

# Physics 127b: Statistical Mechanics

## Boltzmann Equation I: Scattering off fixed impurities

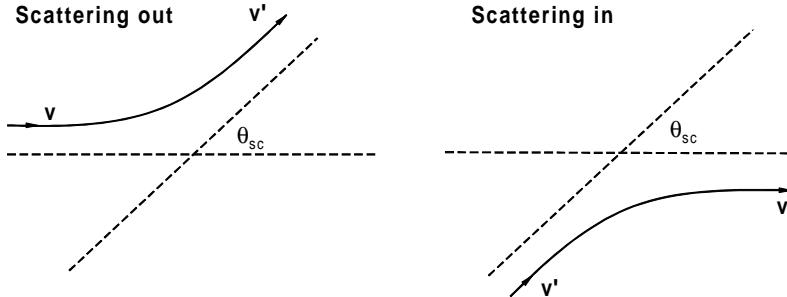


Figure 1: Scattering off a fixed impurity: the *scattering out* and *scattering in* processes.

The particle distribution  $f(\vec{v})$  at  $\vec{v}$  is changed by two types of processes: scattering from  $\vec{v}$  to any other velocity, which decreases  $f(\vec{v})$ —the *scattering out processes*—and scattering from other velocities to  $\vec{v}$ , which increases  $f(\vec{v})$ —*scattering in*. These processes can be collected in symmetry related pairs, Fig. 1. The scattering in and out processes shown in the figure are related by the combined process of space and time inversion, and so *have the same scattering cross section*<sup>1</sup>. Thus we can write the collision term

$$\frac{df(\vec{v})}{dt} \Big|_{coll} = \int (f'_1 - f_1) v n_s \sigma(v, \theta_{sc}) d\Omega_{sc}$$

where  $n_s$  is the density of scattering centers,  $\vec{v}'$  and  $\vec{v}$  are related by the scattering angle  $\theta_{sc}$ ,  $|\vec{v}| = |\vec{v}'|$  (elastic scattering), and  $f'_1$  is used as a shorthand for  $f_1(\vec{v}')$  (see figure). Note that we arrived at this by writing the terms in brackets as  $(f' - f)$  and noticing the  $f_0$  pieces cancel, since the scattering does not change  $|v|$ . The second term in the integral corresponds to “scattering out” and is given by the flux of particles at velocity  $\vec{v}$  i.e.  $f_1 v$  multiplied by the scattering cross-section for scattering into the solid angle  $d\Omega_{sc}$  i.e.  $n \sigma(v, \theta_{sc}) d\Omega_{sc}$  with  $\sigma(v, \theta_{sc})$  the differential scattering cross-section of a single scattering center. The first term gives the “scattering in” process  $\vec{v}' \rightarrow \vec{v}$ , which is governed by the same scattering cross section (by time and space inversion symmetry).

### Example: Electrical conductivity

Lets apply this to the conductivity calculation. We arrive at the equation (for small  $\vec{E}$ ,  $f_1$ )

$$\frac{q}{m} \vec{E} \cdot \hat{v} \frac{\partial f_0}{\partial v} = -f_1 n_s v \int \sigma(v, \theta_{sc}) d\Omega_{sc} + n_s v \int f'_1 \sigma(v, \theta_{sc}) d\Omega_{sc} \quad (1)$$

denoting unit vectors by e.g.  $\hat{v}$ . If we define polar angles for  $\vec{v}$  relative to the field direction  $\vec{E}$ , the left hand side is proportional to  $\cos \theta$ , and since the equation is linear we also expect this dependence for  $f_1(\vec{v})$ , i.e.

$$f_1(\vec{v}) = \cos \theta g_1(v) = \hat{v} \cdot \hat{E} g_1(v)$$

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<sup>1</sup>I presume you are familiar with describing scattering probabilities in terms of *cross sections*. If not, your favorite Quantum textbook can help out.

with  $g_1(v)$  to be found. (You can, if you prefer, expand  $f_1(\vec{v})$  in Legendre polynomials of  $\cos \theta$ , and use the usual orthogonality relationships.) The second integral in Eq. (1) is

$$\int f'_1 \sigma(v, \theta_{sc}) d\Omega_{sc} = g_1(v) \hat{E} \cdot \int \hat{v}' \sigma(v, \theta_{sc}) d\Omega_{sc} \quad (2)$$

with  $\hat{v} \cdot \hat{v}' = \cos \theta_{sc}$ . Now since the integral on the rhs of Eq. (2) is over all  $\hat{v}'$ , the only vector appearing in this integral to set a direction is  $\hat{v}$ , and so we must have

$$\int \hat{v}' \sigma(v, \theta_{sc}) d\Omega_{sc} = A(v) \hat{v} \quad (3)$$

Finally taking the dot product with  $\hat{v}$  gives

$$A = \int \cos \theta_{sc} \sigma(v, \theta_{sc}) d\Omega_{sc} \quad (4)$$

and putting Eqs. (1-4) together gives

$$\frac{qE}{m} \frac{\partial f_0}{\partial v} = -n_s v g_1(v) \int (1 - \cos \theta_{sc}) \sigma(v, \theta_{sc}) d\Omega_{sc}$$

so that

$$f_1(\vec{v}) = \frac{q}{m} \frac{\partial f_0}{\partial v} \frac{1}{n_s v \bar{\sigma}(v)} \hat{v} \cdot \vec{E}$$

Note it is  $\bar{\sigma}(v)$ , a *weighted* integral of the differential scattering cross-section at speed  $v$

$$\bar{\sigma}(v) = \int (1 - \cos \theta_{sc}) \sigma(v, \theta_{sc}) d\Omega_{sc},$$

rather than the total cross-section  $\sigma_t = \int \sigma(v, \theta_{sc}) d\Omega_{sc}$  that appears in this expression. The appearance of the extra term in  $\cos \theta_{sc}$  comes from the scattering in process. The effect of the weighting factor is to reduce the effect of forward scattering—these of course do not much change the current flow, so this makes physical sense. There may be some speed dependence of  $\bar{\sigma}$ : putting  $f_1$  into the expression for the current Eq. (??)

$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E} = \int q \vec{v} f_1(\vec{v}) \quad (5)$$

will show us what velocity average we need to calculate. The final result can be written in the form derived in the previous lecture using the relaxation time approximation  $\sigma = nq^2\tau/m$ , where  $\tau^{-1} = n_s \bar{v} \sigma_{tr}$  with the transport cross-section given by the appropriate speed average of  $\bar{\sigma}(v)$

$$\sigma_{tr} = \frac{\int v^3 \frac{\partial f_0}{\partial v} \bar{\sigma}(v) dv}{\int v^3 \frac{\partial f_0}{\partial v} dv}.$$

### Example: thermal conductivity of an accretion disk

For another example consider the energy transport by light (photons) scattering off nonrelativistic electrons, an important application in astrophysics, e.g. to study accretion onto a black hole (see Ph136 notes, §2.8 for more details).

For photon energies much less than the electron rest mass  $k_B T \ll m_e c^2$  the cross section is the *Thomson cross section* given by simple electromagnetic considerations (Jackson §14.7)

$$\sigma(\theta_{sc}) = \frac{3}{16\pi} \sigma_t [1 + \cos^2 \theta_{sc}] \quad (6)$$

with  $\sigma_t = (8\pi/3)(e^2/m_e c^2)^2$  the total (integrated) cross section. So in this case we have an analytic expression for the scattering.

We proceed as before now with the photon distribution (as a function of momentum or wave vector not velocity!)  $f(\vec{x}, \vec{k}) = f_0(\vec{x}, \vec{k}) + f_1(\vec{x}, \vec{k})$  with  $f_0$  the Planck distribution

$$f_0 = \frac{1}{e^{\beta \hbar c k} - 1}. \quad (7)$$

We now know the collision term of the Boltzmann equation: where does the “driving” on the left hand side come from? The problem we are considering is the heat transport in a temperature gradient, so now  $f_0$  is spatially dependent through  $T(\vec{x})$ . (This is a situation where the collisions relax to a “local equilibrium” fixed by the local temperature, rather than to some global equilibrium.) Again assuming  $f_1$  is small we have

$$c f'_0 \hat{k} \cdot \vec{\nabla} T = \left. \frac{df}{dt} \right|_{coll} \quad (8)$$

$$= -n_e c \int (f_1(\vec{k}) - f_1(\vec{k}')) \sigma(\hat{k} \cdot \hat{k}') d\Omega' \quad (9)$$

writing  $f'_0$  for  $df_0/dT$ .

The argument proceeds exactly as in the electrical conductivity case with  $\vec{\nabla} T$  setting the direction rather than  $\vec{E}$ . So repeating those arguments we get

$$f_1(\vec{k}) = -\frac{f'_0}{n_e \bar{\sigma}} \hat{k} \cdot \vec{\nabla} T \quad (10)$$

where

$$\bar{\sigma} = \int (1 - \cos \theta_{sc}) \sigma(\theta_{sc}) d\Omega_{sc}. \quad (11)$$

In this case, the integral just gives the total cross section  $\sigma_t$ , since the term in  $\cos \theta_{sc}$  cancels by symmetry.

Now calculate the heat transport

$$\vec{J}_Q = 2 \sum_{\vec{k}} f_1(\vec{k}) \hbar c k \hat{k} \quad (12)$$

(2 for two polarizations). Transforming the sum to an integral in the usual way, this gives

$$\vec{J}_Q = \left( \int \frac{d\Omega}{4\pi} \hat{k} \hat{k} \cdot \vec{\nabla} T \right) \frac{\hbar c^2}{\pi^2 n_e \sigma_t} \int_0^\infty k^3 \frac{df_0(k)}{dT} dk. \quad (13)$$

The integral over the Planck distribution can be done to finally give  $\vec{J}_Q = -\kappa \vec{\nabla} T$  with

$$\kappa = \frac{4}{3} \frac{a c T^3}{\sigma_t n_e} \quad \text{with} \quad a = \frac{8\pi^5 k_B^4}{15 h^3 c^3}, \quad (14)$$

( $aT^4$  is the energy density).