Suppose that the electrons and protons have Maxwellian velocity distributions but possibly with different temperatures. Because of their far greater inertia, the protons will have a far smaller mean thermal speed than the electrons, $\sqrt{k_B T_p/m_p} \ll \sqrt{k_B T_e/m_e}$, so the net one-dimensional distribution function $F(v) = F_e(v) + (m_e/m_p)F_p(v)$ [Eq. (22.22)] that appears in the kinetic-theory dispersion relation has the form shown in Fig. 22.3. Now, if $T_e \sim T_p$, then the contributions of the electron pressure and proton pressure to the waves' restoring force will be comparable, and the waves' phase velocity will therefore be $\omega_r/k \sim \sqrt{k_B(T_e+T_p)/m_p} \sim \sqrt{k_B T_p/m_p} = v_{\text{th},p}$, which is the thermal proton velocity and also is the speed at which the proton contribution to $F(v)$ has its steepest slope (see the left tick mark on the horizontal axis in Fig. 22.3); so $|F'(v=\omega_r/k)|$ is large. This means there will be large numbers of protons that can surf on the waves and a large disparity between the number moving slightly slower than the waves (which extract energy from the waves) and the number moving slightly faster (which give energy to the waves). The result will be strong Landau damping by the protons.

This strong Landau damping is avoided if $T_e \gg T_p$. Then the waves' phase velocity will be $\omega_r/k \sim \sqrt{k_B T_e/m_p}$ which is large compared to the proton thermal velocity $v_{\text{th},p} = \sqrt{k_B T_p/m_p}$ and so is way out on the tail of the proton velocity distribution where there are very few protons that can surf and damp the waves; see the right tick mark on the horizontal axis in Fig. 22.3. Thus, Landau damping by protons has been shut down by raising the electron temperature.

What about Landau damping by electrons? The phase velocity $\omega_r/k \sim \sqrt{k_B T_e/m_p}$ is small compared to the electron thermal velocity $v_{\text{th},e} = \sqrt{k_B T_e/m_e}$, so the waves reside near the peak of the electron velocity distribution, where $F_e(v)$ is large so many electrons can surf with the waves, but $F_e'(v)$ is small so there are nearly equal numbers of faster and slower electrons and the surfing produces little net Landau damping. Thus, $T_e/T_p \gg 1$ leads to successful propagation of ion acoustic waves.

A detailed computation, based on our small-ω_i kinetic-theory dispersion relation, Eqs. (22.38), makes this physical argument quantitative. The details are carried out in Ex. 22.5 under the assumptions that $T_e \gg T_p$ and $\sqrt{k_B T_p/m_p} \ll \omega_r/k \ll \sqrt{k_B T_e/m_e}$ (corresponding to the above discussion); and the result is:

$$\boxed{\frac{\omega_r}{k} = \sqrt{\frac{k_B T_e/m_p}{1 + k^2\lambda_D^2}}}\,, \tag{22.42a}$$

$$\boxed{\frac{\omega_i}{\omega_r} = -\frac{\sqrt{\pi/8}}{(1 + k^2\lambda_D^2)^{3/2}}\left[\sqrt{\frac{m_e}{m_p}} + \left(\frac{T_e}{T_p}\right)^{3/2} \exp\left(\frac{-T_e/T_p}{2(1 + k^2\lambda_D^2)}\right)\right]}\,. \tag{22.42b}$$

The real part of this dispersion relation was plotted in Fig. 21.1; as is shown there and in the above formulas, for $k\lambda_D \ll 1$ the waves' phase speed is $\sqrt{k_B T_e/m_p}$, and the waves are only weakly damped: they can propagate for roughly $\sqrt{m_p/m_e} \sim 43$ periods before damping has a strong effect. This damping is independent of the proton temperature, so it must be due to surfing electrons. When the wavelength is decreased (k is increased) into the regime $k\lambda_D \gtrsim 1$, the waves' frequency asymptotes toward $\omega_r = \omega_{pp}$, the proton plasma frequency, and the phase velocity decreases, so more protons can surf the waves and the Landau damping increases. Equation (22.42) shows us that the damping becomes very strong when $k\lambda_D \sim \sqrt{T_e/T_p}$, and that this is also the point at which ω_r/k has decreased to the proton thermal velocity $\sqrt{k_B T_p/m_p}$—which is in accord with our physical arguments about proton surfing.

When T_e/T_p is decreased from $\gg 1$ toward unity, the ion damping becomes strong regardless of how small may be k [cf. the second term of ω_i/ω_r in Eq. (22.42)]. This is also in accord with our physical reasoning.

Ion acoustic waves are readily excited at the earth's bow shock, where the earth's magnetosphere impacts the solar wind. It is observed that these waves are not able to propagate very far away from the shock, by contrast with Alfvén waves, which are much less rapidly damped.

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EXERCISES

Exercise 22.5 *Derivation: Ion Acoustic Dispersion Relation*
Consider a plasma in which the electrons have a Maxwellian velocity distribution with temperature T_e, the protons are Maxwellian with temperature T_p, and $T_p \ll T_e$; and consider a mode in this plasma for which $\sqrt{k_B T_p/m_p} \ll \omega_r/k \ll \sqrt{k_B T_e/m_e}$ (i.e., the wave's phase velocity is between the left and right tick marks in Fig. 22.3). As was argued in the text, for such a mode it is reasonable to expect weak damping, $|\omega_i| \ll \omega_r$. Making approximations based on these "$\ll$" inequalities, show that the small-$|\omega_i|$ dispersion relation (22.38) reduces to Eqs. (22.42).

Exercise 22.6 *Problem: Dispersion Relations for a Non-Maxwellian Distribution Function.*
Consider a plasma with cold protons [whose velocity distribution can be ignored in $F(v)$] and hot electrons with a one-dimensional distribution function of the form

$$F(v) = \frac{n v_0}{\pi(v_0^2 + v^2)} \ . \tag{22.43}$$

(a) Derive the dielectric function $\epsilon(\omega, k)$ for this plasma and use it to show that the dispersion relation for Langmuir waves is

$$\omega = \omega_{pe} - ikv_0 \ . \tag{22.44}$$

(b) Compute the dispersion relation for ion acoustic waves assuming that their phase speeds are much less than v_0 but large compared to the cold protons' thermal velocities (so the contribution from proton surfing can be ignored). Your result should be

$$\omega = \frac{kv_0(m_e/m_p)^{1/2}}{[1 + (kv_0/\omega_{pe})^2]^{1/2}} - \frac{ikv_0(m_e/m_p)}{[1 + (kv_0/\omega_{pe})^2]^{1/2}} \ . \tag{22.45}$$

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22.4 Stability of Electrostatic Waves in Unmagnetized Plasmas

Our small-ω_i dispersion relation (22.38) implies that the sign of F' at resonance dictates the sign of the imaginary part of ω. This raises the interesting possibility that distribution functions that increase with velocity over some range of positive v might be unstable to the exponential growth of electrostatic waves. In fact, the criterion for instability turns out to be a little more complex than this [as one might suspect from the fact that the sign of the denominator of Eq. (22.38b) is non-obvious], and deriving it is an elegant exercise in complex variable theory.

We shall carry out our derivation in two steps. We shall first introduce a very general method, due to Harry Nyquist, for diagnosing instabilities of dynamical systems. Then we shall apply Nyquist's method explicitly to electrostatic waves in an unmagnetized plasma and thereby deduce an instability criterion due to Oliver Penrose.

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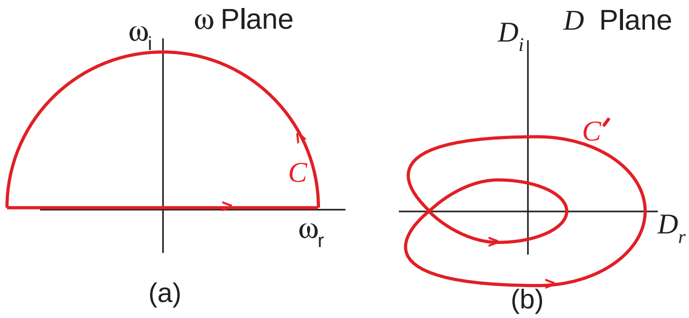

Fig. 22.4: Nyquist diagram for stability of a dynamical system. (a) The curve $\mathcal{C}$ in the complex-ω plane extends along the real frequency axis from $\omega_r = -\infty$ to $\omega_r = +\infty$, then closes up along the semicircle at $|\omega| = \infty$, so it encloses the upper half frequency plane. (b) As ω travels along $\mathcal{C}$, $\mathcal{D}(\omega)$ travels along the closed curve $\mathcal{C}'$ in the complex-$\mathcal{D}$ plane, which counterclockwise encircles the origin twice. This means that the number of zeros of the analytic function $\mathcal{D}(\omega)$ in the upper half frequency plane minus the number of poles is $N_z - N_p = 2$.

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22.4.1 Nyquist's Method

Consider any dynamical system, whose modes of oscillation have complex eigenfreqencies ω that are zeros of some function $\mathcal{D}(\omega)$. [In our case the dynamical system is electrostatic waves in an unmagnetized plasma with some chosen wave number k, and because the waves' dispersion relation is $\epsilon(k,\omega) = 0$, Eq. (22.31), the function $\mathcal{D}$ can be chosen as $\mathcal{D}(\omega) = \epsilon(k,\omega)$.] Unstable modes are zeros of $\mathcal{D}(\omega)$ that lie in the upper half complex-ω plane.

Assume that $\mathcal{D}(\omega)$ is analytic in the upper half ω plane. Then a well-known theorem in complex-variable theory[3] says that that *the number N_z of zeros of $\mathcal{D}(\omega)$ in the upper half plane, minus the number N_p of poles, is equal to the number of times that $\mathcal{D}(\omega)$ encircles the origin clockwise, in the complex-$\mathcal{D}$ plane, as ω travels counterclockwise along the closed path $\mathcal{C}$ that encloses the upper half frequency plane*; see Fig. 22.4.

If one knows the number of poles of $\mathcal{D}(\omega)$ in the upper half frequency plane, then one can infer, from the Nyquist diagram, the number of unstable modes of the dynamical system.

In the next section, we use this Nyquist method to derive the Penrose criterion for instability of electrostatic modes of an unmagnetized plasma. As a second example, in Box 22.2 we show how it can be used to diagnose the stability of a feedback control system.

22.4.2 Penrose's Instability Criterion

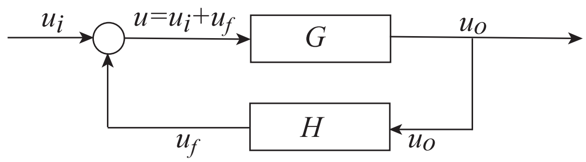
The straightforward way to apply Nyquist's method to electrostatic waves would be to set $\mathcal{D}(\omega) = \epsilon(k,\omega)$. However, to reach our desired instability criterion more quickly, we shall set $\mathcal{D} = k^2\epsilon$; then the zeros of $\mathcal{D}$ are still the electrostatic waves' modes. From Eq. (22.31) for

[3] This is variously called "the principle of the argument", or "Cauchy's theorem", and it follows from the theorem of residues; see, e.g., Chap. 11 of Arfken, Weber and Harris (2013), or Sec. 6.2 of Copson (1935).

Box 22.2

$\boxed{\text{T2}}$ Stability of a Feedback Control System: Analysis by Nyquist's Method

A control system can be described quite generally by the following block diagram:



An input signal $u_i(t)$ and the feedback signal $u_f(t)$ are added, then fed through a filter G to produce an output signal $u_o(t) = \int_{-\infty}^{+\infty} G(t-t')u(t')dt'$; or, in the Fourier domain, $\tilde{u}_o(\omega) = \tilde{G}(\omega)\tilde{u}(\omega)$. [See Sec. 6.7 for filtering of signals. Here, for consistency with this plasma-physics chapter, we adopt the opposite sign convention for Fourier transforms from that in Sec. 6.7.] Then the output signal is fed through a filter H to produce the feedback signal $u_f(t)$.

As an example, consider the following simple model for an automobile's cruise control. The automobile's speed v is to be locked to some chosen value V by measuring v and applying a suitable feedback acceleration. To simplify the analysis, we focus on the difference $u \equiv v - V$, which is to be locked to zero. The input to the control system is the speed $u_i(t)$ that the automobile would have in the absence of feedback, plus the speed change $u_f(t)$ due to the feedback acceleration. Their sum $u = u_i + u_f$ is measured by averaging over a short time interval, with the average exponentially weighted toward the present (in our simple model), so the output of the measurement is $u_o(t) = (1/\tau)\int_{-\infty}^{t} e^{(t'-t)/\tau}u(t')dt'$. By comparing with $u_o = \int_{-\infty}^{+\infty} G(t-t')u(t')dt'$ to infer the measurement filter's Kernel $G(\mathsf{t})$, then Fourier transforming, we find that $\tilde{G}(\omega) \equiv \int_{-\infty}^{+\infty} G(\mathsf{t})e^{i\omega\mathsf{t}} = 1/(1-i\omega\tau)$. If $u_o(t)$ is positive, we apply to the automobile a negative feedback acceleration $a_f(t)$ proportional to it; if u_o is negative, we apply a positive feedback acceleration; so in either case, $a_f = -Ku_o$ for some positive constant K. The feedback speed u_f is the time integral of this acceleration: $u_f(t) = -K\int_{-\infty}^{t} u_o(t')dt'$. Setting this to $\int_{-\infty}^{\infty} H(t-t')u_o(t')$, reading off the Kernel H, and computing its Fourier transform, we find $\tilde{H}(\omega) = -K/(i\omega)$.

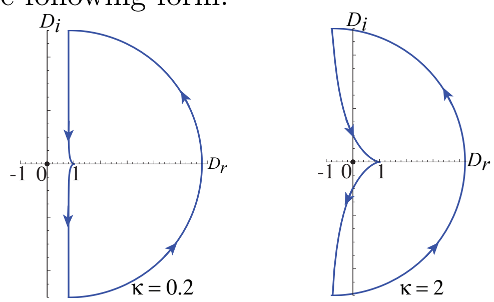
From the block diagram, we see, fully generally, that in the Fourier domain $\tilde{u}_o = \tilde{G}(\tilde{u}_i + \tilde{u}_f) = \tilde{G}(\tilde{u}_i + \tilde{H}\tilde{u}_o)$; so the output in terms of the input is

$$\tilde{u}_o = \frac{\tilde{G}}{1+\tilde{G}\tilde{H}}\tilde{u}_i \; . \tag{1}$$

Evidently, the feedback system will undergo self-excited oscillations, with no input, at any complex frequency ω that is a zero of $\mathcal{D}(\omega) \equiv 1 + \tilde{G}(\omega)\tilde{H}(\omega)$. If that ω is in the lower half complex frequency plane, the oscillations will die out and so are not a problem; but if it is in the upper half ω plane, they will grow exponentially with time. Thus, the zeros of $\mathcal{D}(\omega)$ in the upper half ω plane represent unstable modes of self excitation, and must be avoided in the design of any feedback control system.

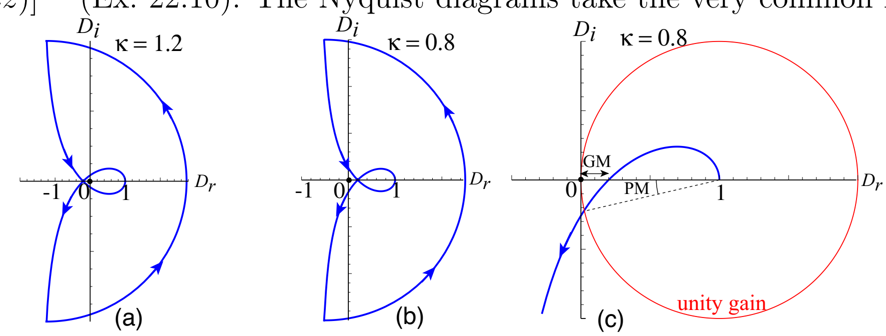
T2 Box 22.2, Continued

For our Cruise-control example, $\mathcal{D}$ is $1 + \tilde{G}\tilde{H} = 1 - K[i\omega(1 - i\omega\tau)]^{-1}$, which can be brought into a more convenient form by introducing the dimensionless frequency $z = \omega\tau$ and dimensionless feedback strength $\kappa = K\tau$: $\mathcal{D} = 1 - \kappa[iz(1 - iz)]^{-1}$. The Nyquist diagram for this $\mathcal{D}$ has the following form:



As $z = \omega\tau$ travels around the upper half frequency plane (curve $\mathcal{C}$ in Fig. 22.4a), $\mathcal{D}$ travels along the left curve (for feedback strength $\kappa = 0.2$), or the right curve (for $\kappa = 2$). These curves do not encircle the origin at all—nor does the curve for any other $\kappa > 0$, so the number of zeros minus the number of poles in the upper half plane is $N_z - N_p = 0$. Moreover, $\mathcal{D} = 1 - \kappa[iz(1 - iz)]^{-1}$ has no poles in the upper half plane, so $N_p = 0$ and $N_z = 0$: our cruise-control feedback system is stable. For further details, see Ex. 22.9.

In designing control systems, it is important to have a significant margin of protection against instability. As an example, consider a control system for which $\tilde{G}\tilde{H} = -\kappa(1 + iz)[iz(1 - iz)]^{-1}$ (Ex. 22.10). The Nyquist diagrams take the very common form:



There are no poles in the upper half plane; and for $\kappa > 1$ (drawing a) the origin is encircled twice, while for $\kappa < 1$ it is not encircled at all (drawing b). Therefore, the control system is unstable for $\kappa > 1$ and stable for $\kappa < 1$. One often wants to push κ as high as possible to achieve one's stabilization goals, but must maintain a margin of safety against instability. That margin is quantified by either or both of two quantities: (i) The *phase margin* (labeled PM in diagram c): the amount by which the phase of $\tilde{G}\tilde{H}$ exceeds 180° at the unity gain point, $|\tilde{G}\tilde{H}| = 1$ (red curve). (ii) The *gain margin* GM: the amount by which the gain $|\tilde{G}\tilde{H}|$ is less than one when the phase of $\tilde{G}\tilde{H}$ reaches 180^0. As κ is increased toward onset of instability, $\kappa = 1$, both PM and GM approach zero.

For the theory of control systems, see, e.g., Franklin, Powell and Emami-Naeini (2005), or Dorf and Bishop (2012).

ϵ, we see that

$$\mathcal{D}(\omega) = k^2 - Z(\omega/k) , \quad \text{where} \tag{22.46a}$$

$$Z(\zeta) \equiv \frac{e^2}{m_e\epsilon_0}\int_{\mathcal{L}} \frac{F'}{v-\zeta}dv , \tag{22.46b}$$

with $\zeta = \omega/k$ the waves' phase velocity.

These equations have several important consequences. (i) For ζ in the upper half plane—the region that concerns us—we can choose the Landau contour $\mathcal{L}$ to travel along the real-v axis from $-\infty$ to $+\infty$, and the resulting $\mathcal{D}(\omega)$ is analytic in the upper half frequency plane, as required for Nyquist's method. (ii) For all ζ in the upper half plane, and for all distribution functions $F(v)$ that are nonnegative and normalizable and thus physically acceptable, the velocity integral is finite, so there are no poles of $\mathcal{D}(\omega)$ in the upper half plane; this means there is an unstable mode, for fixed k, if and only if $\mathcal{D}(\omega)$ encircles the origin at least once, as ω travels around the curve $\mathcal{C}$ of Fig. 22.4. Note that the encircling is guaranteed to be counterclockwise, since there are no poles. (iii) ω traveling along the curve $\mathcal{C}$ in the complex-frequency plane is equivalent to ζ traveling along the same curve in the complex-phase-velocity plane; and $\mathcal{D}$ encircling the origin of the complex-$\mathcal{D}$ plane is equivalent to $Z(\zeta)$ encircling the point $Z = k^2$ on the positive real axis of the complex-Z plane. (iv) For every point on the semicircle segment of the curve $\mathcal{C}$ at $|\zeta| \to \infty$, $Z(\zeta)$ vanishes, so the curve $\mathcal{C}$ can be regarded as going just along the real axis from $-\infty$ to $+\infty$, during which $Z(\zeta)$ emerges from the origin, travels around some curve, and returns to the origin.

In view of these facts, Nyquist's method tells us the following: *There will be an unstable mode, for one or more values of k, if and only if, as ζ travels from $-\infty$ to $+\infty$, $Z(\zeta)$ encloses one or more points on the positive real Z axis. In addition, the wave numbers for any resulting unstable modes are $k = \pm\sqrt{Z}$, for all Z on the positive real axis that are enclosed.*

In Fig. 22.5, we show three examples of $Z(\zeta)$ curves. For diagram (a), no points on the

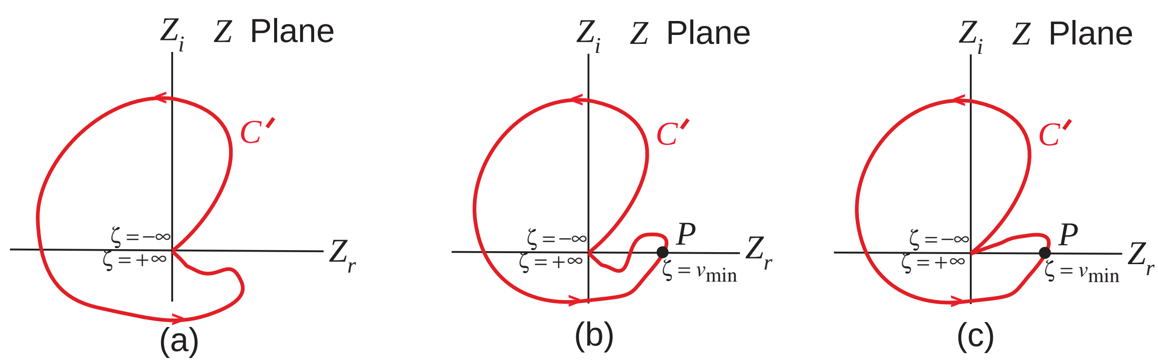


Fig. 22.5: Nyquist diagrams for electrostatic waves. As a mode's real phase velocity ζ increases from $-\infty$ to $+\infty$, $Z(\zeta)$ travels, in the complex-Z plane, around the closed curve $\mathcal{C}'$, which always begins and ends at the origin. For diagram (a), the curve $\mathcal{C}'$ encloses no points on the positive real axis, so there are no unstable electrostatic modes. For diagrams (b) and (c), the curve does enclose a set of points on the positive real axis, so there are unstable modes.

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positive real axis are enclosed, so all electrostatic modes are stable—for all wave numbers k. For diagrams (b) and (c), a segment of the positive real axis is enclosed, so there are unstable modes; and those unstable modes have $k = \pm\sqrt{Z}$ for all Z in the enclosed line segment.

Notice that in diagrams (b) and (c), the rightmost crossing of the real axis is at positive Z, and the curve $\mathcal{C}'$ moves upward as it crosses. A little thought reveals that this must always be the case: $Z(\zeta)$ will encircle, counter-clockwise, points on the positive-Z axis if and only if it somewhere crosses the positive-Z axis traveling upward.

Therefore, *there will be an unstable electrostatic mode in an unmagnetized plasma if and only if, as ζ travels along its real axis from $-\infty$ to $+\infty$, $Z(\zeta)$ crosses some point $\mathcal{P}$ on its positive real axis, traveling upward.* This version of the Nyquist criterion enables us to focus on the small-ω_i domain (while still treating the general case) — for which $\epsilon(k,\zeta)$ is given by Eq. (22.47). From that expression and real ζ (our case), we infer that $Z(\zeta) = k^2[1-\epsilon]$ is given by

$$Z(\zeta) = \frac{e^2}{m_e\epsilon_0}\left[\int_{\rm P} \frac{F'\,dv}{v-\zeta} + i\pi F'(\zeta)\right] . \tag{22.47}$$

This means that $Z(\zeta)$ crosses its real axis at any ζ where $F'(\zeta) = 0$, it crosses moving upward if and only if $F''(\zeta) > 0$ at that crossing point, and these two conditions together say that, at the crossing point $\mathcal{P}$, $\zeta = v_{\rm min}$, a particle speed at which $F(v)$ has a minimum. Moreover, Eq. (22.47) says that $Z(\zeta)$ crosses its positive real axes (rather than negative) if and only if $\int_{\rm P}[F'/(v - v_{\rm min})]dv > 0$. We can evaluate this integral using an integration by parts:

$$\begin{aligned}
\int_{\rm P} \frac{F'}{v-v_{\rm min}}dv &= \int_{\rm P} \frac{d[F(v)-F(v_{min})]/dv}{v-v_{min}}dv \\
&= \int_{\rm P} \frac{[F(v)-F(v_{min})]}{(v-v_{min})^2}dv + \lim_{\delta\to 0}\left[\frac{F(v_{min}-\delta)-F(v_{min})}{-\delta} - \frac{F(v_{min}+\delta)-F(v_{min})}{\delta}\right] .
\end{aligned}$$

The $\lim_{\delta\to 0}$ terms inside the square bracket vanish since $F'(v_{\rm min}) = 0$, and in the first $\int_{\rm P}$ term we do not need the Cauchy principal value because F is a minimum at $v_{\rm min}$. Therefore, our requirement is that

$$\boxed{\int_{-\infty}^{+\infty} \frac{[F(v)-F(v_{min})]}{(v-v_{min})^2}dv > 0} . \tag{22.48}$$

Thus, a necessary and sufficient condition for an unstable mode is that there exist some velocity $v_{\rm min}$ at which the distribution function $F(v)$ has a minimum, and that in addition the minimum be deep enough that the integral (22.48) is positive. This is called the *Penrose criterion* for instability (Penrose 1960).

For a more in-depth, pedagogical derivation and discussion of the Penrose criterion, see, e.g., Sec. 9.6 of Krall and Trivelpiece (1973).

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EXERCISES

Exercise 22.7 *Example: Penrose Criterion*
Consider an unmagnetized electron plasma with a one dimensional distribution function

$$F(v) \propto \{[(v-v_0)^2+u^2]^{-1} + [(v+v_0)^2+u^2]^{-1}\} , \tag{22.49}$$

where v_0 and u are constants. Show that this distribution function possesses a minimum if $v_0 > 3^{-1/2}u$, but the minimum is not deep enough to cause instability unless $v_0 > u$.

Exercise 22.8 *Problem: Range of Unstable Wave Numbers*
Consider a plasma with a distribution function $F(v)$ that has precisely two peaks, at $v = v_1$ and $v = v_2$ [with $F(v_2) \geq F(v_1)$], and a minimum between them at $v = v_{\rm min}$, and assume that the minimum is deep enough to satisfy the Penrose criterion for instability, Eq. (22.48). Show that there will be at least one unstable mode for every wave number k in the range $k_{\rm min} < k < k_{\rm max}$, where

$$k^2_{\rm min} = \frac{e^2}{\epsilon_0 m_e}\int_{-\infty}^{+\infty}\frac{F(v)-F(v_1)}{(v-v_1)^2}dv\ , \quad k^2_{\rm max} = \frac{e^2}{\epsilon_0 m_e}\int_{-\infty}^{+\infty}\frac{F(v)-F(v_{\rm min})}{(v-v_{\rm min})^2}dv\ . \tag{22.50}$$

Show, further, that the marginally unstable mode at $k = k_{\rm max}$ has phase velocity $\omega/k = v_{\rm min}$; and the marginally unstable mode at $k = k_{\rm min}$ has $\omega/k = v_1$. [Hint: Use a Nyquist diagram like those in Fig. 22.5.]

Exercise 22.9 $\boxed{\textbf{\textit{T2}}}$ *Example and Derivation: Cruise-Control System*

(a) Show that the cruise-control feedback system described at the beginning of Box 22.2 has $\tilde{G}(z) = 1/(1 - iz)$ and $\tilde{H} = -\kappa/iz$, with $z = \omega\tau$ and $\kappa = K\tau$, as claimed in the Box.

(b) Show that the Nyquist diagram has the forms shown in the second second set of diagrams in Box 22.2, and that this control system is stable for all feedback strengths $\kappa > 0$.

(c) Solve explicitly for the zeros of $\mathcal{D} = 1 + \tilde{G}(z)\tilde{H}(z)$ and verify that none are in the upper half frequency plane.

(d) To understand the stability from a different viewpoint, imagine that the automobile's speed v is oscillating with an amplitude δv and a real frequency ω around the desired speed V, $v = V + \delta v\,\sin(\omega t)$, and that the feedback is turned off. Show that the output of the control system is $u_o = [\delta v/\sqrt{1+\omega^2\tau^2}]\,\sin(\omega t - \Delta\varphi)$, with a phase delay $\Delta\varphi = \arctan(\omega\tau)$ relative to the oscillations of v. Now turn on the feedback, but at a low strength, so it only weakly changes the speed's oscillations in one period. Show that, because $\Delta\varphi < \pi/2$, $d(\delta v^2)/dt$ is negative, so the feedback damps the oscillations. Show that an instability would arise if the phase delay were in the range $\pi/2 < |\Delta\varphi| < 3\pi/2$. For high-frequency oscillations, $\omega\tau \gg 1$, $\Delta\varphi$ approaches $\pi/2$, so the cruise-control system is only marginally stable.

Exercise 22.10 $\boxed{\textbf{\textit{T2}}}$ *Derivation: Phase Margin and Gain Margin for a Feedback Control System*
Consider the control system discussed in the last long paragraph of Box 22.2. It has $\tilde{G}\tilde{H} = -\kappa(1 + iz)[iz(1 - iz)]^{-1}$, with $z = \omega\tau$ a dimensionless frequency and τ some time constant.

(a) Show that there are no poles of $\mathcal{D} = 1 + \tilde{G}\tilde{H}$ in the upper half frequency plane.

(b) Construct the Nyquist diagram for various feedback strengths κ. Show that for $\kappa > 1$ the curve encircles the origin twice (diagram a of the Box), so the control system is unstable, while for $\kappa < 1$, it does not encircle the origin (diagram b), so the control system is stable.

(c) Show that the phase margin and gain margin, defined in diagram c, approach zero as κ increases toward the instability point, $\kappa = 1$.

(d) Compute explicitly the zeros of $\mathcal{D} = 1+\tilde{G}\tilde{H}$ and plot their trajectories, in the complex frequency plane, as κ increases from zero through one to ∞. Verify that two zeros enter the upper half frequency plane as κ increases through one, and they remain in the upper half plane for all $\kappa > 1$, as is guaranteed by the Nyquist diagrams.

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22.5 Particle Trapping

We now return to the description of Landau damping. Our treatment so far has been essentially linear in the wave amplitude, or equivalently in the perturbation to the distribution function. What happens when the wave amplitude is not infinitesimally small?

Consider a single Langmuir wave mode as observed in a frame moving with the mode's phase velocity. In this frame the electrostatic field oscillates spatially, $E = E_0 \sin kz$, but has no time dependence. Figure 22.6 shows some phase-space orbits of electrons in this oscillatory potential. The solid curves are orbits of particles that move fast enough to avoid being trapped in the potential wells at $kz = 0, 2\pi, 4\pi, \ldots$. The dashed curves are orbits of trapped particles. As seen in another frame, these trapped particles are surfing on the wave, with their velocities performing low-amplitude oscillations around the wave's phase velocity ω/k.

The equation of motion for an electron trapped in the minimum $z = 0$ has the form

$$\begin{aligned}\ddot{z} &= \frac{-eE_0 \sin kz}{m_e} \\ &\simeq -\omega_b^2 z\ , \end{aligned} \tag{22.51}$$

where we have assumed small-amplitude oscillations and approximated $\sin kz \simeq kz$, and where

$$\omega_b = \left(\frac{eE_0 k}{m_e}\right)^{1/2} \tag{22.52}$$

is known as the *bounce frequency*. Since the potential well is actually anharmonic, the trapped particles will mix in phase quite rapidly.

The growth or damping of the wave is characterized by a growth or damping of E_0, and correspondingly by a net acceleration or deceleration of *untrapped* particles, when averaged over a wave cycle. It is this net feeding of energy into and out of the untrapped particles that causes the wave's Landau damping or growth.

Now suppose that the amplitude E_0 of this particular wave mode is changing on a time scale τ due to interactions with the electrons, or possibly (as we shall see in Chap. 23) due to interactions with other waves propagating through the plasma. The potential well will then change on this same timescale and we expect that τ will also be a measure of the maximum length of time a particle can be trapped in the potential well. Evidently, nonlinear wave trapping effects should only be important when the bounce period $\sim \omega_b^{-1}$ is short compared with τ, i.e. when $E_0 \gg m_e/ek\tau^2$.

Electron trapping can cause particles to be bunched together at certain preferred phases of a growing wave. This can have important consequences for the radiative properties of the plasma. Suppose, for example, that the plasma is magnetized. Then the electrons gyrate around the magnetic field and emit cyclotron radiation. If their gyrational phases are random then the total power that they radiate will be the sum of their individual particle powers. However, if N electrons are localized at the same gyrational phase due to being trapped in a potential well of a wave, then, they will radiate like one giant electron with a charge Ne. As the radiated power is proportional to the square of the charge carried by the radiating particle, the total power radiated by the bunched electrons will be N times the power radiated by the same number of unbunched electrons. Very large amplification factors are thereby possible both in the laboratory and in Nature, for example in the Jovian magnetosphere.

This brief discussion suggests that there may be much more richness in plasma waves than is embodied in our dispersion relations with their simple linear growth and decay, even when the wave amplitude is small enough that the particle motion is only slightly perturbed by its interaction with the wave. This motivates us to discuss more systematically nonlinear plasma physics, which is the topic of our next chapter.

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EXERCISES

Exercise 22.11 *Challenge: BGK Waves*

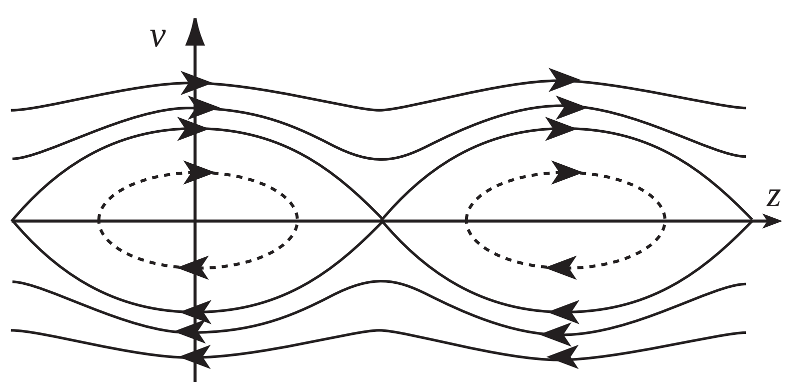

Fig. 22.6: Phase-space orbits for trapped (dashed lines) and untrapped (solid lines) electrons.

Consider a steady, one dimensional, large amplitude electrostatic wave in an unmagnetized, proton-electron plasma. Write down the Vlasov equation for each particle species in a frame moving with the wave, i.e. a frame in which the electrostatic potential is a time-independent function of z, $\Phi = \Phi(z)$, not necessarily precisely sinusoidal.

(a) Use Jeans' theorem to argue that proton and electron distribution functions that are just functions of the energy,

$$F_s = F_s(W_s) \ , \quad W_s = m_s v^2/2 + q_s\Phi(z) \ , \tag{22.53a}$$

satisfy the Vlasov equation; here $s = p, e$ and as usual $q_p = e$, $q_s = -e$. Then show that Poisson's equation for the potential Φ can be rewritten in the form

$$\frac{1}{2}\left(\frac{d\Phi}{dz}\right)^2 + V(\Phi) = \text{const} \ , \tag{22.53b}$$

where the *potential* V is $-2/\epsilon_0$ times the kinetic energy density of the particles

$$V = \frac{-2}{\epsilon_0}\sum_s \int \frac{1}{2} m_s v^2 F_s \, dv \tag{22.53c}$$

(which depends on Φ).

(b) It is possible to find many electrostatic potential profiles $\Phi(z)$ and distribution functions $F_s[W_s(v,\Phi)]$ that satisy Eqs. (22.53). These are called BGK waves after Bernstein, Greene and Kruskal (1957), who first analyzed them. Explain how, in principle, one can solve for (non unique) BGK distribution functions F_s in a large amplitude wave of given electrostatic potential profile $\Phi(z)$.

(c) Carry out this procedure, assuming that the potential profile is of the form $\Phi(z) = \Phi_0 \cos kz$ with $\Phi_0 > 0$. Assume also that the protons are monoenergetic with $W_p = W_+ > e\Phi_0$ and move along the positive z-direction, and that there are both monoenergetic (with $W_e = W_-$), untrapped electrons (also moving along the positive z-direction), and trapped electrons with distribution $F_e(W_e), -e\Phi_0 \leq W_e < e\Phi_0$; see Fig. 22.7 . Show that the density of trapped electrons must vanish at the wave troughs [at $z = (2n+1)\pi/k; n = 0,1,2,3\ldots$]. Let the proton density at the troughs be n_{p0}, and assume that there is no net current as well as no net charge density. Show that the total electron density can then be written as

$$n_e(z) = \left[\frac{m_e(W_+ - e\Phi_0)}{m_p(W_- + e\Phi)}\right]^{1/2} n_{p0} + \int_{-e\Phi_0}^{e\Phi_0} \frac{dW_e F_e(W_e)}{[2m_e(W_e + e\Phi)]^{1/2}} \ . \tag{22.54}$$

(d) Use Poisson's equation to show that

$$\int_{-\xi_0}^{\xi_0} \frac{dW_e F_e(W_e)}{[2m_e(W_e+\xi)]^{1/2}} = \frac{\epsilon_0 k^2 \Phi}{e^2} + n_{p0}\left[\left(\frac{W_+ - \xi_0}{W_+ - \xi}\right)^{1/2} - \left(\frac{m_e(W_+ - \xi_0)}{m_p(W_- + \xi)}\right)^{1/2}\right] , \tag{22.55}$$

where $e\Phi_0 = \xi_0$.

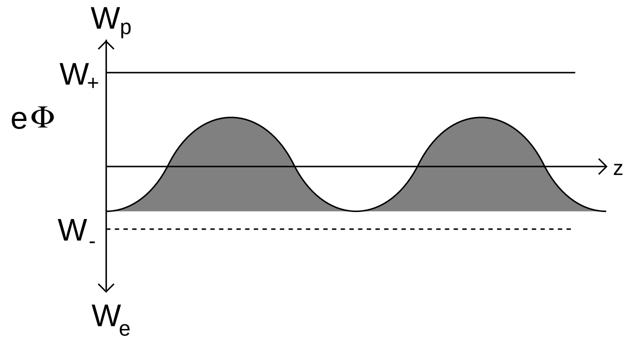

Fig. 22.7: BGK waves. The ordinate is $e\Phi$, where $\Phi(z)$ is the one dimensional electrostatic potential. The proton total energies, W_p, are displayed increasing upward; the electron energies, W_e, increase downward. In this example, corresponding to Challenge 22.11, there are monoenergetic proton (solid line) and electron (dashed line) beams plus a bound distribution of electrons (shaded region) trapped in the potential wells formed by the electrostatic potential.

(e) Solve this integral equation for $F_e(W_e)$. (Hint: it is of Abel type.)

(f) Exhibit some solutions graphically.

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22.6 $\boxed{\text{T2}}$ N-Particle Distribution Function

Before turning to nonlinear phenomena in plasmas (the next chapter), let us digress briefly and explore ways to study correlations between particles, of which our Vlasov formalism is oblivious.

The Vlasov formalism treats each particle as independent, migrating through phase space in response to the local mean electromagnetic field and somehow unaffected by individual electrostatic interactions with neighboring particles. As we discussed in Chap. 20, this is likely to be a good approximation in a collisionless plasma because of the influence of Debye screening—except in the tiny "independent-particle" region of Fig. 20.1. However, we would like to have some means of quantifying this and of deriving an improved formalism that takes into account the correlations between individual particles.

One environment where this may be relevant is the interior of the sun. Here, although the gas is fully ionized, the Debye number is not particularly large (i.e., one is near the independent particle region of Fig. 20.1). As a result, Coulomb corrections to the perfect gas equation of state may be responsible for measurable changes in the sun's internal structure as deduced, for example, using helioseismological analysis (cf. Sec. 16.2.4). In this application our task is simpler than in the general case because the gas will locally be in thermodynamic equilibrium at some temperature T. It turns out that the general case, where the plasma departs significantly from thermal equilibrium, is extremely hard to treat.

The one-particle distribution function that we use in the Vlasov formalism is the first member of a hierarchy of k-particle distribution functions, $f^{(k)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, t)$; e.g., $f^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2, t) d\mathbf{x}_1 d\mathbf{v}_1 d\mathbf{x}_2 d\mathbf{v}_2 \equiv$[the probability that particle 1 will be found in a volume $d\mathbf{x}_1 d\mathbf{v}_1 \equiv dx_1 dy_1 dz_1 dv_{x_1} dv_{y_1} dv_{z_1}$ of its phase space, and that particle 2 will be found in volume $d\mathbf{x}_2 d\mathbf{v}_2$ of its phase space].[4] Our probability interpretation of these distribution functions dictates for $f^{(1)}$ a different normalization than we use in the Vlasov formalism, $f^{(1)} = f/n$ where n is the number density of particles, and dictates that

$$\int f^{(k)} d\mathbf{x}_1 d\mathbf{v}_1 \cdots d\mathbf{x}_k d\mathbf{v}_k = 1. \tag{22.56}$$

[Note that these $f^{(k)}$ are analogous to the probability distributions p_k that we introduced in the theory of random processes, Eq. (6.1).]

It is useful to relate the distribution functions $f^{(k)}$ to the concepts of statistical mechanics, which we developed in Chap. 4. Suppose we have an ensemble of N-electron plasmas and let the probability that a member of this ensemble is in a volume $d^{3N}\mathbf{x}\, d^{3N}\mathbf{v}$ of the $6N$ dimensional phase space of all its particles be $f_{\rm all} d^{3N}\mathbf{x}\, d^{3N}\mathbf{v}$. ($N$ is a very large number!) Of course, $f_{\rm all}$ satisfies the Liouville equation

$$\frac{\partial f_{\rm all}}{\partial t} + \sum_{i=1}^{N} [(\mathbf{v}_i \cdot \boldsymbol{\nabla}_i) f_{\rm all} + (\mathbf{a}_i \cdot \boldsymbol{\nabla}_{\mathbf{v}i}) f_{\rm all}] = 0 \, , \tag{22.57}$$

where $\mathbf{a}_i$ is the electromagnetic acceleration of the i'th particle, and $\boldsymbol{\nabla}_i$ and $\boldsymbol{\nabla}_{\mathbf{v}i}$ are gradients with respect to the position and velocity of particle i. We can construct the k-particle "reduced" distribution function from the statistical-mechanics distribution function $f_{\rm all}$ by integrating over all but k of the particles:

$$\begin{aligned} &f^{(k)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, t) \\ &\quad = N^k \int d\mathbf{x}_{k+1} \ldots d\mathbf{x}_N \mathbf{v}_{k+1} \ldots d\mathbf{v}_N f_{\rm all}(\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N) \, . \end{aligned} \tag{22.58}$$

(Note: k is typically a very small number, by contrast with N; below we shall only be concerned with $k = 1, 2, 3$.) The reason for the prefactor N^k in Eq. (22.58) is that, whereas $f_{\rm all}$ refers to the probability of finding particle 1 in $d\mathbf{x}_1 d\mathbf{v}_1$, particle 2 in $d\mathbf{x}_2 d\mathbf{v}_2$ etc, the reduced distribution function $f^{(k)}$ describes the probability of finding *any* of the N identical, though distinguishable particles in $d\mathbf{x}_1 d\mathbf{v}_1$ and so on. (As long as we are dealing with non-degenerate plasmas we can count the electrons classically.) As $N \gg k$, the number of possible ways we can choose k particles for k locations in phase space is approximately N^k.

For simplicity, suppose that the protons are immobile and form a uniform, neutralizing background of charge, so we need only consider the electron distribution and its correlations. Let us further suppose that the forces associated with *mean* electromagnetic fields produced

[4] *Note:* In this section—and only in this section—we adopt the volume notation commonly used in this multi-particle subject: we use $d\mathbf{x}_j \equiv dx_j dy_j dz_j$ and $d\mathbf{v}_j \equiv dv_{x_j} dv_{y_j} dv_{z_j}$ to denote the volume elements for particle j. *Warning*: Despite the boldface notation, $d\mathbf{x}_j$ and $d\mathbf{v}_j$ are not vectors! Also, in this section we do *not* study waves, so k represents the number of particles in a distribution function, rather than a wave number.

by *external* charges and currents can be ignored. We can then restrict our attention to direct electron-electron electrostatic interactions. The acceleration of an electron is then

$$\mathbf{a}_i = \frac{e}{m_e}\sum_j \boldsymbol{\nabla}_i \Phi_{ij} \ , \tag{22.59}$$

where $\Phi_{ij}(x_{ij}) = -e/4\pi\epsilon_0 x_{ij}$ is the electrostatic potential between two electrons i, j, and $x_{ij} \equiv |\mathbf{x}_i - \mathbf{x}_j|$.

22.6.1 $\boxed{\text{T2}}$ BBKGY Hierarchy

We can now derive the so-called BBGKY[5] hierarchy of kinetic equations, which relate the k-particle distribution function to integrals over the $k+1$ particle distribution function. The first equation in this hierarchy is given by integrating Liouville's Eq. (22.57) over $d\mathbf{x}_2 \ldots d\mathbf{x}_N d\mathbf{v}_2 \ldots d\mathbf{v}_N$. If we assume that the distribution function decreases to zero at large distances, then integrals of the type $\int d\mathbf{x}_i \boldsymbol{\nabla}_i f_{\rm all}$ vanish and the one particle distribution function evolves according to

$$\begin{aligned}
\frac{\partial f^{(1)}}{\partial t} + (\mathbf{v}_1 \cdot \boldsymbol{\nabla}) f^{(1)} &= \frac{-eN}{m_e}\int d\mathbf{x}_2 \ldots d\mathbf{x}_N d\mathbf{v}_2 \ldots d\mathbf{v}_N \Sigma_j \boldsymbol{\nabla}_1 \Phi_{1j} \cdot \boldsymbol{\nabla}_{\mathbf{v}_1} f_{\rm all} \\
&= \frac{-eN^2}{m_e}\int d\mathbf{x}_2 \ldots d\mathbf{x}_N d\mathbf{v}_2 \ldots d\mathbf{v}_N \boldsymbol{\nabla}_1 \Phi_{12} \cdot \boldsymbol{\nabla}_{\mathbf{v}_1} f_{\rm all} \\
&= \frac{-e}{m_e}\int d\mathbf{x}_2 d\mathbf{v}_2 \left(\boldsymbol{\nabla}_{\mathbf{v}1} f^{(2)} \cdot \boldsymbol{\nabla}_1\right) \Phi_{12} \ ,
\end{aligned} \tag{22.60}$$

where we have replaced the probability of having *any* particle at a location in phase space by N times the probability of having *one* specific particle there. The left hand side of Eq. (22.60) describes the evolution of independent particles and the right hand side takes account of their pairwise mutual correlation.

The evolution equation for $f^{(2)}$ can similarly be derived by integrating the Liouville equation (22.57) over $d\mathbf{x}_3 \ldots d\mathbf{x}_N d\mathbf{v}_3 \ldots d\mathbf{v}_N$

$$\begin{aligned}
&\frac{\partial f^{(2)}}{\partial t} + (\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1) f^{(2)} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2) f^{(2)} + \frac{e}{m_e}\left[\left(\boldsymbol{\nabla}_{\mathbf{v}1} f^{(2)} \cdot \boldsymbol{\nabla}_1\right)\Phi_{12} + \left(\boldsymbol{\nabla}_{\mathbf{v}2} f^{(2)} \cdot \boldsymbol{\nabla}_2\right)\Phi_{12}\right] \\
&\quad = \frac{-e}{m_e}\int d\mathbf{x}_3 d\mathbf{v}_3 \left[\left(\boldsymbol{\nabla}_{\mathbf{v}1} f^{(3)} \cdot \boldsymbol{\nabla}_1\right)\Phi_{13} + \left(\boldsymbol{\nabla}_{\mathbf{v}2} f^{(3)} \cdot \boldsymbol{\nabla}_2\right)\Phi_{23}\right] \ .
\end{aligned} \tag{22.61}$$

Similarly in general, allowing for the presence of a mean electromagnetic field (in addition to the inter-electron electrostatic field) causing an acceleration $\mathbf{a}^{ext} = -(e/m_e)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, we obtain the BBGKY hierarchy of kinetic equations

$$\begin{aligned}
&\frac{\partial f^{(k)}}{\partial t} + \sum_{i=1}^{k}\left[(\mathbf{v}_i \cdot \boldsymbol{\nabla}_i) f^{(k)} + (\mathbf{a}_i^{ext} \cdot \boldsymbol{\nabla}_{\mathbf{v}i}) f^{(k)} + \frac{e}{m_e}(\boldsymbol{\nabla}_{\mathbf{v}_i} f^{(k)} \cdot \boldsymbol{\nabla}_i)\sum_{j \neq i}^{k} \Phi_{ij}\right] \\
&\qquad = \frac{-e}{m_e}\int d\mathbf{x}_{k+1} d\mathbf{v}_{k+1} \sum_{i=1}^{k}\left(\boldsymbol{\nabla}_{\mathbf{v}_i} f^{(k+1)} \cdot \boldsymbol{\nabla}_i\right)\Phi_{ik+1} \ .
\end{aligned} \tag{22.62}$$

[5]Bogolyubov (1962), Born and Green (1949), Kirkwood (1946), Yvon (1935).

This k'th equation in the hierarchy shows explicitly how we require knowledge of the $(k+1)$-particle distribution function in order to determine the evolution of the k-particle distribution function.

22.6.2 T2 Two-Point Correlation Function

It is convenient to define the *two-point correlation function*, $\xi_{12}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t)$ for particles 1,2, by

$$\boxed{f^{(2)}(1,2) = f_1 f_2 (1 + \xi_{12})}\ , \tag{22.63}$$

where we introduce the notation $f_1 = f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t)$ and $f_2 = f^{(1)}(\mathbf{x}_2, \mathbf{v}_2, t)$. We now restrict attention to a plasma in thermal equilibrium at temperature T. In this case, f_1, f_2 will be Maxwellian distribution functions, independent of $\mathbf{x}, t$. Now, let us make an *ansatz*, namely that ξ_{12} is just a function of the electrostatic interaction energy between the two electrons and therefore it does not involve the electron velocities. (It is, actually, possible to justify this directly for an equilibrium distribution of particles interacting electrostatically, but we shall make do with showing that our final answer for ξ_{12} is just a function of x_{12}, in accord with our anszatz.) As Debye screening should be effective at large distances, we anticipate that $\xi_{12} \to 0$ as $x_{12} \to \infty$.

Now turn to Eq. (22.61), and introduce the simplest imaginable closure relation, $\xi_{12} = 0$. In other words, completely ignore all correlations. We can then replace $f^{(2)}$ by $f_1 f_2$ and perform the integral over $\mathbf{x}_2, \mathbf{v}_2$ to recover the collisionless Vlasov equation (22.6). We therefore see explicitly that particle-particle correlations are indeed ignored in the simple Vlasov approach.

For the 3-particle distribution function, we expect that, when electron 1 is distant from both electrons 2,3, then $f^{(3)} \sim f_1 f_2 f_3 (1 + \xi_{23})$, etc. Summing over all three pairs we write,

$$f^{(3)} = f_1 f_2 f_3 (1 + \xi_{23} + \xi_{31} + \xi_{12} + \chi_{123})\ , \tag{22.64}$$

where χ_{123} is the *three point correlation function* that ought to be significant when all three particles are close together. χ_{123} is, of course, determined by the next equation in the BBGKY hierarchy.

We next make the closure relation $\chi_{123} = 0$, that is to say, we ignore the influence of third bodies on pair interactions. This is reasonable because close, *three body* encounters are even less frequent than close *two body* encounters. We can now derive an equation for ξ_{12} by seeking a steady state solution to Eq. (22.61), i.e. a solution with $\partial f^{(2)}/\partial t = 0$. We substitute Eqs. (22.63) and (22.64) into (22.61) (with $\chi_{123} = 0$) to obtain

$$\begin{aligned} & f_1 f_2 \left[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1)\xi_{12} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)\xi_{12} - \frac{e(1+\xi_{12})}{k_B T} \{ (\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1)\Phi_{12} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)\Phi_{12} \} \right] \\ & = \frac{e f_1 f_2}{k_B T} \int d\mathbf{x}_3 d\mathbf{v}_3 f_3 \, (1 + \xi_{23} + \xi_{31} + \xi_{12}) \left[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1)\Phi_{13} + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)\Phi_{23} \right]\ , \end{aligned} \tag{22.65}$$

where we have used the relation

$$\boldsymbol{\nabla}_{\mathbf{v}1} f_1 = -\frac{m_e \mathbf{v}_1 f_1}{k_B T}\ , \tag{22.66}$$

valid for an unperturbed Maxwellian distribution function. We can rewrite this equation using the relations

$$\boldsymbol{\nabla}_1\Phi_{12} = -\boldsymbol{\nabla}_2\Phi_{12} \ , \quad \boldsymbol{\nabla}_1\xi_{12} = -\boldsymbol{\nabla}_2\xi_{12} \ , \quad \xi_{12} \ll 1 \ , \quad \int d\mathbf{v}_3 f_3 = n \ , \tag{22.67}$$

to obtain

$$(\mathbf{v}_1 - \mathbf{v}_2)\cdot\boldsymbol{\nabla}_1\left(\xi_{12} - \frac{e\Phi_{12}}{k_BT}\right) = \frac{ne}{k_BT}\int d\mathbf{x}_3(1+\xi_{23}+\xi_{31}+\xi_{12})[(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)\Phi_{13} + (\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)\Phi_{23}] \ . \tag{22.68}$$

Now, symmetry considerations tell us that

$$\int d\mathbf{x}_3(1+\xi_{31})\boldsymbol{\nabla}_1\Phi_{13} = 0 \ , \quad \int d\mathbf{x}_3(1+\xi_{23})\boldsymbol{\nabla}_2\Phi_{23} = 0 \ , \tag{22.69}$$

and, in addition,

$$\int d\mathbf{x}_3\xi_{12}\boldsymbol{\nabla}_1\Phi_{13} = -\xi_{12}\int d\mathbf{x}_3\boldsymbol{\nabla}_3\Phi_{13} = 0 \ , \quad \int d\mathbf{x}_3\xi_{12}\boldsymbol{\nabla}_2\Phi_{23} = -\xi_{12}\int d\mathbf{x}_3\boldsymbol{\nabla}_3\Phi_{23} = 0. \tag{22.70}$$

Therefore, we end up with

$$(\mathbf{v}_1 - \mathbf{v}_2)\cdot\boldsymbol{\nabla}_1\left(\xi_{12} - \frac{e\Phi_{12}}{k_BT}\right) = \frac{ne}{k_BT}\int d\mathbf{x}_3[\xi_{23}(\mathbf{v}_1\cdot\boldsymbol{\nabla}_1)\Phi_{13} + \xi_{31}(\mathbf{v}_2\cdot\boldsymbol{\nabla}_2)\Phi_{23}] \ . \tag{22.71}$$

As this equation must be true for arbitrary velocities, we can set $\mathbf{v}_2 = 0$ and obtain

$$\boldsymbol{\nabla}_1(k_BT\xi_{12} - e\Phi_{12}) = ne\int d\mathbf{x}_3\xi_{23}\boldsymbol{\nabla}_1\Phi_{13} \ . \tag{22.72}$$

We take the divergence of Eq. (22.72) and use Poisson's equation, $\boldsymbol{\nabla}_1^2\Phi_{12} = e\delta(\mathbf{x}_{12})/\epsilon_0$, to obtain

$$\boldsymbol{\nabla}_1^2\xi_{12} - \frac{\xi_{12}}{\lambda_D^2} = \frac{e^2}{\epsilon_0 k_BT}\delta(\mathbf{x}_{12}) \ , \tag{22.73}$$

where $\lambda_D = (k_BT\epsilon_0/ne^2)^{1/2}$ is the Debye length [Eq. (20.10)]. The solution of Eq. (22.73) is

$$\boxed{\xi_{12} = \frac{-e^2}{4\pi\epsilon_0 k_BT}\frac{e^{-x_{12}/\lambda_D}}{x_{12}}} \ . \tag{22.74}$$

Note that the sign is negative because the electrons repel one another. Note also that, to order of magnitude, $\xi_{12}(x_{12} = \lambda_D) \sim -N_D^{-1}$ which is $\ll 1$ in magnitude if the Debye number is much greater than unity. At the mean interparticle spacing, $\xi_{12}(x_{12} = n^{-1/3}) \sim -N_D^{-2/3}$. Only for distances $x_{12} \lesssim e^2/\epsilon_0 k_BT$ will the correlation effects become large and our expansion procedure and truncation ($\chi_{123} = 0$) become invalid. This analysis justifies the use of the Vlasov equation when $N_D \gg 1$; see the discussion at the end of the next subsection.

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EXERCISES

Exercise 22.12 $\boxed{T2}$ *Problem: Correlations in a Tokamak Plasma*
For a Tokamak plasma compute, in order of magnitude, the two point correlation function for two electrons separated by

(a) a Debye length,

(b) the mean interparticle spacing.

****

22.6.3 $\boxed{\text{T2}}$ Coulomb Correction to Plasma Pressure

Let us now turn to the problem of computing the Coulomb correction to the pressure of an ionized gas. It is easiest to begin by computing the Coulomb correction to the internal energy density. Once again ignoring the protons, this is simply given by

$$U_c = \frac{-e}{2} \int d\mathbf{x}_1 n_1 n_2 \xi_{12} \Phi_{12} \ , \tag{22.75}$$

where the factor 1/2 compensates for double counting the interactions. Substituting Eq. (22.74) and performing the integral, we obtain

$$U_c = \frac{-ne^2}{8\pi\epsilon_0 \lambda_D} \ . \tag{22.76}$$

The pressure can be obtained from this energy density using elementary thermodynamics. From the definition (5.32) of the physical free energy converted to a per-unit-volume basis and the first law of thermodynamics, the volume density of Coulomb free energy, $\mathcal{F}_c$, is given by integrating

$$U_c = -T^2 \left(\frac{\partial (\mathcal{F}_c / T)}{\partial T} \right)_n \ . \tag{22.77}$$

From this, we obtain $\mathcal{F}_c = -ne^2/12\pi\epsilon_0\lambda_D$. The Coulomb contribution to the pressure is then given by Eq. (5.33)

$$P_c = n^2 \left(\frac{\partial (\mathcal{F}_c / n)}{\partial n} \right)_T = \frac{-ne^2}{24\pi\epsilon_0\lambda_D} = \frac{1}{3} U_c \ . \tag{22.78}$$

Therefore, including the Coulomb interaction *decreases* the pressure at a given density and temperature.

We have kept the neutralizing protons fixed so far. In a real plasma they are mobile and so the Debye length must be reduced by a factor $2^{-1/2}$ [cf. Eq. (20.9)]. In addition, Eq. (22.75) must be multiplied by a factor 4 to take account of the proton-proton and proton-electron interactions. The end result is

$$\boxed{P_c = \frac{-n^{3/2} e^3}{2^{3/2} 3\pi \epsilon_0^{3/2} T^{1/2}}} \ , \tag{22.79}$$

where n is still the number density of electrons. Numerically the gas pressure for a perfect electron-proton gas is

$$P = 1.6 \times 10^{13} (\rho/1000\mathrm{kg\ m}^{-3})(T/10^6\mathrm{K})\mathrm{N\ m}^{-2}\ , \tag{22.80}$$

and the Coulomb correction to this pressure is

$$P_c = -7.3 \times 10^{11} (\rho/1000\mathrm{kg\ m}^{-3})^{3/2}(T/10^6\mathrm{K})^{-1/2}\mathrm{N\ m}^{-2}. \tag{22.81}$$

In the interior of the sun this is about one percent of the total pressure. In denser, cooler stars, it is significantly larger.

By contrast, for most of the plasmas that one encounters, our analysis implies that the order of magnitude of the two point correlation function ξ_{12} is $\sim N_D^{-1}$ across a Debye sphere and only $\sim N_D^{-2/3}$ at the distance of the mean inter-particle spacing (see end of Sec. 22.6.2). Only those particles that are undergoing large deflections, through angles ~ 1 radian, are close enough together for $\xi_{12} = O(1)$. This is the ultimate justification for treating plasmas as collisionless and for using mean electromagnetic fields in the Vlasov description.

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EXERCISES

Exercise 22.13 $\boxed{\textbf{\textit{T2}}}$ *Derivation: Thermodynamic identities*
Verify Eq. (22.77), (22.78).

Exercise 22.14 $\boxed{\textbf{\textit{T2}}}$ *Problem: Thermodynamics of Coulomb Plasma*
Compute the entropy of a proton-electron plasma in thermal equilibrium at temperature T including the Coulomb correction.

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Bibliographic Note

The kinetic theory of warm plasmas and its application to electrostatic waves and their stability are treated in most all texts on plasma physics. For maximum detail and good pedagogy, we particularly like Chaps. 7, 8 and of the ancient book by Krall and Trivilpiece (1973); but beware of the this books large number of typographical errors. We also recommend, in Bellan (2006): early parts of Chap. 2, and all Chap. 5, and for the extension to magnetized plasmas, Chap. 8. Also useful are Chap. 4 of Swanson (2003), Chaps. 8 and 10 of Stix (1992), Chap. 8 of Boyd and Sandersson (2003), Chap. 3 of Lifshitz and Pitaevski (1981), and Chaps. 3 and 7 of Schmidt (1966).

For brief discussions of the BBGKY hierarchy of N-particle distribution functions and their predicted correlations in a plasma, see Chap. 12 of Boyd and Sandersson (2003) and Sec. 4.1.3 of Swanson (2003). For detailed discussions, see the original literature cited in footnote 5.

Box 22.3
Important Concepts in Chapter 22

- Kinetic theory concepts: distribution function $f_s(\mathbf{v}, \mathbf{x}, t)$ and Vlasov equation for it, Sec. 22.2.1
- Relation of kinetic theory to two-fuid formalism, Sec. 22.2.2
- Jeans' theorem for solutions of Vlasov equation, Sec. 22.2.3
- Electrostatic Waves treated via kinetic theory, Secs. 22.3–22.5
  - Distribution function reduced to one dimension (that of the wave propagation), $F_s(v, z, t)$; its split into an equilibrium part $F_{s0}(v)$ and perturbation F_{s1}, and the unified equilibrium distribution for electrons and protons, $F_0 = F_{e0} + (m_e/m_p)F_{p0}$, Sec. 22.3.1
  - Dispersion relation as vanishing of the plasma's dielectric function $\epsilon(\omega, k) = 0$, Sec. 22.3.1
  - Landau contour and its derivation via Laplace transforms, Sec. 22.3.3 and Ex. 22.4
  - Dispersion relation $\epsilon(\omega, k) = 0$ as an integral along the Landau contour, Sec. 22.3.3
  - Dispersion relation specialized to case of weak damping or growth, $|\omega_i| \ll \omega_r$: ω_i proportional to dF_0/dv evaluated at the waves' phase velocity, and interpretation of this in terms of surfing particles, Secs. 22.3.4, 22.3.5, and 22.5
  - Landau damping of Langmuir waves, Sec. 22.3.5
  - Landau damping of ion acoustic waves, Sec. 22.3.6
  - Two-stream instability, Sec. 22.3.2
  - Nyquist method for analyzing stability of waves, Sec. 22.4; application of Nyquist method to stability of feedback control systems, Box 22.2
  - Penrose criterion for instability of a double humped velocity distribution, Sec. 22.4
  - Particle trapping in a wave, Sec. 22.5
- N-particle distribution functions, Sec. 22.6
  - BBGKY hierarchy of equations for their evolution, Sec. 22.6.1
  - Two-point and three-point correlation functions, Sec. 22.6.2
  - Coulomb correction to plasma pressure, Sec. 22.6.3

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